University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Tutorial 9 — solutions guide

Question 1

(a) $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\,\mathbf{y} = \dfrac{8\cdot 7 - 6\cdot 1}{8^2+6^2}\begin{bmatrix}8\\-6\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}7\\1\end{bmatrix} - \begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$;

(b) $\text{dist}(\mathbf{y},\,\mathbf{u}) = \|\mathbf{z}\| = 5$.

Question 2

(a) $\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \sqrt{14}, \quad \|\mathbf{y}\| = \sqrt{26}.$

(b) $\mathbf{v}_1\cdot\mathbf{v}_2 = 0, \quad \mathbf{v}_1\cdot\mathbf{y} = 6, \quad \mathbf{v}_2\cdot\mathbf{y} = 7,$ so $\mathbf{v}_1 \perp \mathbf{v}_2$.

(c) Thus $\{\mathbf{v}_1,\,\mathbf{v}_2\}$ is an orthogonal basis for W.

(d) $\hat{\mathbf{y}} = \dfrac{\mathbf{v}_1\cdot\mathbf{y}}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \dfrac{\mathbf{v}_2\cdot\mathbf{y}}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \dfrac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \dfrac{7}{14}\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}$.

(e) $\text{dist}(\mathbf{y},\,W) = \|\mathbf{z}\| = \sqrt{\dfrac{21}{2}}$.

Question 3

By row-reducing $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3\end{bmatrix}$ we check that the vectors are linearly independent.

(a) Then

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}; \qquad \mathbf{v}_2 = \begin{bmatrix}4\\2\\2\\0\end{bmatrix} - \frac{4-2+2-0}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\3\\1\\1\end{bmatrix};$$

$$\mathbf{v}_3 = \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \frac{4-3+2-1}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} - \frac{12+9+2+1}{20}\begin{bmatrix}3\\3\\1\\1\end{bmatrix} = \begin{bmatrix}-0.1\\-0.1\\0.3\\0.3\end{bmatrix}; \quad \mathbf{v}_3' = \begin{bmatrix}-1\\-1\\3\\3\end{bmatrix}.$$

(b) Normalising these vectors we get

$$\mathbf{u}_1 = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}; \qquad \mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}3/2\\3/2\\1/2\\1/2\end{bmatrix}; \qquad \mathbf{u}_3 = \frac{1}{\sqrt{5}}\begin{bmatrix}-1/2\\-1/2\\3/2\\3/2\end{bmatrix}.$$

Question 4

(a) Orthonormality of $\mathbf{u}_1$ and $\mathbf{u}_2$ can be quickly checked by confirming that $\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{I}$:

$$\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$\text{proj}_W \mathbf{y} = \mathbf{U}\mathbf{U}^\mathsf{T}\mathbf{y} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Question 5

If matrix $\mathbf{U}$ is orthogonal, then $\sum\limits_{k=1}^{n} u_{ki}u_{kj} = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Thus, it must be $\sum\limits_{k=1}^{n} u_{ki}u_{kj} = 0$ for $i \neq j$ (for any two different columns), which is impossible to satisfy when all u_{ki} and u_{kj} are positive.

There must be negative values among either u_{ki} or u_{kj}. The same is true for the rows.