UNIVERSITY OF TECHNOLOGY SYDNEY

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES

37233 LINEAR ALGEBRA

Solutions 9

Question 1

Vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal so they form an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) $$\hat{\mathbf{y}}_1 = \text{proj}_W \mathbf{y}_1 = \frac{-75+6+20}{225+4+16}\begin{bmatrix}-15\\2\\4\end{bmatrix} + \frac{0+6-5}{4+1}\begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}3\\0\\-1\end{bmatrix}, \qquad \mathbf{z}_1 = \begin{bmatrix}2\\3\\6\end{bmatrix};$$

$$\hat{\mathbf{y}}_2 = \text{proj}_W \mathbf{y}_1 = \frac{-45-4+0}{225+4+16}\begin{bmatrix}-15\\2\\4\end{bmatrix} + \frac{0-4+0}{4+1}\begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}3\\-2\\0\end{bmatrix}, \qquad \mathbf{z}_2 = \mathbf{0};$$

(b) $\text{dist}(\mathbf{y}_1, W) = \|\mathbf{z}_1\| = 7$ and $\text{dist}(\mathbf{y}_2, W) = 0$.

Question 2

First, we need to make sure the vectors are linearly independent; they obviously are.

Then

$$\mathbf{v}_1 = \begin{bmatrix}0\\4\\3\end{bmatrix}; \qquad \mathbf{v}_2 = \begin{bmatrix}12\\5\\10\end{bmatrix} - \frac{\mathbf{v}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}12\\5\\10\end{bmatrix} - \frac{50}{25}\begin{bmatrix}0\\4\\3\end{bmatrix} = \begin{bmatrix}12\\-3\\4\end{bmatrix}.$$

Normalising these vectors we get $\mathbf{u}_1 = \begin{bmatrix}0\\4/5\\3/5\end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix}12/13\\-3/13\\4/13\end{bmatrix}$.

Question 3

(a) After checking the vectors are linearly independent, apply Gram-Schmidt process:

$$\mathbf{v}_1 = \begin{bmatrix}2\\2\\1\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}2\\1\\0\\0\end{bmatrix} - \frac{4+2+0+0}{4+4+1+0}\begin{bmatrix}2\\2\\1\\0\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-2/3\\0\end{bmatrix}, \qquad \mathbf{v}_2' = \begin{bmatrix}2\\-1\\-2\\0\end{bmatrix},$$

where $\mathbf{v}_2'$ can be introduced (but this is not required) for simplifications, so then

$$\mathbf{v}_3 = \begin{bmatrix}18\\0\\0\\8\end{bmatrix} - \frac{36+0+0+0}{4+4+1+0}\begin{bmatrix}2\\2\\1\\0\end{bmatrix} - \frac{36+0+0+0}{4+1+4+0}\begin{bmatrix}2\\-1\\-2\\0\end{bmatrix} = \begin{bmatrix}2\\-4\\4\\8\end{bmatrix}$$

It makes sense to verify (not shown) that the obtained vectors $\mathbf{v}_i$ are orthogonal.

(b) Normalising these vectors we get

$$\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 0.2 \\ -0.4 \\ 0.4 \\ 0.8 \end{bmatrix}$$

Question 4

To extend the basis for W towards entire $\mathbb{R}^3$, we require a vector of unit length, which is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$. Such a vector is easily found as $\mathbf{u}_3 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

(a) Vectors $\mathbf{q}_i$ are of unit length each and are mutually orthogonal, as can be quickly checked for each pair, or alternatively by calculating $\mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}$. Thus they are three linearly independent orthonormal vectors in $\mathbb{R}^3$ and therefore form a basis for $\mathbb{R}^3$.

(b) Multiplication shows that $\mathbf{Q}\mathbf{Q}^\mathsf{T} = \mathbf{I}$. Given that $\mathcal{Q}$ is an orthonormal basis for $\mathbb{R}^3$, the projection of an arbitrary vector $\mathbf{y} \in \mathbb{R}^3$ onto $Q = \mathbb{R}^3$ is given by

$$\operatorname{proj}_Q \mathbf{y} = \mathbf{Q}\mathbf{Q}^\mathsf{T}\mathbf{y} = \mathbf{I}\mathbf{y} = \mathbf{y}$$

which is certainly true, as the projection of any $\mathbf{y} \in \mathbb{R}^3$ onto $\mathbb{R}^3$ equals to $\mathbf{y}$ itself.

(c) The change of basis matrix can be calculated by row-reducing $[\,\mathbf{q}_1\,\mathbf{q}_2\,\mathbf{q}_3\,|\,\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3\,] \to$

$$\cdots \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -2/3 & -1/3 \\ 0 & 4/5 & 3/5 & 1/3 & 2/3 & -2/3 \\ 0 & 0 & -5/4 & 5/12 & -1/6 & 7/6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & 2/3 & 11/15 & -2/15 \\ 0 & 0 & 1 & -1/3 & 2/15 & -14/15 \end{array}\right]$$

which yields the change of basis orthogonal matrix

$$\mathbf{P}_{\mathcal{Q}\leftarrow\mathcal{U}} \equiv [\,\mathbf{p}_1\,\mathbf{p}_2\,\mathbf{p}_3\,] = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ 2/3 & 11/15 & -2/15 \\ -1/3 & 2/15 & -14/15 \end{bmatrix}$$

(d) The orthogonality can be verified by checking for orthonormality of its columns:

$$\begin{array}{lll} \mathbf{p}_1 \cdot \mathbf{p}_1 = 1, & \mathbf{p}_1 \cdot \mathbf{p}_1 = 1, & \mathbf{p}_1 \cdot \mathbf{p}_1 = 1, \\ \mathbf{p}_1 \cdot \mathbf{p}_2 = 0, & \mathbf{p}_2 \cdot \mathbf{p}_3 = 0, & \mathbf{p}_3 \cdot \mathbf{p}_1 = 0, \end{array}$$

or, likewise, for the rows, which requires essentially the same calculation.

Orthogonality makes it easy to get the matrix for the reverse basis change as

$$\mathbf{P}_{\mathcal{U}\leftarrow\mathcal{Q}} = \mathbf{P}^{-1}_{\mathcal{Q}\leftarrow\mathcal{U}} = \mathbf{P}^\mathsf{T}_{\mathcal{Q}\leftarrow\mathcal{U}} = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -2/3 & 11/15 & 2/15 \\ -1/3 & -2/15 & -14/15 \end{bmatrix}$$

which is much faster than finding that by row-reduction from $[\,\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3\,|\,\mathbf{q}_1\,\mathbf{q}_2\,\mathbf{q}_3\,] \to$

$$\to \left[\begin{array}{ccc|ccc} 2/3 & -2/3 & -1/3 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 4/5 & 3/5 \\ 0 & 0 & 3/2 & -1/2 & -1/5 & -7/5 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -2/3 & 11/15 & 2/15 \\ 0 & 0 & 1 & -1/3 & -2/15 & -14/15 \end{array}\right].$$