
LEBESGUE INTEGRATION AND FOURIER ANALYSIS ASSIGNMENT 3

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1 Question 1

1.1 a - Weierstrass function

Let $W(x) = \sum_{n=1}^{\infty} a^n \sin(b^n x)$ where $0 < a < 1$ and b is an odd positive integer such that $ab > 1 + \frac{3\pi}{2}$. Show that W is a well-defined continuous function which is nowhere differentiable on $[0, 2\pi]$.

1.1.1 Well definedness

Take $W_k(x) = \sum_{n=1}^k a^n \sin(b^n x)$ to be a sequence of functions such that $W_k \rightarrow W$

The n th term of the Weierstrass sequence can be bounded by a^n for any $x \in [0, 2\pi]$.

$$a^n \sin(b^n x) \leq a^n$$

The sum of these terms is geometric, and since $a < 1$ is convergent.

$$\sum_{n=1}^{\infty} a^n < \infty$$

Therefore by the Weierstrass M test, $W_k \rightarrow W$ uniformly on $[0, 2\pi]$. This means that a sufficient index of W_k can approximate W to an arbitrarily small error, and the series W converges for any value within the domain of discourse $[0, 2\pi]$.

1.1.2 Continuity

Since W_k is sum of continuous functions (sine is continuous, and continuity is closed under addition) that converges uniformly to W , the Uniform Limit Theorem (ULT) asserts that W is continuous.

1.1.3 Nowhere differentiability

This proof will show that the following limit does not exist

$$W'(c) = \lim_{n \rightarrow \infty} \frac{W(x_n) - W(c)}{x_n - c} = \sum_{k=1}^{\infty} a^k \frac{\sin(b^k x_n) - \sin(b^k c)}{x_n - c}$$

Note the following:

1. $|\sin(x) - \sin(y)| \leq |x - y|$ (Integral inequality)
2. $|a + b + c| \geq |a| - |b| - |c|$ (Corollary of triangle inequality)
3. $\exists x_n : x_n \rightarrow c, \frac{\pi}{b^n} \leq x_n - c \leq \frac{3\pi}{b^n}, |\sin(b^n x_n) - \sin(b^n c)| \geq 1$ (Periodicity of terms of W)

Consider partitioning the numerator of the Newtonian quotient into 3 terms (reminiscent of observation 2)

$$\begin{aligned} |W(x_n) - W(c)| &= \left| \sum_{k=1}^{\infty} a^k [\sin(b^k x_n) - \sin(b^k c)] \right| \\ &= |a^n [\sin(b^n x_n) - \sin(b^n c)] + \sum_{k=1}^{n-1} a^k [\sin(b^k x_n) - \sin(b^k c)] + \sum_{k=n+1}^{\infty} a^k [\sin(b^k x_n) - \sin(b^k c)]| \end{aligned}$$

Bound the first term using the sine bound

$$a^n |\sin(b^n x_n) - \sin(b^n c)| \geq a^n$$

Applying the triangle inequality, applying the second sine inequality, the sequence bound

$$\begin{aligned}
\left| \sum_{k=1}^{n-1} a^k (\sin(b^k x_n) - \sin(b^k c)) \right| &\leq \sum_{k=1}^{n-1} a^k |\sin(b^k x_n) - \sin(b^k c)| \leq \sum_{k=1}^{n-1} (ab)^k |x_n - c| \\
&\leq 3\pi \sum_{k=1}^{n-1} a^k = 3\pi \frac{a^n - 1}{a - 1} < \frac{3\pi a^n}{a - 1} \\
\left| \sum_{k=n+1}^{\infty} a^k (\sin(b^k x_n) - \sin(b^k c)) \right| &\geq \sum_{k=n+1}^{\infty} a^k |\sin(b^k x_n) - \sin(b^k c)| \\
&\geq \sum_{k=n+1}^{\infty} a^k |\sin(b^k x_n)| + a^k |\sin(b^k c)| \\
&\geq 2 \sum_{k=n+1}^{\infty} a^k \\
&= \frac{2a^{n+1}}{a - 1}
\end{aligned}$$

