Modern Analysis

Problem sheet one.

(1) Prove that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is uniformly convergent on $\mathbb{R}$.

(2) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on $X \subseteq \mathbb{R}$. How may we use the Weierstrass M test to prove that the sequence converges uninformly?

(3) Prove Riemann's Criterion for the existence of the Riemann integral.

(4) Prove that if f and g are Riemann integrable on $[a, b]$, then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}.$$

(5) Prove Theorem 2.2 in the lecture notes.

(6) From the definition of the Riemann-Stieltjes integral, prove that RS $\int_0^1 x d(x^2) =$ R $\int_0^1 2x^2 dx$.

(7) Calculate RS $\int_{-1}^1 x^2 d(x|x|)$, RS $\int_{-1}^1 x^2 d(x^2)$ and RS $\int_0^1 \cos x d(\sin x)$.

(8) Prove that if f is continuous and monotone increasing, then

$$\text{RS} \int_a^b f(x) d(f(x)) = \frac{1}{2}(f(b)^2 - f(a))^2.$$

(9) Use the Euler-Maclaurin formula to estimate the following quantities for large N.

(i) $\sum_{n=1}^{N} \frac{\sin(\sqrt{n})}{n}$

(ii) $\gamma_N = \sum_{n=1}^{N} \frac{1}{n} - \ln N.$

(10) Use the Euler-Maclaurin formula to prove the integral test for series convergence: If f is a continuous function on $\mathbb{R}$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if R $\int_1^{\infty} f(x)dx < \infty$.

(11) Calculate the sums $\sum_{k=1}^{n} k^2$, and $\sum_{k=1}^{n} k^3$.

(12) Find $\lim_{n\to\infty}$ RS $\int_0^{\frac{\pi}{2}} (1 - \frac{x}{n})^n d(\cos x)$.

Modern Analysis

Problem sheet two.

(1) Prove Theorem 3.11 in the lecture notes.

(2) Prove that

$$m((0,1]) = \sum_{k=1}^{\infty} m\left(\left(\frac{1}{k+1}, \frac{1}{k}\right]\right).$$

(3) Prove that any countable set is measurable and has measure zero.

(4) What is the measure of the irrational numbers in $[0,1]$?

(5) Prove that if $m^*(A) = 0$ then A is measurable.

(6) Prove Theorem 3.13 in the lecture notes.

(7) Let $A \subset \mathbb{R}$ be a measurable set. For $h \in \mathbb{R}$, define

$$A + h = \{x + h | x \in A\}.$$

Prove that $A + h$ is measurable and $m(A + h) = m(A)$.

(8) Let E, F be measurable and assume $E \subset F$, with $m(F) < \infty$. Prove that $m(F - E) = m(F) - m(E)$.

(9) Show that for any two sets A, B with $A \cup B = [0, 1]$,

$$m^*(A) \geq 1 - m^*(B).$$

(10) Let

$$A = \{x \in [0,1] : \text{no 5s occur in the decimal expansion of } x\}.$$

Find $m^*(A)$.

(11) Suppose that A is a bounded set and $m^*(A \cap I) \leq \frac{1}{2} m^*(I)$ for every interval I. Prove that $m^*(A) = 0$.

(12) (Hard) Prove that intervals are measurable by verifying that the Caratheodory condition is satisfied.

Problem sheet three.

(1) Show that if A_1, A_2 are measurable then
$$m(A_1) + m(A_2) = m(A_1 \cup A_2) + m(A_1 \cap A_2).$$

(2) Let X be a nonempty set, and let $f : X \to [0, \infty)$ be a function. Let $P(X)$ be the collection of all subsets of X. Define $\mu : P(X) \to [0, \infty)$ by $\mu(A) = \sum_{x \in A} f(x)$ if A is a nonempty, countable set, $\mu(A) = \infty$ is A is uncountable and $\mu(\emptyset) = 0$. Show that μ is a measure.

(3) Consider the Cantor set. This is formed by taking the interval $C_1 = [0, 1]$ and removing the middle third $(0, 1)$. So $C_2 = [0, 1/3] \cup [2/3, 1]$. Then remove the middle third from each of these intervals. So $C_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 5/9] \cup [8/9, 1]$. Continue this process indefinitely. The Cantor set is defined to be
$$C = \cap_{n=1}^{\infty} C_n.$$
Prove that the Cantor set is nonempty, with infinitely many points and that $m(C) = 0$. In fact the Cantor set is uncountable. So it provides an example of an uncountable set of Lebesgue measure zero.

(4) Show that a countable union of sets of measure zero has measure zero.

(5) If m^* is Lebesgue outer measure on $\mathbb{R}$ and A is a null set (one with outer measure zero), then
$$m^*(B) = m^*(A \cup B) = m^*(B \setminus A)$$
holds for every subset B of $\mathbb{R}$.

(6) Let m^* be outer measure on $\mathbb{R}$. If a sequence of subsets $\{A_n\}$ of $\mathbb{R}$ satisfies $\sum_{n=1}^{\infty} m^*(A_n) < \infty$, then the set
$$E = \{x \in X : x \in A_n \text{ for infinitely many } n\},$$
is a null set.

(7) Prove that χ_A is measurable if and only if A is measurable.

(8) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is measurable. Prove that the composition $f \circ g$ is also measurable.

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the derivative f' is Lebesgue measurable.

Problem sheet four.

(1) Consider a sequence of functions (f_n), where each $f_n : \mathbb{R} \to \mathbb{R}$ is measurable. Let f be a measurable function. The sequence (f_n) is said to converge in measure to f if, for any $\epsilon > 0$,

$$\lim_{n\to\infty} m[\{x : |f_n(x) - f(x)| \geq \epsilon\}] = 0.$$

Prove that if $f_n \to f$ uniformly, then (f_n) converges in measure to f.

(2) Let (f_n) and (g_n) be sequences of almost everywhere real, measurable functions, that converge in measure to f and g respectively. Let a, b be real numbers. Prove the following.

(i) $(af_n + bg_n)$ converges in measure to $af + bg$.

(ii) $(|f_n|)$ converges in measure to $|f|$.

(iii) $(f_n g_n)$ converges in measure to (fg), on X with $m(X) < \infty$.

(iv) $(f_n g)$ converges in measure to (fg), on X with $m(X) < \infty$.

(3) Prove that if $k \leq f \leq K$ a.e. on a measurable set E, then

$$km(E) \leq \int_E f \leq Km(E).$$

(4) Let f be an integrable function that is positive everywhere on a measurable set E. If $\int_E f = 0$ prove that $m(E) = 0$.

(5) Prove that $\int_0^\infty \sin(x^2)dx$ is not Lebesgue integrable, but exists as an improper Riemann integral.

(6) Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Assume that f is differentiable at $x = 0$ and $f(0) = 0$. Show that the function defined by $g(x) = x^{-\frac{3}{2}} f(x)$ for all $x \in (0, 1]$ and $g(0) = 0$ is Lebesgue integrable.

(7) Find the Lebesgue integral over $[0, 1]$ of the function

$$f(x) = \begin{cases} x^3 + 5x, & x \in [0, 1] - \mathbb{Q} \\ 2x^2, & x \in [0, 1] \cap \mathbb{Q}. \end{cases}$$

Problem sheet five.

(1) Prove that if $\varphi(x)$ is a continuous nondecreasing function in $[a, b]$, then $\varphi'(x)$ is Lebesgue integrable and

$$\int_a^b \varphi'(x)dx \leq \varphi(b) - \varphi(a).$$

(2) Show that

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

(3) Show that for $t \geq 0$

$$\int_0^\infty e^{-xt} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} t.$$

(4) Prove that if f is continuously differentiable, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0.$$

(5) Assume that $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a, \infty)$. Prove that $\int_a^\infty f(x)dx$ exists as an improper Riemann integral, if and only if for every $\epsilon > 0$ there exists an M such that $\left|\int_s^t f(x)dx\right| < \epsilon$ for all $s, t > M$.

The previous result is useful because of the following theorem which you may assume. Let $f : [a, \infty) \to \mathbb{R}$ be Riemann integrable on every closed subinterval of $[a, \infty)$. Then f is Lebesgue integrable if and only if the improper Riemann integral $\int_a^\infty |f(x)|dx$ exists. In this case

$$L \int f dm = R \int_a^\infty f(x)dx.$$

(6) Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

(Integrate by parts and use the convergence theorems).

(7) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function with the left and right limits of the derivatives defined at the end points. If the derivative f' is bounded on $[a,b]$, prove that f' is Lebesgue integrable and

$$\int_{[a,b]} f' dm = f(b) - f(a).$$

(8) Show that $f(x) = \frac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and that $\int f dm = 1$.

(9) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \delta$. Show that

$$\lim_{n\to\infty} \int_0^a f(nx)dx = a\delta$$

for each $a > 0$.

37438 Problem sheet six.

(1) Calculate $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$.

(2) Calculate $\displaystyle\int_0^1 \frac{x-1}{\ln x} dx$. Hint, look at $\int_0^1 \frac{x^p - 1}{\ln x} dx$.

(3) Evaluate $\displaystyle\int_0^\infty e^{-\alpha^2 x^2 - \beta^2/x^2} dx$.

(4) Show that if $d(x,y)$ is a metric on a space X, then

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric on X.

(5) Assume that two vectors $x, y \in X$ where X is a normed linear space satisfy the relation $\|x + y\| = \|x\| + \|y\|$. Show that for all non negative scalars α, β we have

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|.$$

(6) It is true that every norm defines a metric by $d(x,y) = \|x - y\|$. Show by example that not every metric defines a norm.

(7) Let $f : X \to \mathbb{R}$.

$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ holds for almost all } x\}$.

Prove the following.

(i) If $f = g$ a.e., then $\|f\|_\infty = \|g\|_\infty$.

(ii) $\|f\|_\infty \geq 0$ for each function f, and $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.

(iii) $\|af\| = |a|\|f\|_\infty$ for all scales a.

(iv) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(v) If $|f| \leq |g|$ then $\|f\|_\infty \leq \|g\|_\infty$.

(8) Two norms $\|x\|_1$ and $\|x\|_2$ on a vector space X are said to be equivalent if there exist constants $K > 0$ and $M > 0$ such that

$$K\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Prove that on a finite dimensional vector space all norms are equivalent.

(9) Let $C^1[0,1]$ be the vector space of all real valued functions on $[0,1]$ with continuous first derivative. Show that $\|f\| = |f(0)| + \|f'\|_\infty$ is a norm and that it is equivalent to the norm

$$\|f\|_A = \|f\|_\infty + \|f'\|_\infty.$$

Hint: $f(x) = f(0) + \int_0^x f'(t)dt$.

(10) let X, Y be normed linear spaces. An operator $T : X \to Y$ is said to be bounded if there exists $K > 0$ such that

$$\|T(x)\|_Y \leq K\|x\|_E.$$

Consider $C[a,b]$ with the norm defined in question 4. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Show that

$$T(f)(x) = \int_a^b K(x,y)f(y)dy$$

is a bounded linear operator. i.e There exists $M > 0$ such that $\|Tf\|_\infty \leq M\|f\|_\infty$ for all $f \in C[a,b]$.

(11) Let $D : C^1[0,1] \to C^1[0,1]$ be given by $Df = f'$. Use the norm of question 4 and show that with respect to this norm, differentiation is an unbounded linear operator. (Hint. Find an example of a function whose norm grows with derivatives).

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be n times differentiable and assume that $f^{(n)}$ is integrable for all n. Prove that

$$\widehat{f^{(n)}}(y) = (iy)^n \widehat{f}(y).$$

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $xf(x)$ is Lebesgue integrable. Prove that

$$\widehat{xf(x)}(y) = i\frac{d}{dy}\widehat{f}(y).$$

(14) Solve the differential equation

$$u'(x) + xu(x) = 0, \; u(0) = \sqrt{2\pi}.$$

Now take the Fourier transform of this equation and hence show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-iyx} dx = \sqrt{2\pi} e^{-\frac{y^2}{2}}.$$

37438 Problem sheet seven.

(1) Show that in a real inner product space $(x, y) = 0$ holds if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Does the same result hold if we allow the inner product to be complex valued?

(2) Assume that the sequence $\{x_n\}$ in an inner product space satisfies $(x_n, x) \to \|x\|^2$ and $\|x_n\| \to \|x\|$. Show that $x_n \to x$.

(3) A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to x in H if $(x_n, y) \to (x, y)$ for all $y \in H$.

 (i) Show that if a sequence is convergent it is also weakly convergent.

 (ii) Show that if a sequence is weakly convergent, then the limit is unique.

 (iii) Show by an example that a sequence may be weakly convergent, but not convergent.

(4) Let H be a Hilbert space with inner product $(\cdot,\cdot) : H \times H \to \mathbb{C}$. Prove that for all $x, y \in H$

$$(x, y) = \frac{1}{4}\left([\|x+y\|^2 - \|x-y\|^2] + i[\|x+iy\|^2 - \|x-iy\|^2]\right).$$

This is known as the polarisation identity and it used to recover the inner product from the norm.

(5) Given an example of a function f such that $f \in L^2(\mathbb{R})$ but $f \notin L^1(\mathbb{R})$. Then find an example of a function such that $f \in L^1(\mathbb{R})$, but $f \notin L^2(\mathbb{R})$.

(6) Let $[a, b]$ be a closed, bounded interval. Define the spaces $L^p([a, b])$ in the obvious way. If $f \in L^1([a, b])$ does it follow that $f \in L^2([a, b])$? What about the converse?

(7) Let $p > 1$, $p \neq 2$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

(8) Let $f \in L^2([0, 1])$ satisfy $\|f\|_2 = 1$ and $\int_0^1 f(x)dx \geq \alpha > 0$. For each $\beta \in \mathbb{R}$, define $E_\beta = \{x \in [0, 1] : f(x) \geq \beta\}$. If $0 < \beta < \alpha$ show that $m(E_\beta) \geq (\beta - \alpha)^2$. (Hint: This uses Hölder's inequality. Note that $f - \beta \leq (f - \beta)\chi_{E_\beta} \leq f\chi_{E_\beta}$.)

