

**35005 Lebesgue Integration and Fourier Analysis Spring 2023**  
**Take Home Examination**

*All questions are of equal value.*

*Correct solutions to the equivalent of four of these problems will suffice for a High Distinction grade.*

*Available 9am 6 November; due 9 pm 9 November.*

Q1. (a) Let  $f(x) = \sum_{n=1}^{\infty} a^n \sin b^n x$  where  $0 < a < 1$  and  $b$  is an odd positive integer such that  $ab > 1 + \frac{3\pi}{2}$ .

Show that  $f$  is a well-defined continuous function which is nowhere differentiable on  $[0, 2\pi]$ .

(b) Let  $C$  be the Cantor set in  $[0, 1]$ .

Define  $c(x) = \sum_{n=0}^{\infty} \frac{a_n}{2^n}$  if  $x = \sum_{n=0}^{\infty} \frac{2a_n}{3^n} \in C$ , and  $c(x) = \sup_{y \leq x, y \in C} c(y)$  if  $x \notin C$ .

Show that  $c$  is continuous everywhere, and takes on every value between 0 and 1 as  $x$  goes from 0 to 1, but that  $c$  is almost everywhere differentiable with derivative zero.

Q2. Suppose that  $\{E_n\}$  is a sequence of measurable sets and  $m$  is any fixed positive integer. Let  $G$  be the set of points which belong to  $E_n$  for at least  $m$  different values of  $n$ . Show that  $G$  is measurable and

$$m\mu(G) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Q3. Let  $X$  be the space of all real polynomials of one variable with  $\|f\| = \int_0^1 |f(t)| dt$ .

Put  $B(f, g) = \int_0^1 f(t)g(t) dt$ . Show that  $B$  is a bilinear form on  $X$  which is separately continuous but not jointly continuous.

Q4. Let  $X = \prod_{i=1}^{\infty} \{0, 1\}$  be the infinite product space, equipped with the measure  $\otimes^{\infty} \mu$ , where  $\mu$  is a given fixed probability on the finite set  $\{0, 1\}$ . Let  $\mathcal{E}$  be the **exchangeable**  $\sigma$ -algebra of all measurable sets invariant under the set of all permutations of the index set, ie the mappings  $\pi : (x_i) \mapsto (x_{\pi(i)})$ , where  $\pi$  is a finite permutation of the integers.

(a) Let  $\mathcal{P}_n$  be the  $\sigma$ -algebra of all measurable sets which are invariant under permutations of the integers  $\{1, \dots, n\}$ , ie  $\pi(A) = A$  if  $\pi$  permutes the numbers  $\{1, \dots, n\}$ . Show that  $(\mathcal{P}_n)_{n=1}^{\infty}$  is a *decreasing* family of  $\sigma$  algebras.

(b) If  $f$  is  $\mathcal{P}_n$  measurable, show that  $f(\pi(x)) = f(x)$  whenever  $\pi$  is a permutation of  $\{1, \dots, n\}$

(c) Let  $(f_n)$  be a family of functions such that  $f_n$  is  $\mathcal{P}_n$  measurable and  $f_{n+1} = E(f_n | \mathcal{P}_{n+1})$ . (Called a reverse martingale.) Show that  $f_n$  converges a.e. to a function  $f_{\infty}$  which is invariant under all finite permutations  $\pi$  of the integers, ie  $f_{\infty}(\pi x) = f_{\infty}(x)$  for all  $\pi$  for almost all  $x$ .

Q5. Let  $f \in L^1(\mathbb{T})$  and  $s_N(f) = \sum_{|n| \leq N} \hat{f}(n) e^{inx}$ .

(a) Show that  $S_N(f) = D_N * f$ , where  $D_N(x) = \sum_{|n| \leq N} e^{inx} = \frac{\sin(N + \frac{1}{2})x}{\sin \frac{1}{2}x}$  is the Dirichlet kernel.

(b) Show that  $\frac{1}{2\pi} \int D_N(x) dx = 1$  and that  $|D_N(x)| \leq \csc(\frac{1}{2}\delta)$  for  $0 < \delta \leq |x| \leq \pi$ .

(c) Show that  $\|D_N\|_1 = \frac{4}{\pi^2} \log N + O(1)$ .

Deduce that the Fourier series  $s_N(f)$  diverges for some  $f \in L^1$ .

Q6. Let  $X$  be a random variable on the probability space  $(\Omega, \mathcal{F}, P)$ . Associate to  $X$  the probability  $F(x) = Pr(X \leq x) = P(X^{-1}\{(-\infty, x]\})$   $x \in \mathbb{R}$ .

(a) Suppose that  $F$  is differentiable, and let  $f(x)$  be the derivative  $F'(x)$ . Show that for every  $h \in L^1(\Omega)$

$$E(h(X)) = \int_{\mathbb{R}} h(x) f(x) dx.$$

(b) Define the **characteristic function** of  $X$  to be  $\varphi_X(t) = E(e^{-itX}) = \int_{-\infty}^{\infty} e^{-itx} f(x) dx$ . Show that if for an integer  $p$ ,  $X \in L^p(\mathbb{R})$ , then

$$E(X^p) = (i)^p \frac{d^p}{dt^p} \varphi_X(t) |_{t=0}.$$

(c) Two random variables  $X, Y$  are called **independent** if  $E(h(X)g(Y)) = E(h(X))E(g(Y))$  for all Borel functions  $h$  and  $g$ . Show that if  $X$  and  $Y$  are independent random variables with probability densities  $f_X$  and  $f_Y$  respectively, then the probability density of  $Z = X + Y$  is the convolution.

$$f_Z(x) = f_X * f_Y$$

*You may consult online resources, books etc, (giving appropriate reference) but not each other. The examiner reserves the right to do a follow-up oral exam.*