Now apply (2) to show unboundedness of this sequence

$$\begin{aligned}
|W(x_n) - W(c)| &\geq a^n - \frac{3\pi a^n}{a - 1} - \frac{2a^{n+1}}{a - 1} \\
&= \frac{a^{n+1} - a^n + 3\pi a^n + 2a^{n+1}}{a - 1} = \frac{3a^{n+1} + (3\pi - 1)a^n}{a - 1} \\
&> \frac{(3\pi + 2)a^{n+1}}{a - 1} > \frac{11a^{n+1}}{a - 1} > 11a^n
\end{aligned}$$

Now let's consider the whole Newtonian quotient and recalling previous bounds set for $x_n - c$

$$\left| \frac{W(x_n) - W(c)}{x_n - c} \right| > \frac{11(ab)^n}{\pi}$$

Since the Weierstrass function is defined to have $ab > 1 + \frac{3\pi}{2}$, this sequence is bound below by a monotone increasing function and hence divergent, proving that the Weierstrass function is nowhere differentiable.

1.2 b - Cantor function

Let \mathcal{C} be the Cantor set in $[0, 1]$. Define $c(x) = \sum_{n=0}^{\infty} \frac{a_n}{2^n}$ if $x = \sum_{n=0}^{\infty} \frac{2a_n}{3} \in \mathcal{C}$, and $c(x) = \sup_{y \leq x, y \in \mathcal{C}} c(y)$ if $x \notin \mathcal{C}$. Show that c is continuous everywhere, and takes on every value between 0 and 1 as x goes from 0 to 1, but that c is almost everywhere differentiable with derivative zero.

1.2.1 Continuity

Continuity on the Cantor set can be hinted towards by analysis by finding a bound when two elements share their first $m - 1$ digits, as well as the fact on non-Cantor intervals c is constant and hence continuous on non-Cantor intervals. $x \neq y \implies \exists m : m = \min\{n \in \mathbb{N} : |x_n - y_n| = 1\}$.

$$|c(x) - c(y)| = \sum_{n=m}^{\infty} \left| \frac{x_n - y_n}{2^n} \right| < \sum_{n=m}^{\infty} \frac{1}{2^n} = 2^{1-m}$$

However there is a much cleaner approach to proving this rather than using δ dependent on the shared binary digits of the outputs of c ; similar to the Weierstrass function, the idea is to use a continuous, uniformly convergent sequence of functions converging to c

$$\begin{aligned}
c_0(x) &= \begin{cases} x & x \in [0, 1] \end{cases} \\
c_1(x) &= \begin{cases} \frac{3x}{2} & x \in [0, \frac{1}{3}] \\ \frac{1}{2} & x \in (\frac{1}{3}, \frac{2}{3}) \\ \frac{3x}{2} - \frac{1}{2} & x \in [\frac{2}{3}, 1] \end{cases} \\
c_n(x) &= \begin{cases} \frac{1}{2}c_{n-1}(3x) & x \in [0, \frac{1}{3}] \\ \frac{1}{2} & x \in (\frac{1}{3}, \frac{2}{3}) \\ \frac{1}{2}c_{n-1}(3x-2) + \frac{1}{2} & x \in [\frac{2}{3}, 1] \end{cases}
\end{aligned}$$

Clearly these are continuous (except for $\{0, 1\}$, since they have no neighborhood on $[0, 1]$), as all pieces are continuous and at the limit points at the ends of the first and third thirds equal that of the constant value of the middle third. This logic can be recursively applied to each sequence due to its fractal nature.

Since the Cantor function (and set) is a fractal when zooming into the thirds to the side of $[0, 1]$ and constant (in the set case, deleted) in the middle third, this recursive sequence reflects this fundamental property of the Cantor set. Hence $c_n \rightarrow c$. Now to show uniform convergence...

One can see by noting the largest differences that $|c_1(x) - c_0(x)| \leq \frac{1}{6} \leq \frac{1}{2^1}$ and $|c_2(x) - c_1(x)| \leq \frac{1}{12} \leq \frac{1}{2^2}$, by POI, let $|c_n(x) - c_{n-1}(x)| \leq \frac{1}{2^n}$ and let's prove that $|c_{n+1}(x) - c_n(x)| \leq \frac{1}{2^{n+1}}$ by considering each of the 3 piecewise parts of the function.