(9) Suppose that $\{\phi_n\}$ is an orthonormal set in the Hilbert space $L^2([-1, 1])$. Show that the sequence $\{\psi_n\}$ defined by

$$\psi_n(x) = \left(\frac{2}{b-a}\right)^{1/2} \phi_n\left(\frac{2}{b-a}\left(x - \frac{b+a}{2}\right)\right)$$

is an orthonormal set in $L^2([a,b])$.

(10) Let $1 \leq p < \infty$ and suppose $f \in L^p([a,b])$, where $a < b$, $a, b \in \mathbb{R}^*$ and let $\epsilon > 0$. Show that

$$m^*(\{x \in [a,b] : |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int_a^b |f(x)|^p dx.$$

where m^* is Lebesgue outer measure.

(11) Let $1 \leq p < \infty$ and suppose $\{f_n\}$ is a sequence in $L^p([a,b])$, where $a < b$, $a, b \in \mathbb{R}^*$. Prove that if $\|f_n - f\|_p \to 0$ as $n \to \infty$, then $f_n \to f$ in measure.

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37438 Problem sheet eight.

(1) Solve the PDE
$$u_t = u_{xx} + h(x), \quad h \in L^1(\mathbb{R}), \quad x \in \mathbb{R}$$
subject to the initial condition $u(x, 0) = f(x)$ and the assumption that $u(x, t), u_x(x, t) \to 0$ as $|x| \to \infty$.

(2) Solve the Poisson equation
$$u_{xx} + u_{yy} = h(x), \quad h \in L^1(\mathbb{R}) \quad x \in \mathbb{R}, \ y \geq 0$$
subject to the condition $u(x, 0) = f(x)$ and the assumption that $u(x, y), u_x(x, y) \to 0$ as $|x| \to \infty$.

(3) Solve the integral equation
$$u(x) = h(x) + \int_{-\infty}^{\infty} k(x - y)u(y)dy,$$
where h and k and their Fourier transforms are integrable.

(4) Use Parseval's identity to evaluate the integrals
(a) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

(b) $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$

(5) Define the function h_λ by
$$h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|y|} e^{iyx} dy.$$
Prove that
(a) $h_\lambda(x) = \dfrac{2\lambda}{\lambda^2 + x^2}$,

(b) $\displaystyle\int_{-\infty}^{\infty} h_\lambda(x)dx = 2\pi.$

(c) $h_\lambda(x) = \dfrac{1}{\lambda} h_1\left(\dfrac{x}{\lambda}\right).$

The convolution of two functions f and g is defined by
$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy.$$
(d) Prove that if f is integrable, then for every $\lambda > 0$
$$f * h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|y|} \hat{f}(y) e^{ixy} dy.$$

(e) Prove that for all $f \in L^1(\mathbb{R})$, $\lim_{\lambda \to 0} f * h_\lambda = 2\pi f$. Deduce from this the Fourier inversion Theorem. (Hint: Consider $f * h_\lambda - 2\pi f$).

The function h_λ is known as an approximation of the identity.

(6) Suppose that $f \in L^1(\mathbb{R})$ and that both f', f'' exist and are continuous and integrable. Prove that the Fourier transform $\widehat{f} \in L^1(\mathbb{R})$.

(7) Suppose that $f, f_n \in L^1([-\pi, \pi])$ and that $f_n \to f$. Prove that the Fourier coefficients satisfy $\widehat{f_n} \to \widehat{f}$.

(8) Prove the Weierstrass approximation theorem: If $f \in C([-\pi, \pi])$, then, given any $\epsilon > 0$, there is a polynomial $p(x)$ such that

$$\sup_{x \in [-\pi, \pi]} |f(x) - p(x)| < \epsilon.$$

(Hint: Use a Fourier series).

(9) Let f be 2π periodic and integrable on $[-\pi, \pi]$. Let $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixn} dx$.

(a) Show that $\widehat{f}(n) = -\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) e^{-inx} dx$.

(b) Use the result of (a) and the DCT to prove that if f is continuous, then $\widehat{f}(n) \to 0$ as $n \to \infty$. (Hint: Find an expression for $2\widehat{f}(n)$).

(c) Prove that if for all x there is a $C > 0$ and $0 < \alpha \leq 1$, such that $|f(x+h) - f(x)| \leq C|h|^\alpha$, then $\widehat{f}(n) = O(1/|n|^\alpha)$.

(10) Let the Theta function be defined by

$$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-t\pi k^2}.$$

Prove that $\Theta(t) = \dfrac{1}{\sqrt{t}} \Theta(\dfrac{1}{t})$.

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Problem sheet nine.

(1) Prove that the product measure of two measures μ and ν is a measure. That is, prove that $(\mu \times \nu)(A \times B) \geq 0$ and

$$(\mu \times \nu)(\cup_{i=1}^{\infty}(A_i \times B_i) = \sum_{i=1}^{\infty}(\mu \times \nu)(A_i \times B_i).$$

(2) Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ with $f(0, 0) = 0$. Evaluate

$$\int_0^1 \int_0^1 f(x, y)dxdy \text{ and } \int_0^1 \int_0^1 f(x, y)dydx.$$

Explain your answer.

(3) If λ, μ and ν are finite measures such that $\lambda << \nu$ and $\nu << \mu$, show that

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}.$$

(4) Suppose that μ and ν are finite measures and that $\mu << \nu$ and $\nu << \mu$. Prove that

$$\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1.$$

(5) Calculate the moment generating function of a normal random variable.

(6) Let λ be Lebesgue measure. Suppose that $\mu(E) = \int_E f d\lambda$ and that the measure μ satisfies $\mu(aE) = \mu(E)$ for all $a > 0$ and each measurable subset E of $(0, \infty)$. The aim of this question is to compute the Radon-Nikodym derivative f.

(i) Let $E = [1, x]$. What is aE?

(ii) Show that f satisfies $\int_1^x f(t)dt = \int_a^{ax} f(t)dt$

(iii) Show that $f(x) = c/x$ for some constant c.

(7) Let X be a Banach space. Show that if L is a linear functional on X and L is continuous at $a \in X$, then L is uniformly continuous on the whole of X.

(8) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of subsets of Ω. Define

$$\begin{aligned}\overline{\lim_{n\to\infty}} A_n &= \bigcap_{m=1}^{\infty} \bigcup_{n\geq m} A_n \\ &= \{\omega : \omega \in A_n \text{for infinitely many } n\}.\end{aligned}$$

Now suppose that each A_n is measurable. Prove the Borel-Cantelli Lemma:

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(\overline{\lim}_{n\to\infty} A_n) = 0.$$

Conversely, if the A_n are P independent sets: $P(A_n \cap A_m) = P(A_n)P(A_m)$; then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P(\overline{\lim}_{n\to\infty} A_n) = 1.$$

(9) If X and Y are independent prove that $Var(X+Y) = Var(X) + Var(Y)$.

(10) If X_n are independent random variables on $(\Omega, \mathcal{F}, P)$, with $E(X_n) = \mu$, $Var(X_n) \leq K < \infty$ prove the weak law of large numbers: $\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu$ in L^2.

(11) Show that if X and Y are random variables on a probability space, then $d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)$ is a metric and that convergence in d is equivalent to convergence in the probability measure P.

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Problem sheet one solutions.

Question 1.

We use the Weierstrass M test. We let $f_n(x) = \frac{1}{n^2}\sin(nx)$ and we have the inequality $|f_n(x)| \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$ it follows that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Question 2.

The Weierstrass test is for series. So we need a series whose nth term is f_n. Thus we need a telescoping series. So we let $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ for $n \geq 2$. Then

$$\sum_{k=1}^{n} g_n = f_1 + f_2 - f_1 + f_3 - f_2 + \cdots + f_{n-1} + f_n - f_{n-1} = f_n.$$

The Weierstrass M test tells us that if $|g_n(x)| \leq M_n$ for all $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$, then the series converges uniformly on X with limit f. So if $|f_n(x) - f_{n-1}(x)| \leq M_n$ then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent. So if would be sufficient to establish (as an example) that for all $x \in X$

$$|f_n(x) - f_{n-1}(x)| \leq \frac{1}{n^a},\ a > 1$$

in order to guarantee that $f_n \to f$ uniformly.

Question 3.

Suppose that f is Riemann integrable. Then the lower integral

$$\underline{\int_a^b} f = \int_a^b f.$$

Consequently, given $\epsilon > 0$ there exists a partition P_1 of $[a, b]$ such that $L(f, P_1) \geq \underline{\int_a^b} f - \epsilon/2$. Similarly there is a partition P_2 such that $U(f, P_2) = \overline{\int_a^b} f + \epsilon/2$. Here $\overline{\int_a^b} f$ denotes the upper integral. Thus if $P = P_1 \cup P_2$ then

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) \\ &< \overline{\int_a^b} f + \epsilon/2 - \left(\underline{\int_a^b} f - \epsilon/2\right) \\ &< \epsilon, \end{aligned}$$

as the upper and lower integrals are equal. Conversely, let $\epsilon > 0$ and suppose that there is a partition P such that

$$\overline{\int_a^b} f \leq U(f, P) < L(f, P) + \epsilon \leq \underline{\int_a^b} f + \epsilon. \tag{0.3}$$

Then

$$\left| \overline{\int_a^b} f - \underline{\int_a^b} f \right| < \epsilon. \tag{0.4}$$

So f is Riemann integrable.

Question 4.

Let f, g be Riemann integrable on $[a, b]$ and observe that

$$\int_a^b (x|f(t)| + |g(t)|)^2 \, dt = x^2 \int_a^b |f(t)|^2 dt + 2x \int_a^b |f(t)g(t)| dt$$
$$+ \int_a^b |g(t)|^2 dt = Ax^2 + 2Bx + C \geq 0.$$

Since $Ax^2 + 2Bx + C \geq 0$ it follows that $B^2 \leq AC$. Which gives

$$\left(\int_a^b |f(t)g(t)| dt \right)^2 \leq \int_a^b |f(t)|^2 dt \int_a^b |g(t)|^2 dt.$$

Now take the square root of both sides.

Question 5.

Assume that f is continuous on $[a, b]$ and let

$$\phi(x) = \begin{cases} \beta_1 & a \leq x < \widehat{x} \\ \beta & x = \widehat{x} \\ \beta_2 & \widehat{x} < x \leq b. \end{cases}$$

Then ϕ is a step function with one jump at $\widehat{x}$, for $a < \widehat{x} < b/$. Let P be a partition of $[a, b]$. If $\widehat{x}$ is a partition point, $\widehat{x} = x_K$ for $0 < K < n$, then $x_{K-1} \leq c_k \leq \widehat{x} = x_K \leq c_{K+1} \leq x_{K+1}$ and

$$\sum_P f \Delta\phi = [f(c_K) - f(\widehat{x})](\beta - \beta_1) + [f(c_{K+1}) - f(\widehat{x})](\beta_2 - \beta)$$
$$+ f(\widehat{x})(\beta_2 - \beta_1).$$

As we let $|P| \to 0$, by continuity the first two terms vanish and so we have

$$RS \int_a^b f d\phi = f(\widehat{x})(\beta_2 - \beta_1)$$
$$= f(\widehat{x})(\phi(\widehat{x}^+) - \phi(\widehat{x}^-)).$$

If $\widehat{x}$ is not a partition point, then we have $x_{K-1} < \widehat{x} < x_K$ and $x_{K-1} \leq c_K \leq x_K$ for some K. So that

$$\sum_P f \Delta\phi = [f(c_K) - f(\widehat{x})](\beta_1 - \beta_2) + f(\widehat{x})(\beta_2 - \beta_1)$$

and again continuity shows that as $|P| \to 0$, the first term disappears and again

$$RS \int_a^b f d\phi = f(\widehat{x})(\phi(\widehat{x}^+) - \phi(\widehat{x}^-)).$$

Finally we extend to the case when ϕ has n jump points $\widehat{x}_1, ..., \widehat{x}_n$ by using the fact that if $a_1 < a_2 < \cdots a_n$, then

$$RS \int_{a_1}^{a_n} = RS \int_{a_1}^{a_2} + \cdots + RS \int_{a_{n-1}}^{a_n}.$$

Question 6.

Recall that if f is Riemann integrable on $[a,b]$ and $P = \{x_1, ..., x_n\}$ is a partition of $[a,b]$, then the Riemann sum $\sum_{k=1}^n f(x_k^*)(x_k - x_{k-1}) \to \int_a^b f(x)dx$ as the partition length $|P| \to 0$.

