35005 **Lebesgue Integration and Fourier Analysis Spring 2024**
Assignment 2

Q1. Let $1 \leq p < \infty$. Show that the simple functions are dense in L^p.

Further show that the p-th power integrable continuous functions are dense in L^p. What happens when $p = \infty$?

Q2. Recall that we proved Lebesgue's Theorem that every measure on $[0, 1]$ can be written as the sum of two measures, one absolutely continuous w.r.t. Lebesgue measure and one disjoint from Lebesgue measure.

A measure μ defined on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is called **a regular Borel measure** if (i) $\mu(F) < \infty$ for all compact sets F;(ii) $\mu(A) = \inf\{\mu(U) : U \text{ is open and } A \subseteq U\}$ for all Borel sets A; (iii)$\mu(U) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq U\}$ for all open sets U.

(a) Show that for any $x \in \mathbb{R}$, the set function

$$\delta_x(A) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

is a regular Borel measure on $\mathcal{B}$. (It is called the **delta measure at x**.) Deduce that for any sequence $\{x_i\} \subseteq [0, 1]$, and any sequence $\{\alpha_i\}$ of positive real numbers, that $\nu = \Sigma_i \alpha_i \delta_{x_i}$ is a Borel measure on $\mathbb{R}$.

The x_i are called **atoms** of ν, and ν is called an **atomic measure**.

(b) A measure ζ is called **continuous** if it has no atoms, ie $\zeta(x) = 0$ for all $x \in [0, 1]$. Show that every regular Borel measure μ on $\mathbb{R}$ can be written in exactly one way as $\mu = \mu_c + \mu_a$, where μ_c is continuous and μ_d is atomic.

(c) Deduce that every regular Borel measure μ on $\mathbb{R}$ can be written in just one way as

$$\mu = \mu_a + \mu_c + \mu_d$$

where $\mu_a \prec\prec \lambda$, μ_c is continuous and $\mu_c \perp \lambda$ and μ_d is atomic. (Here, as usual λ is Lebesgue measure.)

Q3. Let $(X, \mathcal{B}$ and $(Y, \mathcal{C})$ be measurable spaces and $(X \times Y, \mathcal{B} \times \mathcal{C})$ be the product measurable space. Let μ be a probability measure on $(X \times Y, \mathcal{B} \times \mathcal{C})$.

Define the **marginals** of μ to be the measures ν_1 on X and ν_2 on Y defined by $\nu_1(B) = \mu(B \times Y)$ and $\nu_2(C) = \mu(X \times C)$, for $B \in \mathcal{B}$ and $C \in \mathcal{C}$ respectively. μ is called a **joining** of ν_1 and ν_2.

(a) Show that $\mu \prec\prec \nu_1 \times \nu_2$ and deduce that the Radon-Nikodým derivative $\rho = \frac{d\mu}{d(\nu_1 \times \nu_2)}$ is defined.

(b) If $f(x, y)$ is measurable on $X \times Y$, write a formula using ρ to express $\int_{X \times Y} f d\mu$ in terms of $\int_X d\nu_1$ and $\int_Y d\nu_2$

(c) Under which conditions is $\mu \sim \nu_1 \times \nu_2$? Give an example where this fails.

You may consult online resources, books etc, (giving appropriate reference) but you must write up the solutions independently of each other. The examiner reserves the right to do a follow-up oral exam.

Due midnight on 11 October