When $x \in [0, \frac{1}{3}]$, Rewrite c_{n+1} in terms of c_n and use the previous assumptions.

$$|c_{n+1}(x) - c_n(x)| = \frac{1}{2}|c_n(3x) - c_{n-1}(3x)| \leq \frac{1}{2} \frac{1}{2^n} = \frac{1}{2^{n+1}}$$

When $x \in (\frac{1}{3}, \frac{2}{3})$, the proof is trivial.

$$|c_{n+1}(x) - c_n(x)| = 0 \leq \frac{1}{2^{n+1}}$$

Finally on $x \in [\frac{2}{3}, 1]$, prove similarly to the first piecewise proof.

$$|c_{n+1}(x) - c_n(x)| = \frac{1}{2}|c_n(3x-2) - c_{n-1}(3x-2)| \leq \frac{1}{2} \frac{1}{2^n} = \frac{1}{2^{n+1}}$$

Now with this result, it is clear that c is uniformly Cauchy since without any dependence on x , and for any n, m above a sufficient N , $|c_m - c_n| \leq \sum_{k=n+1}^m \frac{1}{2^k} < \frac{1}{2^n}$, so one can choose $\varepsilon > \frac{1}{2^n}$ and hence by ULT, c is continuous.

1.2.2 Surjective map from \mathcal{C} to $[0, 1]$

This Cantor function surjectively maps \mathcal{C} to $[0, 1]$ by substituting the partial base-3 system used to define the Cantor set to a base-2 system.

Since $\{a_n\} \subset \{0, 1\}$, $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ can represent any real number in $[0, 1]$ and the Cantor set is defined by $\{a_n\} \subset \{0, 1\}$, $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$, this sequence a_n can be employed in a base-2 sum rather than a 'Cantor' sum to represent the interval $[0, 1]$.

Note that each c_n is increasing, so by the Order Limit Theorem c is also increasing, so the c maps $\mathcal{C} \rightarrow [0, 1]$ from 0 to 1.

Note however that this is not a bijective map since $\sum_{n=1}^{\infty} \frac{\delta_{1n}}{2^n} = \sum_{n=1}^{\infty} \frac{(1-\delta_{1n})}{2^n}$.

1.2.3 Almost everywhere differentiability

It is true that $\lambda(\mathcal{C}) = 0$. This is easily seen by applying σ -additivity of the Lebesgue measure to each partial Cantor set \mathcal{C} since with every iteration the middle third interval of each disjoint interval is removed, so $\lambda(\mathcal{C}_n) = (\frac{2}{3})^n$. This has limit 0, hence the Cantor set has zero measure.

Consider differentiability on the set $[0, 1] \setminus \mathcal{C}$ and note how $\lambda([0, 1]) = \lambda([0, 1] \setminus \mathcal{C})$; differentiability of c on this set can be proved, unlocking almost everywhere differentiability.

For non Cantor set elements, one may restrict the definition of c to

$$c(x) = \sup_{x > y, y \in \mathcal{C}} c(y)$$

$[0, 1] \setminus \mathcal{C}$ is a countable union of disjoint open intervals, and since c has a constant value on these sets (merely echoing the last Cantor- $[0, 1]$ mapping until a Cantor element is reached again) $\forall x \in [0, 1] \setminus \mathcal{C}, c'(x) = \lim_{y \rightarrow x} \frac{c(y) - c(x)}{y - x} = \lim_{y \rightarrow x} \frac{0}{y - x} = 0$.

2 Question 2

Suppose that $\{E_n\}$ is a sequence of measurable sets and m is any fixed positive integer. Let G be the set of points which belong to E_n for at least m different values of n . Show that G is measurable and

$$m\mu(G) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

2.1 Measurability

Note that $G = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^m E_{a_{i,j}}$, where $a_{i,j}$ is a sequence where i enumerates the combinations of intersecting m of the E_n and j enumerates the sets themselves. This will prove vital in proving the subsequent conjectures.

Due to the formalized definition of G stated above, measurability is implied due to the closure of measurability under countable intersections and unions.