We take $f(x) = x$, and choose a partition $\{x_1, ..., x_n\}$ of $[0,1]$ and let c_k satisfy $x_{k-1} \le c_k \le x_k$ for all k. The inequality

$$\sum_{k=1}^n x_{k-1}(x_k^2 - x_{k-1}^2) \le \sum_{k=1}^n c_k(x_k^2 - x_{k-1}^2) \le \sum_{k=1}^n x_k(x_k^2 - x_{k-1}^2)$$

is obvious. Simple algebra gives

$$\sum_{k=1}^n x_k(x_k^2 - x_{k-1}^2) = \sum_{k=1}^n x_k^2(x_k - x_{k-1}) + \sum_{k=1}^n x_k x_{k-1}(x_k - x_{k-1}).$$

Similarly for the first sum and so we have

$$\sum_{k=1}^n x_{k-1}^2(x_k - x_{k-1}) + \sum_{k=1}^n x_k x_{k-1}(x_k - x_{k-1}) \le \sum_{k=1}^n c_k(x_k^2 - x_{k-1}^2) \le$$
$$\sum_{k=1}^n x_k^2(x_k - x_{k-1}) + \sum_{k=1}^n x_k x_{k-1}(x_k - x_{k-1}).$$

It is also clear that

$$\sum_{k=1}^n x_k x_{k-1}(x_k - x_{k-1}) < \sum_{k=1}^n x_k^2(x_k - x_{k-1})$$

since $x_k > x_{k-1}$. We therefore have

$$2\sum_{k=1}^n x_{k-1}^2(x_k - x_{k-1}) \le \sum_{k=1}^n c_k(x_k^2 - x_{k-1}^2) \le 2\sum_{k=1}^n x_k^2(x_k - x_{k-1}). \quad (*)$$

Now $2\sum_{k=1}^n x_{k-1}^2(x_k - x_{k-1})$ is a Riemann sum for the integral $2\int_0^1 x^2 dx$ as is $2\sum_{k=1}^n x_k^2(x_k - x_{k-1})$. So that as $|P| \to 0$

$$2\sum_{k=1}^n x_{k-1}^2(x_k - x_{k-1}) \to 2\int_0^1 x^2 dx$$

and

$$2\sum_{k=1}^{n} x_k^2(x_k - x_{k-1}) \to 2\int_0^1 x^2 dx.$$

Since $\sum_{k=1}^n c_k(x_k^2 - x_{k-1}^2) \to RS\int_0^1 xd(x^2)$, the inequality (*) then allows us to conclude that $RS\int_0^1 xd(x^2) = 2\int_0^1 x^2dx$.

Question 7.

Let $g(x) = x|x|$. then $g'(x) = 2x^2$ if $x > 0$ and $g'(x) = -2x^2$ if $x < 0$. So that

$$RS\int_{-1}^{1} xd(x|x|) = \int_0^1 2x^3dx - \int_{-1}^{0} 2x^3dx = 1.$$

The other parts of the question may be done the same way.

Question 8

Integration by parts gives

$$RS\int_a^b f(x)d(f(x)) = f(b)f(b) - f(a)f(a) - RS\int_a^b f(x)d(f(x)).$$

Rearranging this gives the result.

Question 9

The most efficient way to do this question is to evaluate the various terms in the Euler-MacLaurin summation formula in Mathematica.

(i) For the first sum we have

$$\sum_{k=1}^{N}\frac{\sin(\sqrt{k})}{k} = \int_1^N \frac{\sin(\sqrt{x})}{\sqrt{x}}dx + \frac{1}{2}(\sin 1 + \frac{1}{N}\sin(\sqrt{N}))$$
$$+ \frac{1}{12}[f'(N) - f'(1)] + \cdots,$$

where $f(x) = \sin(\sqrt{x})/\sqrt{x}$. So that $f'(x) = \frac{\cos(\sqrt{x})}{2x} - \frac{\sin(\sqrt{x})}{2x^{3/2}}$. We can as many terms as we want in the Euler-MacLaurin formula, depending on how much work we are prepared to do.

With $N = 100$ we get

$$\sum_{k=1}^{100} f(k) \approx 1.42453 + 0.418015 + 0.04 = 1.88802.$$

(ii) For $\gamma_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k} - \ln n$. If $f(x) = 1/x$ we have

$$\gamma_n = -\ln n + \int_1^n f(x)dx + \frac{1}{2}(1 + 1/n) + \frac{1}{12}(-1/n^2 + 1)$$
$$- \frac{1}{720}(-6/n^4 + 6) + \cdots.$$

If we take $n = 1000$ and a decent number of terms in the summation formula we get $\gamma_{1000} \approx .577716$. The limit $\gamma = \lim_{n\to\infty} \gamma_n$ exists and is known as Euler's constant. It is the third most frequently occurring constant in mathematics after e and π. It is not known if γ is rational or irrational.

Question 10.

Suppose that $\sum_{k=1}^{\infty} f(k) < \infty$. Then the Euler-MacLaurin formula implies that $\int_1^{\infty} f(x)dx < \infty$. This is obvious because if $\int_1^N f(x)dx \to \infty$ as $N \to \infty$, then $\sum_{k=1}^{N} f(k) \to \infty$.

In general we let

$$\begin{aligned} \Delta_N = \sum_{k=1}^{N} f(k) - \int_1^N f(x)dx &= \frac{1}{2}(f(1) + f(N)) \\ &+ \int_1^N (x - [x] - 1/2)f'(x)dx. \end{aligned}$$

Observe that if $x \in (N, N+1)$ then $[x] = N$. Suppose that f is positive and decreasing on $[1, \infty)$ and $\int_1^{\infty} f(x)dx < \infty$. Since f is decreasing it follows that

$$f(k+1) \le \int_k^{k+1} f(x)dx \le f(k).$$

And

$$\begin{aligned} \Delta_{N+1} - \Delta_N &= \frac{1}{2}(f(N+1) - f(N)) + \int_N^{N+1} (x - [x] - 1/2)f'(x)dx \\ &= \frac{1}{2}(f(N+1) - f(N)) + \int_N^{N+1} (x - N - 1/2)f'(x)dx \\ &= \frac{1}{2}(f(N+1) - f(N)) + f(N+1)(N+1 - (N+1/2)) \\ &\quad - f(N)(N - (N+1/2)) - \int_N^{N+1} f(x)dx \\ &= f(N+1) - \int_N^{N+1} f(x)dx \le f(N+1) - f(N+1) = 0. \end{aligned}$$

So the sequence $\{\Delta_N\}$ is non-increasing. Hence it is convergent. Thus the series $\sum_{k=1}^{\infty} f(k)$ is convergent.

Question 11.

Take $f(k) = k^2$. Then $f'(k) = 2k$, $f'''(k) = 0$. So by Euler-MacLaurin summation

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \int_1^n x^2 dx + \frac{1}{2}(f(1) + f(n)) + \frac{1}{12}(f'(n) - f'(1)) + \cdots \\ &= \frac{1}{3}(n^3 - 1) + \frac{1}{2}(1 + n^2) + \frac{1}{12}(2n - 2) \\ &= \frac{n}{6}(2n + 1)(n + 1).\end{aligned}$$

Now we take $f(k) = k^3$. Then $f'(k) = 3k^2$, $f''(k) = 6k$, $f'''(k) = 6$. So we have

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \int_1^n x^3 dx + \frac{1}{2}(n^3 + 1) + \frac{1}{12}(3n^2 - 3) - \frac{1}{720}(6 - 6) \\ &= \frac{1}{4}(n^4 - 1) + \frac{1}{2}(n^3 + 1) + \frac{1}{12}(3n^2 - 3) \\ &= \frac{n^2}{4}(n + 1)^2.\end{aligned}$$

Question 12

$$\begin{aligned}\lim_{n\to\infty} \int_0^{\pi/2} \left(1 - \frac{x}{n}\right)^n d(\cos x) &= -\int_0^{\pi/2} e^{-x} \sin x dx \\ &= \frac{1}{2}(e^{-\pi/2} - 1).\end{aligned}$$

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Problem sheet two solutions.

Question 1.

Let $B_n = \bigcup_{k=1}^n A_k$ for $1 \leq n \leq N$. By induction B_n is measurable. Now assume that

$$m^*(E \cap B_n) = \sum_{k=1}^{n} m^*(E \cap A_k).$$

Then by the definition of measurability and the measurability of A_n

$$\begin{aligned} m^*(E \cap B_{n+1}) &= m^*((E \cap B_{n+1}) \cap A_{n+1}^c) + m^*(E \cap B_{n+1} \cap A_{n+1}) \\ &= m^*(E \cap B_n) + m^*(E \cap A_{n+1}) \\ &= \sum_{k=1}^{n+1} m^*(E \cap A_k). \end{aligned}$$

As the result is true for $n = 1$, the general result follows by induction.

Question 2.

This follows from the telescoping series $\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) = 1$.

Question 3.

If $\{x_k\}$ is a set containing a single point, then $m(\{x_k\}) = 0$. By countable additivity, if $A = \cup_{k=1}^{\infty}\{x_k\}$, then $m(A) = 0$.

Question 4

We know $m(\mathbb{Q}) = 0$ since the rationals are countable. Let $X = [0,1] \cap \mathbb{Q}$ and $U = [0,1] - X$. Then $[0,1] = X \cup U$. So that $m([0,1]) = m(U) + m(X)$. But $m(X) = 0$, so $M(U) = 1$.

Question 5

We use $m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^c)$. Now

$$m^*(E \cap A) \leq m^*(A) = 0$$

and $m^*(E \cap A^c) \leq m^*(E)$. So we have

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E).$$

Thus $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$ and A is measurable.

Question 6

Suppose that (B_n) is any sequence of measurable set. Let $A_1 = B_1$, $A_2 = B_2 \cap B_1^c$,..., $A_n = B_n \cap (B_1 \cup B_2 \cup \cdots B_{n-}1)^c$, etc. Then the A_i are pairwise disjoint and

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k.$$

Each A_n is measurable as each B_k is measurable and intersections and unions and compliments of measurable sets are measurable. So

$\cup_{k=1}^{\infty} A_k$ is measurable. For the intersection of the sets we use the relation

$$\bigcap_{n=1}^{\infty} B_n = \left(\bigcup_{n=1}^{\infty} B_n^c \right)^c .$$

Since each B_n is measurable and countable unions are measurable, the measurability of the intersection follows.

Question 7.

We have $A + h = \{x + h | x \in A\}$. If A is covered by I_k then $A + h$ is covered by $I_k + h$ and $l(I_k + h) = l(I_k)$. It follows that $m^*(A + h) = m^*(A)$. To prove measurability we use the relations

$$E \cap (A + h) = ((E - h) \cap A) + h,$$

and

$$E \cap (A + h)^c = ((E - h) \cap A^c) + h.$$

Let us illustrate the first relation using intervals. Let $E = [a, c]$ and $A = [b, d]$ where $a < b < c < d$. Then $A + h = [b + h, d + h]$ and $E \cap (A + h) = [b + h, c]$. Now $E - h = [a - h, c - h]$ and $A \cap (E - h) = [b, c - h]$. So that $h + (A \cap (E - h)) = [b + h, c] = E \cap (A + h)$. The second relation can be demonstrated similarly.

Then by the measurability of A

$$\begin{aligned} m^*(E) &= m^*(E - h) \\ &= m^*((E - h) \cap A) + m^*((E - h) \cap A^c) \\ &= m^*((E - h) \cap A) + h) + m^*((E - h) \cap A^c) + h) \\ &= m^*(E \cap (A + h)) + m^*(E \cap (A + h)^c) \end{aligned}$$

and so $A + h$ is measurable.

Question 8

Clearly $F = E \cup (F - E)$. So $m(F) = m(E) + m(F - E)$

Question 9

As $A \cup B = [0, 1]$ we have

$$[0, 1] = (A - (A \cap B)) \cup (B - (A \cap B)).$$

Now $m^*([0, 1]) = 1$ and

$$\begin{aligned} m^*(A) + m^*(B) &= m^*(A - (A \cap B)) + m^*(B - (A \cap B)) + m^*(A \cap B) \\ &\geq 1. \end{aligned}$$

So $m^*(A) \geq 1 - m^*(B)$.

Question 10

The key is to remove sets which contain a 5 at some place in their decimal expansion. These sets make up intervals whose measures are easily computed. Let the set of numbers in $[0, 1]$ with no 5 in their decimal expansions be A. So we let

$$
\begin{aligned}
A_0^1 &= \{x \in [0,1], x = 0.0a_2a_3...\} \\
A_1^1 &= \{x \in [0,1], x = 0.1a_2a_3...\} \\
&\vdots \\
Q_5^1 &= \{x \in [0,1], x = 0.5a_2a_3...\} \\
&\vdots \\
A_9^1 &= \{x \in [0,1], x = 0.0a_2a_3...\}
\end{aligned}
$$

and $[0,1] = \cup_{i=0}^{9} A_i^1$. Each of these is an interval of length $1/10$. We remove the set A_5^1 because it contains a 5 in the first place of the decimal expansion.

Next we look at the sets

$$
\begin{aligned}
A_{00}^2 &= \{x \in [0,1], x = 0.00a_3a_4\cdots\}, \\
A_{01}^2 &= \{x \in [0,1], x = 0.01a_3a_4\cdots\}, \\
&\vdots \\
A_{05}^2 &= \{x \in [0,1], x = 0.05a_3a_4\cdots\}, \\
&\vdots
\end{aligned}
$$

We need to remove one interval of this form from each of $A_0^1, ..., A_9^1$, but not A_5^1. Each of these subintervals are of length $1/100$ and there are 9 of them. Continuing this process we have to remove 9^2 intervals of length $1/1000$ from the A_{ij}^2 etc. In this way we remove every number with a 5 somewhere in its decimal expansion. The total length removed is

$$
\frac{1}{10} + \frac{9}{100} + \frac{9^2}{1000} + \cdots = \frac{1/10}{1 - 9/10} = 1. \tag{0.5}
$$

It follows then that $m^*(A) = m[0,1] - 1 = 0$. So A has outer measure zero, which also implies that it is measurable.

Question 11

Suppose that A is covered by intervals $I_1, ..., I_n$. Then by question one

$$
m^*(A) \leq m^*\left(A \cap \cup_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} m^*(A \cap I_k) \leq \frac{1}{2} \sum_{k=1}^{n} l(I_k).
$$

Taking the inf over both sides gives

$$
0 \leq m^*(A) \leq \frac{1}{2} m^*(A).
$$

This is only possible if $m^*(A) = 0$.

