35005 **Lebesgue Integration and Fourier Analysis Spring 2024**
Assignment 1

Q1. Let $f : A \to B$ be a bijective function for $A, B \subseteq \mathbb{R}$, so that $f^{-1} : B \to A$ is well defined. Show that if f is continuous and A is compact, then f^{-1} is also continuous.

Q.2 In $\mathbb{R}^n$, the compact sets are exactly the closed and bounded sets. In the metric space $C([0,1])$ of all continuous functions with distance given by $\|f - g\|_\infty = \sup_{0 \le x \le 1} |f(x) - g(x)|$, the compact sets are described by the Arzelà-Ascoli Theorem.

(a) Look up this theorem online or in a book, and describe the compact sets in $C([0,1])$, using the definitions and language from our course.

(b) Explain how the theorem shows that every IVP

$$x'(t) = F(x,t); x(t_0) = x_0$$

where F is continuous in a neighbourhood of (x_0, t_0) has a solution defined in a neigbourhood of t_0.

Be sure to reference the sources you use.

Q.3 (a) Show that if f and g measurable functions, then the functions $f \vee g$ and $f \wedge g$ are measurable, where $f \vee g(x) = \inf\{f(x), g(x)\}$ and $f \wedge g(x) = \sup\{f(x), g(x)\}$.

(b) Find a non-measurable function f so that f^2 is measurable.

You may consult online resources, books etc, (giving appropriate reference) but you must write up the solutions independently of each other. The examiner reserves the right to do a follow-up oral exam.

Due midnight on Friday 6 September