2.2 Inequality

Note that G contains points common with **at least** m of the E_n , meaning that there could potentially be several combinations (enumerations of i) containing a point and hence disjointness is not ensured.

$$\mu(G) = \mu\left(\bigcup_{i=1}^{\infty} \bigcap_{j=1}^m E_{a_{i,j}}\right) \leq \sum_{i=1}^{\infty} \mu\left(\bigcap_{j=1}^m E_{a_{i,j}}\right)$$

Consider the function $s(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x)$, this counts the amount of sets an element x belongs to, and $x \in G \iff s(x) \geq m$. To prove the desired inequality, it would be useful to find a bound on this function.

Since $m\chi_G(x) = m \iff s(x) \geq m$ (similar to the previous lemma), it is deduced that

$$\sum_{n=1}^{\infty} \chi_{E_n}(x) \geq m\chi_G(x)$$

Now integrate over both expressions

$$\int_X \sum_{n=1}^{\infty} \chi_{E_n}(x) d\mu \geq \int_X m\chi_G(x) d\mu$$

applying linearity of the Lebesgue integral results in

$$\sum_{n=1}^{\infty} \int_X \chi_{E_n}(x) d\mu \geq m \int_X \chi_G(x) d\mu$$

Since the Lebesgue integral of characteristic functions equals their measure, we have

$$\sum_{n=1}^{\infty} \mu(E_n) \geq m\mu(G)$$

3 Question 3

Let X be the space of all real polynomials of one variable with $\|f\| = \int_0^1 |f(t)|dt$. Put $B(f, g) = \int_0^1 f(t)g(t)dt$. Show that B is a bilinear form on X which is separately continuous but not jointly continuous.

Due to the commutativity of multiplication, B is symmetric; this will be used to apply the same logic to different arguments of B in proofs.

3.1 Bilinearity

Let c be a scalar and h a real polynomial in X , then

$$B(cf + h, g) = \int_0^1 (cf(t) + h(t))g(t)dt = \int_0^1 cf(t)g(t)dt + \int_0^1 h(t)g(t)dt$$

Linearity of the Lebesgue integral allows for the following

$$\begin{aligned} &= \int_0^1 af(t)g(t) + h(t)g(t)dt = \int_0^1 cf(t)g(t)dt + \int_0^1 h(t)g(t)dt \\ &= c \int_0^1 f(t)g(t)dt + \int_0^1 h(t)g(t)dt = cB(f, g) + B(h, g) \end{aligned}$$

By symmetry of B , the same argument can be applied to the second argument and hence B is bilinear.

3.2 Separate continuity

One seeks to prove the following

$$\lim_{n \rightarrow \infty} g_n = g \implies \lim_{n \rightarrow \infty} B(f, g_n) = B(f, g)$$

Pointwise convergence may also be represented as

$$\forall x, \exists N : n > N \implies \|g_n(x) - g(x)\| < \varepsilon$$

Since f is fixed and real valued polynomials are bounded on $[0, 1]$ let $M = \sup_{x \in [0, 1]} \{f(x)\}$.

Now to apply this bounding to f and apply the triangle inequality

$$|B(f, g_n) - B(f, g)| \leq |B(M, g_n) - B(M, g)| \leq |M| \int_0^1 |g_n(t) - g(t)|dt = |M| \|g_n - g\|$$

We have only pointwise convergence to work with, however since g is a real valued polynomial on $[0, 1]$, one can find some other polynomial to dominate it, hence by DCT $\lim_{n \rightarrow \infty} \|g_n - g\| = 0$ or in epsilon format, $\exists N_2 : n > N_2 \implies \|g_n - g\| < \frac{\varepsilon}{M}$

$$|B(f, g_n) - B(f, g)| \leq |M| \|g_n - g\| \leq \frac{|M|\varepsilon}{M} = \varepsilon$$

. Due to the commutativity of multiplication, the same argument applies when keeping the

3.3 Lack of joint continuity

One seeks to find a counterexample for the following

$$\lim_{n \rightarrow \infty} (f_n, g_n) = (f, g) \implies \lim_{n \rightarrow \infty} B(f_n, g_n) = B(f, g)$$