Question 12

Suppose that A is an interval and $E \subset \mathbb{R}$. We cover E with open intervals $I_1, I_2, ...$ We want to check that the Caratheodory condition is satisfied. Observe that $A \cap I_k$ is an interval and $A^c \cap I_k$ is an interval or the union of two intervals. So there is an open interval J_{1k} such that $A \cap I_k \subset J_{1k}$ and open intervals J_{2k}, J_{3k} such that $A^c \cap I_k \subset J_{1k} \cup J_{2k}$ and

$$l(J_{1k}) + l(J_{2k}) + l(J_{3k}) \leq l(I_k).$$

Therefore $A \cap E \subset \cup_k J_{1k}$ and $A \cap A^c \cup_k (J_{2k} \cup J_{3k})$. We now have

$$\begin{aligned}
m^*(E) &\geq \sum_{k=1}^{\infty} l(I_k) \\
&\geq \sum_{k=1}^{\infty} (l(J_{1k}) + l(J_{2k}) + l(J_{3k})) \\
&\geq m^*(A \cap E) + m^*(A^c \cap E).
\end{aligned}$$

Since the opposite inequality holds, A is measurable.

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Problem sheet three solutions.

Question 1.

We observe that $A_1 \cup A_2 = (A_1 - (A_1 \cap A_2)) \cup A_2$ and these are disjoint. So

$$m(A_1 \cup A_2) = m(A_1 - (A_1 \cap A_2)) + m(A_2).$$

Next $A_1 = (A_1 - (A_1 \cap A_2)) \cup (A_1 \cap A_2)$. So

$$m((A_1 - (A_1 \cap A_2)) = m(A_1) - m(A_1 \cap A_2).$$

Combining gives

$$m(A_1 \cup A_2) = m(A_1) - m(A_1 \cap A_2) + m(A_2).$$

Rearranging gives

$$m(A_1 \cup A_2) + m(A_1 \cap A_2) = m(A_1) + m(A_2).$$

Question 2

We only need to check countable additivity since the other properties hold by construction. So let $\{A_n\}$ be a pairwise disjoint sequence of subsets of X. Let $= \cup_{n=1}^{\infty} A_n$. If some A_n is uncountable, then A is also uncountable, so trivially we have

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) = \infty.$$

If however, each A_n is countable, then A is countable and hence

$$\mu(A) = \sum_{x \in A} f(x) = \sum_{n=1}^{\infty} \left[\sum_{x \in A_n} f(x) \right] = \sum_{n=1}^{\infty} \mu(A_n)$$

and countable additivity holds.

Question 3

The length of all the subintervals removed to form the Cantor set is $1/3 + 2/9 + 4/27 + \cdots = 1$. So the Cantor set has outer measure zero. It is thus measurable and has measure zero. To prove that C is uncountable, we let $x = 0.b_1 b_2 b_3 \cdots$ be the binary expansion of a number in $[0, 1)$. Each b_n is either a zero or one. Let $f(x) = 0.t_1 t_2 t_3 ...$ where $t_n = 2b_n$. This is the ternary expansion of a number. (Expansion in base 3). Each t_n is either 0 or 2, so $f(x)$ is not in $[1/3, 2/3)$, nor is it in $[1/9, 2/9)$ etc. So $f(x)$ is in the Cantor set for every x. This shows that the Cantor set is non-empty. As f is one to one and $[0, 1)$ is uncountable, then C is uncountable, since $f : [0, 1) \to C$.

Question 4. We use the inequality. $m\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$. The result easily follows from this.

Question 5

We have

$$\begin{aligned} m^*(B) &\le m^*(B \cup A) = m^*((B - A) \cup A) \\ &\le m^*(B - A) + m^*(A) \\ &= m^*(B - A) \le m^*(B). \end{aligned}$$

So $m^*(B) = m^*(B - A) = m^*(B \cup A)$.

Question 6

Suppose that $\sum_{n=1}^{\infty} m^*(A_n) < \infty$. This is only possible if

$$\sum_{i=n}^{\infty} m^*(A_i) \to 0 \qquad (*)$$

as $n \to \infty$. Now let $E_n = \cup_{i=1}^{n} A_i$. Then $E \subseteq E_n$ for all n. We then have the inequality

$$0 \le m^*(E) \le m^*(E_n) \le \sum_{n=1}^{\infty} m^*(A_n).$$

By (*) it follows that $m^*(E) = 0$.

Question 7

If A is not measurable then $\chi_A^{-1}(\{1\}) = A$ is not measurable, so χ_A is not measurable. Conversely, if A is measurable, then

$$\{\chi_A > a\} = \begin{cases} A & 0 < a < 1 \\ \mathbb{R} & a \le 0 \\ \emptyset & a \ge 1. \end{cases}$$

These sets are measurable, so χ_A is measurable.

Question 8

Since f is continuous, it follows that $f^{-1}(A)$ is open, whenever A is open. Now $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$. So if A is open, $B = f^{-1}(A)$ is open, and hence $g^{-1}(B)$ is measurable because B is open and g is a measurable function. Thus $f \circ g$ is measurable. It is worth noting that the converse of this result is false.

Question 9

We know that if $f_n \to f$ pointwise and each f_n is measurable, then f is measurable. Now

$$f'(x) = \lim_{n \to \infty} \frac{f(x + 1/n) - f(x)}{1/n}.$$

Since $\frac{f(x+1/n) - f(x)}{1/n}$ is measurable, then f' is measurable.

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Problem sheet four solutions.

Question 1.

Suppose that $f_n \to f$ uniformly on $X \subseteq \mathbb{R}$. Then given $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that for $n \geq N$

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon.$$

Then for $n \geq N$

$$m[\{x : |f_n(x) - f(x)| \geq \epsilon\}] = 0.$$

Thus $f_n \to f$ in measure.

Question 2

This is an exercise in comparing the measure of sets.

(i) Suppose that a, b are nonzero and $f_n \to f$ and $g_n \to g$ in measure. The case $a = b = 0$ is trivial. If for all $\epsilon > 0$, $m[\{x : |f_n(x) - f(x)| \geq \epsilon\}] \to 0$ as $n \to \infty$, then

$$m[\{x : |a||f_n(x) - f(x)| \geq \epsilon\}] = m[\{x : |f_n(x) - f(x)| \geq \epsilon/|a|\}] \to 0.$$

So $af_n \to af$ in measure.

Now by the triangle inequality

$$\begin{aligned}|af_n(x) + bg_n(x) - af(x) - bg(x)| \leq |a||f_n(x) - f(x)| \\ + |b||g_n(x) - g(x)|.\end{aligned}$$

Thus if

$$x \in A_n = \{x : |af_n(x) + bg_n(x) - af(x) - bg(x)| \geq \epsilon\},$$

then certainly

$$x \in B_n = \{x : |a||f_n(x) - f(x)| + |b||g_n(x) - g(x)| \geq \epsilon\}.$$

So $A_n \subseteq B_n$ and $m(A_n) \leq m(B_n)$. Now

$$B_n \subseteq \{x : |a||f_n(x) - f(x)| \geq \epsilon/2\} \cup \{x : |b||g_n(x) - g(x)| \geq \epsilon/2\}$$

So it follows that

$$\begin{aligned}m(A_n) \leq m(B_n) \leq m[\{x : |a||f_n(x) - f(x)| \geq \epsilon/2\}] \\ + m[\{x : |b||g_n(x) - g(x)| \geq \epsilon/2\}] \to 0.\end{aligned}$$

Hence $af_n + bg_n \to af + bg$ in measure.

(ii) We have $||f_n| - |f|| \leq |f_n - f|$. So that if $||f_n(x)| - |f(x)|| \geq \epsilon$, then $|f_n(x) - f(x)| \geq \epsilon$. Thus

$$m[\{x : ||f_n(x)| - |f(x)|| \geq \epsilon\}] \leq m[\{x : |f_n(x) - f(x)| \geq \epsilon\}].$$

so if $f_n \to f$ in measure, then $|f_n| \to |f|$ in measure.

(iii) First we show that if $f_n \to f$ in measure on a set X then there is a subsequence $\{f_{n_k}\}$ which converges to f a.e. It is easy to see that given $n > 0$ we can find a k_n such that

$$m[\{x \in X : |f_k(x) - f(x)| \geq 1/n\}] < 2^{-n}$$

for all $k > k_n$. Set $E_n = \{x \in X : |f_{k_n}(x) - f(x)| \geq 1/n\}$ for each n. Let $E = \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_k$. Then

$$m(E) \leq m\left(\cup_{k=n}^{\infty}\right) \leq \sum_{k=n}^{\infty} m(E_k) \leq 2^{1-n}.$$

This holds for all n so $m(E) = 0$. In addition, if $x \notin E$, then there is an n such that $x \notin \cap_{k=n}^{\infty} E_k$, so $|f_{k_n}(x) - f(x)| \leq 1/m$ holds for each $m \geq n$. Therefore $f_{k_n}(x) \to f(x)$ for each $x \in E^c$, so f_{k_n} converges to f a.e. From this it follows that there is a subsequence $f_{k_n} g_{k_n}$ of $f_n g_n$ which converges a.e. to fg.

Now we show that if $f_n \to f$ a.e. on X, with $m(X) < \infty$, then $f_n \to f$ in measure. To this end let $\epsilon > 0$ and set $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Since $m(X) < \infty$ we can use Eogoroff's Theorem and given $\delta > 0$ there exists a measurable set A such that $m(A) < \delta$ and $f_n \to f$ uniformly to f on A^c. So $f_n \to f$ in measure on A^c. Now choose k such that $|f_n(x) - f(x)| < \epsilon$ holds for all $x \in A^c$ and all $n \geq k$. Then $E_n \subseteq A$ holds for all $n \geq k$. So $m(E_n) \leq m(A) < \delta$ for all $n \geq k$. So $\lim_{n\to\infty} m(E_n) = 0$. And the proof is finished.

Consequently, $f_{k_n} g_{k_n}$ converges to fg in measure.

To finish the proof suppose that $f_n g_n$ does not converge in measure to fg. Then given $\epsilon > 0$ we can find $\delta > 0$ such that for all n.

$$m[\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}] \geq \delta$$

But we know that the subsequence $f_{k_n} g_{k_n}$ converges in measure to fg. This is a contradiction. So $f_n g_n$ to fg in measure. Note this result is only true if $m(X) < \infty$. If $m(X) = \infty$, then there are sequences of functions for which the result is false.

(iv) The proof of this is now trivial. We simply apply the result of (iv) with $g_n = g$ all n.

Question 3

We have $\int_E f = \int f\chi_E$. Now suppose that $k \leq f \leq K$. Then

$$\int |f\chi_E| \leq K \int \chi_E = Km(E).$$

Similarly for the other inequality.

Question 4

If $f > 0$ then there exists $\delta > 0$ such that for all x we have $f(x) > \delta$.. Now $\int_E f = 0$ and $\int_E f \geq \delta m(E)$ by the previous question. So we have

$$0 = \int_E f \geq \delta m(E).$$

Hence $m(E) = 0$.

Question 5

To show that $\int_0^\infty \sin x^2 dx$ does not exist as a Lebesgue integral, we use the substitution $y = x^2$ to obtain

$$\int_0^\infty \sin x^2 dx = \frac{1}{2}\int_0^\infty \frac{\sin y}{\sqrt{y}} dy$$

Now $y^{-1/2}$ is integrable near zero, so we are concerned with what happens as the upper limit of integration increases to infinity. Recall that the Lebesgue integral of f only exists if $\int |f| < \infty$. So we note that

$$\begin{aligned}\int_0^\infty \frac{|\sin y|}{\sqrt{y}} dy &> \int_\pi^\infty \frac{|\sin y|}{\sqrt{y}} dy \\ &= \sum_{k=2}^\infty \int_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{\sqrt{y}} dy \\ &\geq \sum_{k=2}^\infty \frac{\pi}{\sqrt{k\pi}} = \infty.\end{aligned}$$

So the Lebesgue integral diverges.

Now we evaluate the improper Riemann integral. To do this we notice that

$$\int_0^\infty \int_0^r e^{-xy^2} \sin x dx dy = \int_0^r \int_0^\infty e^{-xy^2} \sin x dy dx$$

and

$$\int_0^\infty e^{-xy^2} \sin x dy = \frac{\sqrt{\pi} \sin x}{2\sqrt{x}}.$$

Now integration by parts gives

$$\int_0^r e^{-xy^2} \sin x dx = \frac{1 - e^{-ry^2}(\cos r + y^2 \sin r)}{1 + y^4}.$$

So

$$\int_0^r \frac{\sin x}{2\sqrt{x}} dx = \frac{1}{\sqrt{\pi}} \int_0^r \frac{1 - e^{-ry^2}(\cos r + y^2 \sin r)}{1 + y^4} dy,$$

Hence

$$\lim_{r\to\infty} \int_0^r \frac{\sin x}{2\sqrt{x}} dx = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dy}{1 + y^4}.$$

Now $1+y^4 = (1+\sqrt{2}y+y^2)(1-\sqrt{2}y+y^2)$. Partial fractions now gives

$$\frac{1}{1+y^4} = \frac{\sqrt{2}-y}{2\sqrt{2}\left(y^2-\sqrt{2}y+1\right)} + \frac{y+\sqrt{2}}{2\sqrt{2}\left(y^2+\sqrt{2}y+1\right)}.$$

So that

$$\int \frac{dy}{1+y^4} = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{1+\sqrt{2}y+y^2}{1-\sqrt{2}y+y^2}\right) + 2\tan^{-1}\frac{\sqrt{2}y}{1-y^2}\right].$$

The Fundamental Theorem of Calculus then gives

$$\int_0^\infty \frac{dy}{1+y^4} = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Which gives the improper Riemann integral

$$\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Question 6

We know that $f : [0,1] \to \mathbb{R}$, $f'(0)$ exists and $f(0) = 0$. So by continuity and differentiability there exists $M > 0$ and $0 < \delta < 1$ such that for $0 \le x \le \delta$ $|f(x)| \le Mx$. Since for $0 \le x \le 1$, we have $x^{-3/2} \le \delta^{-3/2}$. We may assume that $M > \delta^{-3/2}$. Now $g(x) = x^{-3/2}f(x)$. So

$$\begin{aligned} |g(x)| &= |x^{-3/2}f(x)| \\ &\le M\begin{cases} x^{-1/2}, & 0 < x < \delta \\ |f(x)|, & \delta \le x \le 1 \end{cases} \end{aligned}$$

Now $h(x) = x^{-1/2}$ is Lebesgue integrable on $[0,\delta]$ and we have

$$\begin{aligned} \int_0^1 |g| &= \int_0^\delta |g| + \int_\delta^1 |g| \\ &\le M\int_0^\delta h + M\int_\delta^1 f < \infty. \end{aligned}$$

So g is Lebesgue integrable.

