35005 **Lebesgue Integration and Fourier Analysis Spring 2024**
**Assignment 2**

Q1. Let $1 \leq p < \infty$. Show that the simple functions are dense in $L^p$.

Further show that the $p$-th power integrable continuous functions are dense in $L^p$. What happens when $p = \infty$?

Q2. Recall that we proved Lebesgue's Theorem that every measure on $[0, 1]$ can be written as the sum of two measures, one absolutely continuous w.r.t. Lebesgue measure and one disjoint from Lebesgue measure.

A measure $\mu$ defined on the Borel $\sigma$-algebra $\mathcal{B}$ of $\mathbb{R}$ is called **a regular Borel measure** if (i) $\mu(F) < \infty$ for all compact sets $F$;(ii) $\mu(A) = \inf\{\mu(U) : U \text{ is open and } A \subseteq U\}$ for all Borel sets $A$; (iii)$\mu(U) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq U\}$ for all open sets $U$.

(a) Show that for any $x \in \mathbb{R}$, the set function

$$\delta_x(A) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

is a regular Borel measure on $\mathcal{B}$. (It is called the **delta measure at x**.) Deduce that for any sequence $\{x_i\} \subseteq [0, 1]$, and any sequence $\{\alpha_i\}$ of positive real numbers, that $\nu = \Sigma_i \alpha_i \delta_{x_i}$ is a Borel measure on $\mathbb{R}$.

The $x_i$ are called **atoms** of $\nu$, and $\nu$ is called an **atomic measure**.

(b) A measure $\zeta$ is called **continuous** if it has no atoms, ie $\zeta(x) = 0$ for all $x \in [0, 1]$. Show that every regular Borel measure $\mu$ on $\mathbb{R}$ can be written in exactly one way as $\mu = \mu_c + \mu_d$, where $\mu_c$ is continuous and $\mu_d$ is atomic.

(c) Deduce that every regular Borel measure $\mu$ on $\mathbb{R}$ can be written in just one way as

$$\mu = \mu_a + \mu_c + \mu_d$$

where $\mu_a \prec\prec \lambda$, $\mu_c$ is continuous and $\mu_c \perp \lambda$ and $\mu_d$ is atomic. (Here, as usual $\lambda$ is Lebesgue measure.)

Q3. Let $(X, \mathcal{B}$ and $(Y, \mathcal{C})$ be measurable spaces and $(X \times Y, \mathcal{B} \times \mathcal{C})$ be the product measurable space. Let $\mu$ be a probability measure on $(X \times Y, \mathcal{B} \times \mathcal{C})$.

Define the **marginals** of $\mu$ to be the measures $\nu_1$ on $X$ and $\nu_2$ on $Y$ defined by $\nu_1(B) = \mu(B \times Y)$ and $\nu_2(C) = \mu(X \times C)$, for $B \in \mathcal{B}$ and $C \in \mathcal{C}$ respectively. $\mu$ is called a **joining** of $\nu_1$ and $\nu_2$.

(a) Show that $\mu \prec\prec \nu_1 \times \nu_2$ and deduce that the Radon-Nikodým derivative $\rho = \frac{d\mu}{d(\nu_1 \times \nu_2)}$ is defined.

(b) If $f(x, y)$ is measurable on $X \times Y$, write a formula using $\rho$ to express $\int_{X \times Y} f d\mu$ in terms of $\int_X d\nu_1$ and $\int_Y d\nu_2$

(c) Under which conditions is $\mu \sim \nu_1 \times \nu_2$? Give an example where this fails.

*You may consult online resources, books etc, (giving appropriate reference) but you must write up the solutions independently of each other. The examiner reserves the right to do a follow-up oral exam.*

*Due midnight on 11 October*