Let f_n be a sequence of functions on $[0, 1]$ of a function constructed such that when in an integral, its limit may not be passed due to noncompliance with BCT, DCT, Uniform Convergence etc. On $[0, 1]$ its limit is 0, which is a polynomial.

$$f_n(x) = nx^n$$

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

$$\|f\| = 0$$

Define f_n as a sequence on $[0, 1)$ that obviously converges uniformly to 1, which is valid as a polynomial on $[0, 1)$.

$$g_n(x) = 1 - \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} g_n(x) = 1$$

$$\|g\| = 0$$

Compute the 'inner product'-like function B on the limits.

$$B(f, g) = \int_0^1 f(x)g(x)dx = \int_0^1 1 \times 0 dx = 0$$

Now compute B on the convergent sequences tending to the limit, and take the limit of this result.

$$\lim_{n \rightarrow \infty} B(f_n, g_n) = \lim_{n \rightarrow \infty} \int_0^1 f_n(x)g_n(x)dx = \lim_{n \rightarrow \infty} \int_0^1 (1 - \frac{1}{n})(nx^n)dx = \lim_{n \rightarrow \infty} \frac{n-1}{n+1} = 1$$

The integrals return different values therefore this is a contradiction as joint continuity states that these should be equivalent.

4 Question 5

Let $f \in L^1(\mathbb{T})$ and $S_N(f) = \sum_{|n| \leq N} \hat{f}(n)e^{inx}$.

4.1 a

Show that $S_N(f) = D_N * f$, where $D_N(x) = \sum_{|n| \leq N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$ is the Dirichlet kernel.

Setting up the Dirichlet kernel to be employed in the convolution gives

$$D_N(x-t) = \sum_{|n| \leq N} e^{inx} e^{-int}$$

Now to set up the convolution.

$$D_N * f = \int_{-\pi}^{\pi} f(t)D_N(x-t)dt$$

$$= \int_{-\pi}^{\pi} f(t) \sum_{|n| \leq N} e^{inx} e^{-int} dt$$

By linearity of the Lebesgue integral, one has the following.

$$= \sum_{|n| \leq N} \int_{-\pi}^{\pi} f(t) e^{inx} e^{-int} dt$$

$$= \sum_{|n| \leq N} e^{inx} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

Using the definition for Fourier coefficients leads to the following, which is also equivalent to the Fourier series of f .

$$= \sum_{|n| \leq N} \hat{f}(n) e^{inx} = S_N(x)$$

4.2 b

Show that $\frac{1}{2\pi} \int D_N(x) dx = 1$ and that $|D_N(x)| \leq \csc(\frac{\delta}{2})$ for $0 < \delta \leq |x| \leq \pi$.

4.2.1 Integral of Dirichlet kernel

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{|n| \leq N} e^{inx} dx \\ &= \sum_{|n| \leq N} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx \end{aligned}$$

Note that

$$\begin{aligned} n \neq 0 &\implies \int_{-\pi}^{\pi} e^{inx} dx = \left[\frac{e^{inx}}{in} \right]_{-\pi}^{\pi} = 0 \\ n = 0 &\implies \int_{-\pi}^{\pi} e^{inx} dx = \int_{-\pi}^{\pi} dx = 2\pi \end{aligned}$$

Now most terms of this series zero out, with the one notable exception.

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i0x} dx \\ &= \frac{2\pi}{2\pi} = 1 \end{aligned}$$

4.2.2 Bounding L^1 norm of Dirichlet kernel

$$|D_N(x)| = \left| \frac{\sin([n + \frac{1}{2}]x)}{\sin(\frac{x}{2})} \right|$$

Since sine goes subzero after the argument surpasses π , and since sine is bounded above by 1, bound it by 1.

$$\leq \left| \frac{1}{\sin(\frac{x}{2})} \right|$$

$\sin(\frac{x}{2})$ is monotone increasing on $[0, \pi]$, so $\sin(\frac{\delta}{2}) \leq \sin(\frac{|x|}{2})$ so therefore when considering reciprocals, we have the following. Also note that the absolute value sign can be removed when considering $\delta \geq 0$.

$$\begin{aligned} &\leq \frac{1}{\sin(\frac{\delta}{2})} \\ &\leq \csc(\frac{\delta}{2}) \end{aligned}$$

5 References

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