Question 7. Recall that if A, B are disjoint, then $\int_{A\cup B} h = \int_A h + \int_B h$. Let $A = [0,1] - \mathbb{Q}$ and $B = [0,1] \cap \mathbb{Q}$. Obviously $[0,1] = A \cup B$ and A and B are disjoint. Further, $\lambda(B) = 0$, where λ is Lebesgue measure. Let $f(x) = x^3 + 5x$ and $g(x) = 2x^2$. Set $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. Therefore $\int_0^1 h(x)dx = \int_A f + \int_B g = \int_A f$, since the integral over B is zero. Now

$$\int_0^1 f(x)dx = \int_A f + \int_B f. \tag{0.6}$$

But again $\int_B f = 0$. So
$$\int_0^1 h(x)dx = \int_A f = \int_0^1 (x^3 + 5x)dx = \frac{11}{4}. \tag{0.7}$$

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Problem sheet five solutions.

Question 1.

Define $\varphi(x) = \varphi(b)$ if $x > b$ and $\varphi(x) = \varphi(a)$ if $x < a$. Since

$$\frac{\varphi(x+h) - \varphi(x)}{h} \geq 0,$$

and

$$\frac{\varphi(x+h) - \varphi(x)}{h} \to \varphi'(x),$$

as $h \to 0$, then Fatou's Lemma gives the inequality

$$\int_a^b \varphi'(x)dx \leq \lim_{h\to 0} \int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx.$$

Now

$$\begin{aligned}
\int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx &= \frac{1}{h}\int_a^b \varphi(x+h)dx - \frac{1}{h}\int_a^b \varphi(x)dx \\
&= \frac{1}{h}\int_{a+h}^{b+h} \varphi(x)dx - \frac{1}{h}\left(\int_a^{a+h} + \int_{a+h}^b\right)\varphi(x)dx \\
&= \frac{1}{h}\left(\int_{a+h}^b + \int_b^{b+h} - \int_a^{a+h} + \int_{a+h}^b\right)\varphi(x)dx \\
&= \frac{1}{h}\int_b^{b+h} \varphi(x)dx - \frac{1}{h}\int_a^{a+h} \varphi(x)dx \\
&\to \varphi(b) - \varphi(a)
\end{aligned}$$

by the Fundamental Theorem of Calculus. Hence

$$\begin{aligned}
\int_a^b \varphi'(x)dx &\leq \lim_{h\to 0} \int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx \\
&= \varphi(b) - \varphi(a).
\end{aligned}$$

Question 2.

We know that $(1 + \frac{x}{n})^n \to e^x$ as $n \to \infty$. So that

$$(1 + \frac{x}{n})^n e^{-2x} \to e^{-x}.$$

We want to show that $(1+\frac{x}{n})^n e^{-2x} \leq e^{-x}$. This is equivalent to showing that $(1 + \frac{x}{n})^n \leq e^x$. Since the logarithm is increasing we want

$$n \ln(1 + \frac{x}{n}) \leq x, \tag{0.8}$$

which is true at $x = 0$. Now if $h(x) = n\ln(1 + \frac{x}{n}) - x$, then

$$h'(x) = \frac{n}{n+x} - 1 < 0, \quad x > 0$$

so h is decreasing. Hence (**??**) holds and $(1+\frac{x}{n})^n \leq e^x$. Now $g(x) = e^{-x}$ is integrable, so by the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^n e^{-2x} dx = \int_0^\infty e^{-x} dx = 1.$$

Question 3

If $t = 0$ we have the improper Riemann integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. Now let

$$F(t) = \int_0^\infty e^{-xt} \frac{\sin x}{x} dx.$$

Then $|\frac{\sin x}{x}| \leq 1$ and so $\left|e^{-xt}\frac{\sin x}{x}\right| \leq e^{-xt}$. Since e^{-xt} is integrable, we may apply the dominated convergence series, and $\frac{d}{dt}\left(e^{-xt}\frac{\sin x}{x}\right) = -e^{-xt}\sin x$ is integrable, so we differentiate under the integral sign to obtain

$$\begin{aligned} F'(t) &= -\int_0^\infty e^{-xt}\sin x dx \\ &= -\frac{1}{1+t^2}. \end{aligned}$$

Integrating with respect to t gives

$$F(t) = C - \tan^{-1} t.$$

The DCT gives $\lim_{t\to\infty} F(t) = C - \pi/2 = 0$ and so

$$F(t) = \frac{\pi}{2} - \tan^{-1} t.$$

Question 4.

We only do one. The second is similar. Integration by parts gives

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[-f(x)\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\frac{\cos(nx)}{n}dx$$

By continuity we have the inequality

$$\left|\int_{-\pi}^{\pi} f(x)\sin(nx)dx\right| \leq C/n$$

for some constant C depending on f. Taking limits as $n \to \infty$ gives the result.

Question 5.

Assume that $I = \int_a^\infty f(x)dx$ exists. Choose $M > 0$ such that

$$\left|I - \int_a^r f(x)dx\right| < \frac{\epsilon}{2}$$

holds for all $r \geq M$. If $s, t \geq M$ then

$$\begin{aligned}\left|\int_s^t f(x)dx\right| &= \left|\int_a^t f(x)dx - \int_a^s f(x)dx\right| \\ &\leq \left|I - \int_a^t f(x)dx\right| + \left|I - \int_a^s f(x)dx\right| \\ &< \epsilon.\end{aligned}$$

Conversely, assume that the condition is satisfied. If $\{a_n\}$ is a sequence in $[, \infty)$ such that $a_n \to \infty$, then $\left\{\int_a^{a_n} f(x)dx\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus $A = \lim_{n\to\infty} \int_a^{a_n} f(x)dx$ exists. Now let $\{b_n\}_{n=1}^{\infty}$ be another sequence in $[a, \infty)$ with $b_n \to \infty$. Then let $B = \lim_{n\to\infty} \int_a^{b_n} f(x)dx$. Now

$$\begin{aligned}|A - B| &\leq \left|A - \int_a^{a_n} f(x)dx\right| + \left|\int_{a_n}^{b_n} f(x)dx\right| + \left|B - \int_a^{b_n} f(x)dx\right| \\ &\to 0.\end{aligned}$$

as $n \to \infty$. So $A = B$. Hence $\int_a^{\infty} f(x)dx$ exists.

Question 7

The function

$$f(x) = \begin{cases} 1 & x = 0 \\ \left(\frac{\sin x}{x}\right)^2 & 0 < x \leq 1 \\ \frac{1}{x^2}, & x > 1, \end{cases}$$

is Lebesgue integrable over $[0, \infty)$. Now $0 \leq \left(\frac{\sin x}{x}\right)^2 \leq f(x)$, so $\left(\frac{\sin x}{x}\right)^2$ is Lebesgue integrable on $[0, \infty)$. Now for each $r, \epsilon > 0$ we have

$$\int_\epsilon^r \left(\frac{\sin x}{x}\right)^2 dx = \frac{\sin^2 \epsilon}{\epsilon} - \frac{\sin^2 r}{r} + \int_\epsilon^r \frac{2\sin x \cos x}{x} dx.$$

The limit as $\epsilon \to 0$ and $r \to \infty$ on the left side exists, since the function is Lebesgue integrable. Now

$$\int_\epsilon^r \frac{2\sin x \cos x}{x} dx = \int_{2\epsilon}^{2r} \frac{\sin x}{x} dx,$$

since $2\sin x \cos x = \sin(2x)$. Taking limits we see that the Lebesgue integral is equal to the improper Riemann integral obtained by taking limits on the right. That is

$$L\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = IR\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Question 8.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and such that for some $M > 0$ we have $|f'(x)| \leq M$ for all $x \in [a, b]$. This is reasonable since f is continuous and continuous functions are bounded. Now for $x < a$ we can set $f(x) = f(x) + f'(a)(x - a)$ and for $x > b$ we set $f(x) =$

$f(b)+f'(b)(x-b)$ and so we can extend f to be a differentiable function on $\mathbb{R}$.

Next we consider the sequence

$$f_n(x) = n[f(x+1/n) - f(x)] = \frac{f(x+1/n) - f(x)}{1/n}.$$

Then $f_n(x) \to f'(x)$ for each $x \in \mathbb{R}$. By the Mean Value Theorem $|f_n(x)| \leq |f'(x)| \leq M$. Consequently by the DCT, f' is Lebesgue integrable over $[a, b]$ and

$$\int_a^b f'(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Now

$$\begin{aligned}
\int_a^b f_n(x)dx &= n\left[\int_a^b f(x+1/n)dx - \int_a^b f(x)dx\right] \\
&= n\left[\int_{a+1/n}^{b+1/n} f(x)dx - \int_a^b f(x)dx\right] \\
&= n\left[\int_b^{b+1/n} f(x)dx - \int_a^{a+1/n} f(x)dx\right] \\
&= \frac{\int_b^{b+1/n} f(x)dx}{1/n} - \frac{\int_a^{a+1/n} f(x)dx}{1/n} \\
&\to f(b) - f(a),
\end{aligned}$$

by the Fundamental Theorem of Calculus, since the limits return the derivatives of the integrals. That is, if $F(x) = \int_{x_0}^x f(x)dx$, then

$$\frac{F\left(b+\frac{1}{n}\right) - F(b)}{\frac{1}{n}} = \frac{\int_b^{b+\frac{1}{n}} f(x)dx}{\frac{1}{n}} \to F'(b) = f(b),$$

as $n \to \infty$. Similarly for the second integral.

Question 9.

Obviously $f(x) = \dfrac{\ln x}{x^2} \geq 0$ holds for each $x \geq 1$. If $r > 1$ then integration by parts gives

$$\begin{aligned}
\int_1^r \frac{\ln x}{x^2}dx &= -\frac{\ln x}{x}\bigg|_1^r + \int_1^r \frac{dx}{x^2} \\
&= 1 - \frac{1}{r} - \frac{\ln r}{r}.
\end{aligned}$$

So

$$\int_1^\infty f(x)dx = \lim_{r\to\infty}\left(1 - \frac{1}{r} - \frac{\ln r}{r}\right) = 1.$$

Question 10

Fix $a > 0$ and define $f_n(x) = f(nx)$. Clearly $\lim_{n\to\infty} f_n(x) = \delta$ for all $x \in [0, a]$. Since f is continuous the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded. As $f \to \delta$, there is an $M > 0$ such that for $x > M$ $|f(x)| < 1 + |\delta|$. That is, $|f(x)|$ is eventually smaller than $1 + |\delta|$. By continuity f is bounded on $[0, M]$. So f is bounded on $[0, M]$ and $[M, \infty)$. Thus there is a constant C such that $|f(x)| \leq C$ for all x. Hence $|f_n(x)| = |f(nx)| \leq C$ and so by the DCT

$$\lim_{n\to\infty} \int_0^a f(nx)dx = \lim_{n\to\infty} \int_0^a f_n(x)dx = \int_0^a \delta dx = a\delta.$$

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Problem sheet six solutions.

Question 1.

We will actually give a more general result, due to Frullani. Let f be differentiable and suppose we can differentiate under the integral sign. Further suppose that $\lim_{x\to\infty} f(x) = 0$. Let

$$F(a,b) = \int_0^\infty \frac{f(ax) - f(bx)}{x} dx$$

and suppose that the integral exists. Then

$$\frac{\partial F}{\partial a} = \int_0^\infty f'(ax)dx.$$

Put $ax = t$ so that $\frac{\partial F}{\partial a} = 1/a \int_0^\infty f'(t)dt = -f(0)/a$. Integration with respect to a gives

$$F(a) = -f(0)\ln a + C(b).$$

Likewise differentiation of F with respect to b gives

$$\frac{\partial F}{\partial b} = \frac{1}{b} f(0) = C'(b).$$

This gives $F(a,b) = f(0)\ln(b/a) + K$, where K is a constant. Now $a = b$ implies $F = 0$, so that $K = 0$. Hence

$$F(a,b) = f(0)\ln(b/a).$$

Taking $f(x) = e^{-x}$ gives the result of the question on the sheet. It is possible to give precise technical conditions on f needed to make this work, but we will not do so here.

Question 2. For each $p \geq 0$, define

$$k(x) = \frac{x^p - 1}{\ln x}, \ x \in (0,1).$$

Notice that $\lim_{x\to1} k(x) = p$ and $\lim_{x\to0+} k(x) = 0$. So we can define $k(1) = p$ and $k(0) = 0$ and the resulting function is continuous and hence integrable on $[0,1]$. Let $F(p) = \displaystyle\int_0^1 \frac{x^p - 1}{\ln x} dx$. Clearly $F(p)$ exists for all $p \geq 0$. We want $F(1)$.

Now $x^p = e^{p\ln x}$ so that $\displaystyle\frac{d}{dp} x^p = x^p \ln x$. Hence $\frac{d}{dp}\frac{x^p-1}{\ln x} = x^p$ which is integrable. Observe that k is continuously differentiable on $[0,1]$. So for h suitably small so by Taylor's Theorem we have

$$\frac{x^{p+h} - x^p}{h\ln x} = x^p + L(x,h),$$

where $L(x,h) \to 0$ as $h \to 0$, for each x. L must be integrable for $p > 0$ (why?) and so $x^p + L(x,h)$ is integrable on $[0,1]$. Thus we may apply the dominated convergence theorem and deduce that

$$
\begin{aligned}
\lim_{h\to 0}\frac{F(p+h)-F(p)}{h} &= \lim_{h\to 0}\int_0^1 \frac{x^{p+h}-x^p}{h\ln x}dx\\
&= \lim_{h\to 0}\int_0^1 x^p\frac{x^h-1}{h\ln x}dx\\
&= \int_0^1 \lim_{h\to 0} x^p\frac{x^h-1}{h\ln x}dx\\
&= \int_0^1 x^p dx.
\end{aligned}
$$

Thus

$$F'(p) = \frac{1}{p+1}.$$

So $F(p) = \ln(p+1) + K$. Taking $p = 0$ gives $F(0) = 0 = \ln 1 + K$. So $K = 0$. Hence $F(1) = \ln 2$.

Question 3. Differentiation under the integral sign is easy to establish here because of the rapid decay of the integrand.

So let $\alpha, \beta > 0$. If we put $y = \alpha x, p = \alpha\beta$, then

$$\int_0^\infty e^{-\alpha^2x^2-\beta^2/x^2}dx = \frac{1}{\alpha}\int_0^\infty e^{-y^2-p^2/y^2}dy.$$

Let

$$J(p) = \int_0^\infty e^{-y^2-p^2/y^2}dy$$

so that

$$J'(p) = -2\int_0^\infty \frac{p}{y^2}e^{-y^2-p^2/y^2}dy.$$

We put $z = p/y$, so that $dz = -p/y^2$ and

$$J'(p) = -2\int_0^\infty e^{-z^2-p^2/z^2}dz = -2J(p).$$

Solving the ODE gives $J(p) = J(0)e^{-2p}$. But $J(0) = \int_0^\infty e^{-y^2}dy = \frac{1}{2}\sqrt{\pi}$. So we have shown that

$$\int_0^\infty e^{-\alpha^2x^2-\beta^2/x^2}dx = \frac{\sqrt{\pi}}{2\alpha}e^{-2p}.$$

Question 4.

Let $\rho(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. It is obvious that $\rho(x,x) = 0$, $\rho(x,y) > 0$ and $\rho(x,y) = \rho(y,x)$ as d is a metric. For the triangle inequality we use the fact that if $0 \le x \le y$, we have

$$\frac{x}{1+x} \le \frac{y}{1+y},$$

which follows from the fact that $f(x) = x/(1+x)$ is increasing on $[0, \infty)$ since $f'(x) = 1/(1+x)^2 > 0$. We will show that

$$\frac{x+y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}, \quad x, y > 0.$$

We have

$$\begin{aligned}(x+y)(1+x)(1+y) &= x(1+x)(1+y) + y(1+x)(1+y) \\ &\leq x(1+x)(1+y) + xy(1+y) + (x+1)y(1+y) \\ &\quad + xy(1+x) \\ &= x(1+y)(1+x+y) + y(1+x)(1+x+y)\end{aligned}$$

Now divide through both sides by $(1+x)(1+y)(1+x+y)$. We can therefore conclude that

$$\begin{aligned}\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} &\leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(z,y)} \\ &= \rho(x,z) + \rho(z,y).\end{aligned}$$

Question 5.

Suppose that $\alpha \geq \beta \geq 0$. We know that $\|x+y\| = \|x\| + \|y\|$ for some x, y. Now

$$\|\alpha x + \beta y\| \leq |\alpha|\|x\| + |\beta|\|y\| = \alpha\|x\| + \beta\|y\|$$

as $\alpha, \beta > 0$. For the reverse inequality we observe that

$$\begin{aligned}\|\alpha x + \beta y\| &= \|\alpha(x+y) + (\beta - \alpha)y\| \\ &\geq \|\alpha(x+y)\| - \|(\beta-\alpha)y\| \\ &= \alpha\|x+y\| - (\alpha-\beta)\|y\| \\ &= \alpha(\|x\| + \|y\|) - \alpha\|y\| + \beta\|y\| \\ &= \alpha\|x\| + \beta\|y\|.\end{aligned}$$

So we conclude that $\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$.

Question 6.

Given a metric d we would seek to define a norm by setting $d(x, 0) = \|x\|$. This does not in general give a norm. To see why let d be defined by $d(x,x) = 0$ and $d(x,y) = 1$. Then $d(\lambda x, 0) = 1 \neq |\lambda| d(x, 0)$. So $\|\lambda x\| \neq |\lambda|\|x\|$. Thus this metric does not define a norm.

Question 7.

Let $f : X \to \mathbb{R}$. Set $\|f\|_\infty = \inf\{M : |f(x)| \leq M, \text{almost all } x\}$.

(i) If $f = g$ a.e. then $|f(x)| \leq M$ a.e. implies $|g(x)| \leq M$ a.e. So $\|f\|_\infty = \|g\|_\infty$.

(ii) This is obvious.

(iii) If $|f(x)| \leq M$ then $|af(x)| \leq aM$. So $|af(x)| = |a||f(x)|$.

(iv) If $|f(x)| \leq M_1$ a.e. and $|g(x)| \leq M_2$ a.e. then

$$|f(x) + g(x)| \leq M_1 + M_2 \text{ a.e.}$$

So that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(v) It is clear that if $|f(x)| \leq |g(x)|$ then if $|g(x)| \leq M$ a.e then $|f(x)| \leq M$ a.e. so that $\|f\|_\infty \leq \|g\|_\infty$.

Question 8.

All finite dimensional vector spaces are isomorphic to $\mathbb{R}^n$, so that without loss of generality we can prove the result on $\mathbb{R}^n$. Let $\|\cdot\|_2$ denote the usual Euclidean norm given by $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$. If $\|\cdot\|$ is another norm on $\mathbb{R}^n$ we prove that this is equivalent to the Euclidean norm. If all norms are equivalent $\|\cdot\|_2$ then we are done.

Let $e_1, ..., e_n$ be the standard basis for $\mathbb{R}^n$ and let $x = \sum_{k=1}^n x_k e_k$. Then the triangle inequality gives

$$\begin{aligned}\|x\| = \left\| \sum_{k=1}^n x_k e_k \right\| &\leq \sum_{k=1}^n |x_k| \|e_k\| \\ &\leq \left(\sum_{k=1}^n \|e_k\| \right) \|x\|_2 = M\|x\|_2.\end{aligned}$$

The last step essentially following from Cauchy-Schwartz. So with $M = \sum_{k=1}^n \|e_k\|$ we have $\|x\| \leq M\|x\|_2$ holds for all $x \in \mathbb{R}^n$. This is the first part.

Now $|\|x\| - \|y\|| \leq \|x - y\| \leq M\|x - y\|_2$, so the map $x \to \|x\|$ is a continuous function on $\mathbb{R}^n$. Since continuous functions attain their maximum values on compact sets, we let x_0 be the point on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ where $\|x\|$ has its maximum. So $\|x\| \geq \|x_0\|$ all $x \in S^{n-1}$.

Let $K = \|x_0\|$. Now $\|x_0\|_2 = 1$, so $x_0 \neq 0$ and hence $K = \|x_0\| > 0$. Now if $x \in \mathbb{R}^n$ is nonzero, then $\left\| \dfrac{x}{\|x\|_2} \right\| \geq K$. Hence

$$K\|x\|_2 \leq \|x\|$$

as required. So all norms on $\mathbb{R}^n$ are equivalent to the Euclidean norm and hence each other. Since there is a one to one correspondence between finite dimensional vector spaces, the result holds for an arbitrary finite dimensional vector space.

Question 9.

Since $\|\cdot\|_\infty$ is a norm, then clearly

$$\|f+g\| = |f(0)+g(0)| + \|f'+g'\|_\infty \leq |f(0)| + \|f'\|_\infty + |g(0)| + \|g'\|_\infty$$

so the triangle inequality is satisfied. Similarly, if $\|f\| = 0$, then $|f(0)| = \|f'\|_\infty = 0$. Hence $f' = 0$ so f is constant, but $f(0) = 0$, so $f = 0$. Finally $\|af\| = |a||f(0)| + |a|\|f'\|_\infty = |a|\|f\|$. So this is a norm.

The Fundamental Theorem of Calculus gives

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

So $|f(x)| \leq |f(0)| + \|f'\|_\infty$ for each $x \in [0,1]$. Taking the supremum gives

$$\|f\| \leq |f(0)| + \|f'\|_\infty.$$

Now

$$\begin{aligned}\|f\|_\infty + \|f'\|_\infty &\leq |f(0)| + 2\|f'\|_\infty \\ &\leq 2(|f(0)| + \|f'\|_\infty) \\ &= 2\|f\|_A \leq 2(\|f\|_\infty + \|f'\|_\infty).\end{aligned}$$

Question 10.

The operator is given by $Tf(x) = \displaystyle\int_a^b f(y)K(x,y)dy$. So that

$$\begin{aligned}|T(f)| &\leq \int_a^b |f(y)||K(x,y)|dy \\ &\leq M \sup_{a\leq x\leq b} |f(y)| = M\|f\|_\infty,\end{aligned}$$

where $M = \sup_{a\leq x\leq b}\int_a^b |K(x,y)|dy$, which is finite since K is continuous and hence bounded. So $\|Tf\|_\infty \leq M\|f\|_\infty$.

Question 11.

Take $f_n(x) = x^n$. Then $\|f_n\|_\infty = 1$ on $[0,1]$, but

$$\|Df_n\|_\infty = \sup\{nx^{n-1}; x \in [0,1]\} = n$$

for each n. Thus there is no constant K for which $\|Df\|_\infty \leq K\|f\|_\infty$ for every K. Hence D is unbounded.

Question 12.

We have

$$\int_{-\infty}^{\infty} f'(x)e^{-ixy}dx = iy\int_{-\infty}^{\infty} f(x)e^{-ixy}dx = iy\widehat{f}(y)$$

where we used integration by parts and assumed that

$$\lim_{x\to\pm\infty} f(x) = 0.$$

Question 13.

$$\int_{-\infty}^{\infty} xf(x)e^{-ixy}dx = i\frac{d}{dy}\int_{-\infty}^{\infty} f(x)e^{-ixy}dx$$
$$= i\frac{d}{dy}\widehat{f}(y).$$

Question 14.

Taking the Fourier transform in x in the given DE gives $iy\widehat{u}+i\widehat{u}_y = 0$. Cancelling the is gives $y\widehat{u} + \widehat{u}_y = 0$. Thus the Fourier transformed equation is the same as the original equation. Thus u and $\widehat{u}$ satisfy the same equation and so by uniqueness of solutions they must be the same up to a constant. Solving the original equation gives $u(x) = \sqrt{2\pi}e^{-x^2/2}$.

Now the only solution of the equation for $\widehat{u}$ is $\widehat{u}(y) = Ce^{-y^2/2}$. This must be the Fourier transform of u so that

$$\int_{-\infty}^{\infty} e^{-x^2/2-ixy}dx = \frac{C}{\sqrt{2\pi}}e^{-y^2/2}.$$

This holds for all y, so taking $y = 0$ gives

$$\int_{-\infty}^{\infty} e^{-x^2/2}dx = \frac{C}{\sqrt{2\pi}} = \sqrt{2\pi}.$$

So that $C = 2\pi$ and

$$\int_{-\infty}^{\infty} e^{-x^2/2-ixy}dx = \sqrt{2\pi}e^{-y^2/2}.$$

37438 Modern Analysis

Problem sheet seven solutions.

Question 1.

Let x, y be vectors in a real inner product space. If $x \perp y$ then

$$\begin{aligned}\|x+y\|^2 &= (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y) \\ &= (x,x) + (y,y) = \|x\|^2 + \|y\|^2.\end{aligned}$$

Conversely suppose that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then since the vector space is real, $(x,y) = (y,x)$ and

$$\begin{aligned}\|x+y\|^2 &= (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y) \\ &= \|x\|^2 + 2(x,y) + \|y\|^2\end{aligned}$$

so that $(x,y) = 0$.

This is false in a complex inner product space. Take $x \neq 0$ and $y = ix$. Then $(y,y) = \|y\|^2 = i\bar{i}(x,x) = \|x\|^2$ and $(x,y) \neq 0$. But

$$\begin{aligned}\|x+y\|^2 &= (x+ix, x+ix) \\ &= (x,x) + (x,ix) + (ix,x) + (ix,ix) \\ &= \|x\|^2 - i(x,x) + i(x,x) + \|x\|^2 = 2\|x\|^2.\end{aligned}$$

Question 2.

We know $(x_x, x) \to \|x\|^2$ and $\|x_n\|^2 \to \|x\|^2$. So that $(x, x_n) = \overline{(x_n, x)} \to \overline{\|x\|^2} = \|x\|^2$. Then

$$\begin{aligned}\|x_n - x\|^2 &= (x_n - x, x_n - x) \\ &= \|x_n\|^2 - (x_n, x) - (x_n, x) + \|x\|^2 \\ &\to \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0.\end{aligned}$$

Hence $\|x_n - x\| \to 0$ so that $x_n \to x$.

Question 3.

For (i) we note that $\{x_n\}$ is convergent if $\|x_n - x\| \to 0$. That is $(x_n - x, x_n - x) \to 0$. By the Cauchy-Schwartz inequality

$$|(x_n, y) - (x, y)| = |(x_n - x, y)| \leq \|x_n - x\|\|y\| \to 0.$$

For part (ii) observe that if $x_n \to x$ and $x_n \to z$ weakly, then $(x,y) = (z,y)$ for all $y \in H$. Thus $(x - z, y) = 0$ for all $y \in H$. Take $y = x - z$. We have $(x-z, x-z) = \|x-z\|^2 = 0$. Which implies that $x = z$ since $\|w\| = 0$ only when $w = 0$.

For part (iii) Consider $f_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$ which is in $L^2([-\pi, \pi])$. Let $g \in L^2([-\pi, \pi])$. Then $\widehat{g}(k) = (f_k, g)$, the kth Fourier coefficient of g. By the Riemann-Lebesgue Lemma (which we have seen in lectures and will prove later) $\widehat{g}(k) \to 0$ as $k \to \infty$. Thus f_k converges weakly to zero in $L^2([-\pi, \pi])$. But f_k does not converge in the strong sense.

Question 4.

Just expand both sides. All terms on right cancel except (x, y).

Question 5.

Let $f(x) = \dfrac{1}{\sqrt{1+x^2}}$. Then $f \in L^2(\mathbb{R})$, but $f \notin L^1(\mathbb{R})$. Conversely, let $g(x) = \dfrac{1}{\sqrt{x}}\chi_{[0,1]}$. Then $g \in L^1(\mathbb{R})$ but $g \notin L^2(\mathbb{R})$.

Question 6.

$f \in L^1([a, b])$ does not imply that $f \in L^2([a, b])$ as the previous question shows. However if $f \in L^2([a, b])$ and a, b are finite, then by Hölder's inequality

$$
\begin{aligned}
\int_a^b |f(x)|dx &= \int_a^b |1f(x)|dx \\
&\leq \|1\|_2\|f\|_2 \\
&= \sqrt{(b-a)}\|f\|_2 < \infty.
\end{aligned}
$$

So $f \in L^1([a, b])$.

Question 7.

This is again Hölder's inequality

$$
\int_{-\infty}^{\infty} |f(x)|^2 dx \leq \|f\|_p\|f\|_q.
$$

Question 8.

Let $f \in L^2([0, 1])$ and $0 < \beta < \alpha$. Then $f - \beta \leq (f - \beta)\chi_E \leq f\chi_{E_\beta}$. So by Hölder's inequality

$$
\begin{aligned}
0 < \alpha - \beta &\leq \int_0^1 f(x)dx - \beta \\
&= \int_0^1 (f(x) - \beta)dx \\
&\leq \int_0^1 f(x)\chi_{E_\beta}(x)dx \\
&\leq \|f\|_2[m(E_\beta)]^{1/2} = [m(E_\beta)]^{1/2}.
\end{aligned}
$$

So $m(E_\beta) \geq (\alpha - \beta)^2$.

Question 9.

We have

$$
\begin{aligned}
(\psi_n, \psi_m) &= \int_a^b \psi_n(x)\overline{\psi_m(x)}dx \\
&= \frac{2}{b-a}\int_a^b \phi_n\left(\frac{2}{b-a}\left(x-\frac{b+a}{2}\right)\right)\overline{\phi_n\left(\frac{2}{b-a}\left(x-\frac{b+a}{2}\right)\right)}dx
\end{aligned}
$$

Now let $t = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$. Then $(\psi_n, \psi_m) = \delta_{nm}$.

Question 10

Observe that the set

$$E = \{x \in [a,b] : |f(x)| \geq \epsilon\} = \{x \in [a,b] : |f(x)|^p \geq \epsilon^p\}.$$

Now we have

$$
\begin{aligned}
\int_a^b |f(x)|^p dx &\geq \int \chi_E |f(x)|^p dx \\
&\geq \epsilon^p \int \chi_E dx \\
&= m^*(E)\epsilon^p.
\end{aligned}
$$

The result follows.

Question 11

The previous question gives

$$m^*([x \in [a,b] : |f_n(x) - f(x)| \geq \epsilon]) \leq \epsilon^{-p} \int_a^b |f_n(x) - f(x)|^p dx.$$

Clearly if $\|f_n - f\|^p \to 0$, then $f_n \to f$ in measure.

37438 Modern Analysis

Problem sheet eight solutions.

Question 1.

We assume that $f, h \in L^1(\mathbb{R})$ and solve

$$u_t = u_{xx} + h(x), \ \ x \in \mathbb{R}, t > 0,$$
$$u(x,0) = f(x),$$

by taking the Fourier transform in x. This gives the first order ODE $\widehat{u}_t + y^2\widehat{u} = \widehat{h}(y)$, where $\widehat{u}(y,0) = \widehat{f}(y)$. Here $\widehat{f}$ is the Fourier transform of f. This is the same as

$$\frac{d}{dt}\left(e^{y^2t}\widehat{u}\right) = e^{y^2t}\widehat{h}(y),$$

where we multiplied the equation through by the integrating factor e^{y^2t}.

This gives

$$\widehat{u}(y,t) = \widehat{f}(y)e^{-y^2t} + \int_0^t e^{-y^2(t-s)}\widehat{h}(y)ds.$$

Taking the inverse Fourier transform gives

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(y)e^{-y^2t+iyx}dy + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{-y^2(t-s)+iyx}\widehat{h}(y)dsdy.$$

As in the lecture notes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(y)e^{-y^2t+iyx}dy = \int_{-\infty}^{\infty} f(\xi)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-\xi)^2}{4t}}d\xi.$$

Similarly

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{-y^2(t-s)+iyx}\widehat{h}(y)dtdy &= \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi)e^{-y^2(t-s)+iy(x-\xi)}dyd\xi ds\\ &= \int_0^t\int_{-\infty}^{\infty} h(\xi)\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-\xi)^2}{4(t-s)}}d\xi ds.\end{aligned}$$

So

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-\xi)^2}{4t}}d\xi + \int_0^t\int_{-\infty}^{\infty} h(\xi)\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-\xi)^2}{4(t-s)}}d\xi ds.$$

Question 2.

To solve $u_{xx} + u_{yy} = h(x)$, take the Fourier transform in x. This gives $\widehat{u}_{yy} - \xi^2\widehat{u} = \widehat{h}(\xi)$. Solving the ODE using variation of parameters or undetermined coefficients gives

$$\widehat{u}(\xi,y) = A(\xi)e^{-y|\xi|} + B(\xi)e^{y|\xi|} - \frac{\widehat{h}(\xi)}{\xi^2}$$

We take $B = 0$, since we cannot invert the Fourier transform if B is not zero, the corresponding integral being divergent. Using the initial data we have

$$\widehat{u}(\xi, 0) = A(\xi) - \frac{\widehat{h}(\xi)}{\xi^2} = \widehat{f}(\xi).$$

So

$$u(\xi, y) = \left(\widehat{f}(\xi) + \frac{\widehat{h}(\xi)}{\xi^2}\right) e^{-|\xi| y} - \frac{\widehat{h}(\xi)}{\xi^2}.$$

Inverting this is not at all straightforward. It depends on the behaviour of h, especially at $\xi = 0$. If h does not have sufficient decay at infinity, as well as the right behaviour at the origin, the inverse transform will not exist.

This is the starting point for a major theme of Fourier analysis, the theory of *singular integrals*. The origin of the problem of singular integrals lies in solving the Poisson equation $\Delta u = f$ on some domain $\Omega \subset \mathbb{R}$. The fundamental solution of the Laplace equation for dimensions strictly greater than 2 is $K(x) = c\|x\|^{2-n}$, where c is a constant depending on n. This is the Newton potential and notice that it is singular at the origin. To solve the Poisson equation in principle we have

$$u(x) = c \int_{\Omega} \frac{f(y)}{\|(x-y)\|^{n-2}} dy.$$

But for which functions f does this make sense? This is the basic problem of the theory of singular integrals. How do we make sense of integrals where the kernel has a singularity? It is a major area of research in Fourier analysis.

Question 3.

We take the Fourier transform and use the convolution theorem. Then $\widehat{u} = \widehat{h} + \widehat{k}\widehat{u}$. This gives

$$\widehat{u} = \frac{\widehat{h}}{1 - \widehat{k}}.$$

Given h we will often be able to invert this to obtain the value of u.

Question 4.

(a) We use the Parseval-Plancherel theorem $\|\widehat{f}\|_2^2 = 2\pi \|f\|_2^2$. Notice that

$$\frac{1}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y| - iyx} dy$$

Consequently

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi \int_{-\infty}^{\infty} \left(\frac{1}{2}e^{-|y|}\right)^2 dy$$
$$= \frac{\pi}{2}.$$

(b) Take $f(x) = \chi_{[-1,1]}(x)$, so that

$$\widehat{f}(y) = \int_{-1}^{1} e^{-iyx} dx = \frac{2\sin y}{y}.$$

Hence by Plancherel

$$4\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2} dy = 2\pi \int_{-1}^{1} dx = 4\pi.$$

Question 5.

The purpose of this question is to prove a form of the Fourier inversion theorem. The first few parts establish some useful identities.

(a)

$$\begin{aligned} h_\lambda(x) &= \int_{-\infty}^{\infty} e^{-\lambda|y|+iyx} dy \\ &= \int_{-\infty}^{0} e^{\lambda y+iyx} dy + \int_{0}^{\infty} e^{-\lambda y+iyx} x dy \\ &= \left.\frac{e^{y(y+ix)}}{y+ix}\right|_{-\infty}^{0} + \left.\frac{e^{-y(\lambda-ix)}}{-(\lambda-ix)}\right|_{0}^{\infty} \\ &= \frac{2\lambda}{\lambda^2+x^2}. \end{aligned}$$

(b)

$$\begin{aligned} \int_{-\infty}^{\infty} h_\lambda(x) dx &= 2\tan^{-1}(x/\lambda)\Big|_{-\infty}^{\infty} \\ &= 2\pi. \end{aligned}$$

(c)

$$h_\lambda(x) = \frac{1}{\lambda}\frac{2}{1+(x/\lambda)^2} = \frac{1}{\lambda}h_1\left(\frac{x}{\lambda}\right).$$

(d)

$$\begin{aligned}
f * h_\lambda(x) &= \int_{-\infty}^{\infty} f(x-y) \int_{-\infty}^{\infty} e^{-\lambda|\xi|+iy\xi} d\xi dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) e^{-\lambda|\xi|+iy\xi} dy d\xi \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r) e^{-\lambda|\xi|+i(x-r)\xi} dr y d\xi \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-\lambda|\xi|+ix\xi} d\xi
\end{aligned}$$

as required.

(e) We consider

$$\begin{aligned}
f * h_\lambda(x) - 2\pi f(x) &= f * h_\lambda(x) - \int_{-\infty}^{\infty} h_\lambda(y) f(x) dy \\
&= \int_{-\infty}^{\infty} \left(f(x-y) - f(x)\right) h_\lambda(y) dy \\
&= \int_{-\infty}^{\infty} \left(f(x-y) - f(x)\right) \frac{1}{\lambda} h_1(y/\lambda) dy \\
&= \int_{-\infty}^{\infty} \left(f(x-\lambda z) - f(x)\right) h_1(z) dz.
\end{aligned}$$

Now we apply the DCT to obtain

$$\begin{aligned}
\lim_{\lambda\to 0} \left(f * h_\lambda(x) - 2\pi f(x)\right) &= \lim_{\lambda\to 0} \int_{-\infty}^{\infty} \left(f(x-\lambda z) - f(x)\right) h_1(z) dz \\
&= \int_{-\infty}^{\infty} \lim_{\lambda\to 0} \left(f(x-\lambda z) - f(x)\right) h_1(z) dz \\
&= 0.
\end{aligned}$$

So $\lim_{\lambda\to 0} f * h_\lambda(x) = 2\pi f(x)$. Now if $f, \widehat{f} \in L^1(\mathbb{R})$ then we can apply the DCT to obtain

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \lim_{\lambda\to 0} \int_{-\infty}^{\infty} \widehat{f}(y) e^{ixy} e^{-\lambda|y|} dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{\lambda\to 0} \widehat{f}(y) e^{ixy} e^{-\lambda|y|} dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(y) e^{iyx} dy,
\end{aligned}$$

which is the Fourier inversion theorem.

Question 6.

Integration by parts gives

$$\int_{-\infty}^{\infty} f(x)e^{-iyx}dx = -\int_{-\infty}^{\infty} \frac{1}{y^2} f''(x)e^{-iyx}dx.$$

Thus

$$|\widehat{f}(y)| \leq \frac{1}{|y|^2}\|f''\|_1.$$

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{f}(y)|dy &= \int_{-M}^{M} |\widehat{f}(y)|dy + \int_{|y|>M} |\widehat{f}(y)|dy \\
&\leq \|\widehat{f}\chi_{[-M,M]}\|_1 + \|f''\|_1 \int_{|y|>M} \frac{1}{|y|^2}dy \\
&= \|\widehat{f}\chi_{[-M,M]}\|_1 + \|f''\|_1 \frac{2}{M} \\
&< \infty.
\end{aligned}$$

So $\widehat{f} \in L^1(\mathbb{R})$.

Question 7.

We define

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-kx}dx,$$

and

$$\widehat{f_n}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(x)e^{-kx}dx.$$

Then

$$\begin{aligned}
|\widehat{f_n}(k) - \widehat{f}(k)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}(f_n(x) - f(x))dx\right| \\
&\leq \int_{-\pi}^{\pi} |f_n(x) - f(x)|dx \to 0
\end{aligned}$$

as $n \to \infty$ by the DCT. Since we are on a finite interval then $|f_n(x) - f(x)|$ is bounded by a constant, so swapping limits and integrals is allowed.

Question 8.

There is a proof of Weierstrass's Theorem in the notes. We provide here an alternative proof using Fourier series. Any interval $[a, b]$ can be mapped to the interval $[-\pi, \pi]$ by a linear change of variables, so it is enough to work on $[-\pi, \pi]$ to prove the result. See the notes for more on this.

Let $f \in C([-\pi,\pi])$. Let $\widehat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}dx$. Then define $S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}$. By Fejer's Theorem

$$\sigma_M f = \frac{1}{M}\sum_{N=0}^{M-1} S_N f \to f$$

uniformly. Thus for $\epsilon > 0$ we can find M' large enough so that for $M > M'$

$$\sup_{x\in[-\pi,\pi]} |\sigma_M f(x) - f(x)| < \epsilon.$$

Now $\sigma_M f = \sum_{n=-M}^{M} a_n e^{inx}$ for some numbers $\{a_n\}$. Let M be such that

$$\left| f(x) - \sum_{n=-M}^{M} a_n e^{inx} \right| < \frac{\epsilon}{2}$$

for all $x \in [-\pi,\pi]$. Now $e^{ixn} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}(ixn)^k$ and the convergence is uniform on any interval $[-R,R]$. So we can find $M(n)$ such that

$$\left| \sum_{k=0}^{M(n)} \frac{(ixn)^k}{k!} - e^{ikx} \right| < \frac{\epsilon}{(1+4M(n)|a_n|)}$$

all $|x| \leq \pi$. Now let $P(x) = \displaystyle\sum_{n=-M}^{M} a_n \sum_{k=0}^{M(n)} \frac{(ixn)^k}{k!}$. Clearly P is a polynomial and by the triangle inequality

$$\begin{aligned}
|P(x) - f(x)| &\leq \sup_{x\in[-\pi,\pi]} \left(\left| P(x) - \sum_{n=-M}^{M} a_n e^{inx} \right| + \left| f(x) - \sum_{n=-M}^{M} a_n e^{inx} \right| \right) \\
&< \sum_{n=-M}^{M} |a_n| \left| \sum_{k=0}^{M(n)} \frac{(ixn)^k}{k!} - e^{ikx} \right| + \frac{\epsilon}{2} \\
&< \epsilon \sum_{n=-M}^{M} \frac{1}{4M+2} + \frac{\epsilon}{2} \\
&= \frac{2M\epsilon}{4M+2} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Thus P uniformly approximates f and we are done.

Question 9.

(a) We have

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ixn}dx \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n}\right)e^{-ixn}dx.\end{aligned}$$

So

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x) - f(x+\frac{\pi}{n})\right)e^{-ixn}dx.$$

(b) By the previous part

$$|\widehat{f}(n)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f\left(x+\frac{\pi}{n}\right)|dx \to 0$$

as $n \to \infty$ by the DCT. (Bound the integrand by a constant).

(c) If $|f(x+h) - f(x)| \leq C|h|^{\alpha}$ then

$$\begin{aligned}|\widehat{f}(n)| &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{C\pi}{|n|^{\alpha}}dx \\ &= \frac{C\pi}{2|n|^{\alpha}}.\end{aligned}$$

So

$$\widehat{f}(n) = O\left(|n|^{-\alpha}\right).$$

Question 10.

To simplify we use the Fourier transform in the form

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx.$$

Then the Poisson summation formula takes the form

$$\sum_{n=-\infty}^{\infty}\widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n).$$

Let $f(x) = e^{-\pi tx^2}$. Then $\widehat{f}(y) = \frac{1}{\sqrt{t}}e^{-\pi y^2/t}$. So

$$\begin{aligned}\theta(t) &= \sum_{n=-\infty}^{\infty} e^{-\pi tn^2} \\ &= \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} \\ &= \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).\end{aligned}$$

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Problem sheet nine solutions.

Question 1.

It is obvious that $\mu \times \nu(\emptyset) = 0$. Positivity is also obvious. We need to establish that the product measure is countably additive. To this end, let $A \times B \in S \times \Sigma$ and suppose that $A_n \times B_n$ is a collection of mutually disjoint sets, such that $A \times B = \cup_{n=1}^{\infty} A_n \times B_n$. We want to prove that

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) \qquad (*).$$

If either A or B has measure zero, then $(*)$ holds trivially, so we assume this is not the case.

Using the relation $\chi_{A\times B} = \sum_{n=1}^{\infty} \chi_{A_n \times B_n}$ we have

$$\chi_A(x)\chi_B(y) = \sum_{n=1}^{\infty} \chi_{A_n}(x)\chi_{B_n}(y),$$

for all x, y. Fix $y \in B$ and notice that $\chi_{B_n}(y)$ is either 1 or 0, so that $\chi_A(x) = \sum_{i\in K} \chi_{A_i}(x)$, where $K = \{i \in \mathbb{N}; y \in B_i\}$. The collection $A_i : i \in K$ is disjoint and so $\mu(A) = \sum_{i\in K} \mu(A_i)$ holds. Therefore

$$\mu(A)\chi_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\chi_{B_n}(y) \qquad (**),$$

for all $y \in Y$. Since a term with $\mu(A_n) = 0$ does not alter the sum in (*) or (**), we can assume $\mu(A_n) = 0$ for all n.

If both A and B have finite measure, then we integrate term by term and (*) holds. If either A or B has infinite measure, then

$$\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) = \infty,$$

and equality holds.

Question 2.

We have

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y)dxdy &= \int_0^1 \left[\left(-\frac{x}{x^2+y^2}\right)\right]_{x=0}^{x=1} dy \\ &= -\int_0^1 \frac{dy}{1+y^2} = -\tan^{-1} y|_{y=0}^{y=1} = -\frac{\pi}{4}.\end{aligned}$$

Conversely

$$\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \left[\left(\frac{y}{x^2+y^2} \right) \right]_{y=0}^{y=1} dx$$
$$= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x|_{y=0}^{y=1} = \frac{\pi}{4}.$$

This does not contradict Fubini's Theorem since the integrand is not integrable. That is the integral of $|f(x,y)|$ is not finite.

Question 3.

Put $f = \dfrac{d\lambda}{d\nu}, g = \dfrac{d\nu}{d\mu}$, so that

$$\lambda(A) = \int_A f d\nu = \int f\chi_A d\nu$$
$$= \int fg\chi_A d\mu$$
$$= \int_A fg d\mu \quad \mu \text{ a.e.}$$

So by the Radon-Nikodym Theorem

$$\frac{d\lambda}{d\mu} = fg = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}.$$

Question 4.

This is obvious from the previous question.

Question 5.

Let $\alpha = \mu/\sigma^2 - s$. With $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2} \right)$ we have

$$\int_{-\infty}^{\infty} f(x)e^{-xs}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) e^{-xs}dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} + x(\frac{\mu}{\sigma^2} - s) - \frac{\mu^2}{2\sigma^2} \right) dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\mu^2 - \sigma^4\alpha^2)}{2\sigma^2} \right) \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\sigma^2\alpha^2)^2}{2\sigma^2} \right) dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\mu^2 - \sigma^4\alpha^2)}{2\sigma^2} \right) \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2\sigma^2} \right) dz$$
$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\mu^2 - \sigma^4\alpha^2)}{2\sigma^2} \right) \sqrt{2\pi\sigma^2}$$
$$= \exp\left(\frac{1}{2}\sigma^2 s^2 - \mu s \right).$$

Question 6.

Clearly $aE = [a, ax]$. It follows that

$$\mu(E) = \int_1^x f(t)dt = \mu(aE) \tag{0.9}$$

and

$$\mu(aE) = \int_{aE} f d\lambda = \int_a^{ax} f(t)dt. \tag{0.10}$$

Hence the Radon-Nikodym derivative must satisfy

$$\int_1^x f(t)dt = \int_a^{ax} f(t)dt. \tag{0.11}$$

Differentiating both sides gives

$$\frac{d}{dx}\int_1^x f(t)dt = \frac{d}{dx}\int_a^{ax} f(t)dt. \tag{0.12}$$

Thus $f(x) = af(ax)$. Taking $x = 1$ gives $f(a) = \frac{f(1)}{a}$. Hence $f(x) = c/x$ for some constant c.
Question 7.

A linear functional L is continuous at $a \in X$ if for every sequence $\{a_n\}$ in X with $a_n \to x$ we have $L(a_n) \to L(a)$. This is equivalent to saying that if $\epsilon > 0$ then there exists $\delta > 0$ such that $\|a - y\| < \delta$ implies $|L(a) - L(y)| < \epsilon$. Let $y = a + h$. Then we have $|L(a + h) - L(a)| < \epsilon$ whenever $\|h\| < \delta$. But since L is linear, this gives

$$|L(a + h) - L(a)| = |L(h)| < \epsilon.$$

Now suppose $\|x - y\| < \delta$ for any $x, y \in X$. Write $y = x + h$ for some $h \in X$. Then

$$|L(x) - L(y)| = |L(x + h) - L(h)| = |L(h)| < \epsilon,$$

whenever $\|h\| < \delta$. Hence L is uniformly continuous.
Question 8.

$$\begin{aligned} P\left(\overline{\lim}_{n\to\infty} A_n\right) &= \lim_{m\to\infty} P\left(\bigcup_{n\geq m} A_n\right) \\ &\leq \lim_{m\to\infty} \sum_{n\geq m} P(A_n) = 0, \end{aligned}$$

if $\sum_{n=1}^{\infty} P(A_n) < \infty$.

Next $P\left(\underline{\lim}_{n\to\infty} A_n\right) = 1$ if and only if

$$\begin{aligned} \lim_{m\to\infty} P\left(\bigcap_{n\geq m} A_n^c\right) &= P\left(\bigcup_{m=1}^{\infty}\bigcap_{n\geq m} A_n^c\right) \\ &= P\left(\left(\overline{\lim}_{n\to\infty} A_n\right)^c\right) = 0. \end{aligned}$$

But by countable additivity, for given $m \geq 1$ we have

$$P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = \lim_{N\to\infty} \prod_{n=m}^{\infty}(1 - P(A_n))$$
$$\leq \lim_{N\to\infty} \exp\left(-\sum_{n=m}^{N} P(A_n)\right) = 0$$

if $\sum_{n=1}^{\infty} P(A_n) = \infty$. Here we used the inequality $1 - t \leq e^{-t}$ for $t \geq 0$.

Question 9.

Let X, Y be independent random variables. Then

$$\begin{aligned} Var(X+Y) &= E(((X+Y) - E(X+Y))^2) \\ &= E(X+Y)^2 - (E(X)+E(Y))^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - (E(X))^2 \\ &\quad - 2E(X)E(Y) - (E(Y))^2 \\ &= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 \\ &\quad + 2[E(XY) - E(X)E(Y)] \\ &= Var(X) + Var(Y), \end{aligned}$$

since by independence $E(XY) = E(X)E(Y)$. By induction we can establish the more general result that if $X_i, i = 1, ..., n$ are independent random variables, then $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$.

Question 10.

Let $S_n = \sum_{k=1}^{n} X_k$. Then $E(S_n) = \sum_{k=1}^{n} E(X_k)$. So that we have $E\left(\dfrac{S_n}{n}\right) = \mu$. Thus

$$\begin{aligned} E\left(\left(\frac{S_n}{n} - \mu\right)^2\right) &= Var\left(\frac{S_n}{n}\right) \\ &= \frac{1}{n^2} Var(S_n) \\ &= \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) \\ &\leq \frac{K}{n} \to 0 \end{aligned}$$

as $n \to \infty$. Thus $E\left(\left(\frac{S_n}{n} - \mu\right)^2\right) \to 0$ as $n \to \infty$. Hence $1/nS_n \to \mu$ in the L^2 sense.

Question 11.

The fact that $d(X,Y) = d(Y,X)$ is obvious. If $d(X,Y) = 0$ then $E(|X - Y|) = 0$ so $X = Y$ in L^1. The triangle inequality follows from the same inequality for the metric

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

which was established in an earlier tutorial.

Now suppose that $X_n \to X$ in the probability measure P. Let $\epsilon > 0$. Then

$$\begin{aligned} d(X_n, X) &= \int_{|X_n - X| < \epsilon/2} \frac{|X_n - X|}{1 + |X_n - X|} dP + \int_{|X_n - X| > \epsilon/2} \frac{|X_n - X|}{1 + |X_n - X|} dP \\ &\leq \int_{|X_n - X| < \epsilon/2} dP + \leq \int_{|X_n - X| > \epsilon/2} dP \\ &\leq \frac{\epsilon}{2} + P(|X_n - X| \geq \epsilon/2). \end{aligned}$$

Since we assumed that $P(|X_n - X| \geq \epsilon/2) \to 0$ then $d(X_n, X) \to 0$ as $n \to \infty$.

Conversely, let $E_{\epsilon,n} = \{x : |X_n(x) - X(x)| > \epsilon\}$ and suppose that $0 < \epsilon < 1$. Let $A_n = \{x : |X_n(x) - X(x)| < 1\}$. Then write

$$d(X_n, X) = \int_{A_n} \frac{|X_n - X|}{1 + |X_n - X|} dP + \int_{A_n^c} \frac{|X_n - X|}{1 + |X_n - X|} dP.$$

We estimate from below these terms.

$$\begin{aligned} \int_{A_n} \frac{|X_n - X|}{1 + |X_n - X|} dP &\geq \int_{A_n \cap E_{\epsilon,n}} \frac{|X_n - X|}{1 + |X_n - X|} dP \\ &\geq \frac{1}{2} \int_{A_n \cap E_{\epsilon,n}} \epsilon dP \\ &= \frac{\epsilon}{2} P(A_n \cap E_{\epsilon,n}), \end{aligned}$$

since $\dfrac{a}{1+a} > \dfrac{a}{2}$ if $a < 1$. For the second integral

$$\begin{aligned} \int_{A_n^c} \frac{|X_n - X|}{1 + |X_n - X|} dP &\geq \int_{A_n^c} \frac{1}{2} dP + \int_{A_n^c \cap E_{\epsilon,n}} \frac{1}{2} dP \\ &\geq \frac{\epsilon}{2} P(A_n \cap E_{\epsilon,n}) \end{aligned}$$

since $\epsilon < 1$. Hence $d(X_n, X) \geq \frac{\epsilon}{2} P(E_{\epsilon,n}) \to 0$ as $n \to \infty$. So X_n converges to X in the measure P.