MODERN ANALYSIS AND ITS APPLICATIONS

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Abstract. These are the lecture notes for the subject 37438 Modern Analysis and its applications, taught at the University of Technology, Sydney. They are based on a variety of sources and are not to be sold or to be distributed to students outside of UTS. The Sources include, Introduction to Real Analysis, by G. Mcllenand, UTS; Real Analysis by Royden and Fitzpatrick 4th Ed, (Pearson); A Garden of Integrals, F.E. Burk (MAA); Analysis. An Introduction R. Beals, Cambridge; Principles of Real Analysis, Aliprantis and Burkinshaw, North Holland; W. Rudin, Functional Analysis, as well as my own notes. The notes are not for profit.

Contents

1. Introductory Real Analysis

Let us review a few important ideas introduced in a typical first course in analysis. Analysis is largely concerned with the behaviour of functions, which we usually want to be continuous. However in order to develop a useful theory of the behaviour of functions, we first need to study sequences. An investigation of sequences and functions leads to the major tools of elementary calculus, namely the derivative and integral.

We begin with one of the most basic concepts in mathematics.

1.0.1. *Sets.* Particularly in the theory of measure, we are required to manipulate sets. So here we review some elementary facts.

Definition 1.1. Let A and B be sets.

(i) The union of A and B is denoted $A \cup B$ and is given by
$$A \cup B = \{x : x \in A \text{ or } B\}.$$
The union of a collection of sets $A_i, i \in \mathbb{N}$ is defined inductively and denoted $\cup_{i\in\mathbb{N}} A_i$.

(ii) The intersection of A and B is denoted $A \cap B$ and is given by
$$A \cap B = \{x : x \in A \text{ and } B\}.$$
The intersection of a collection of sets $A_i, i \in \mathbb{N}$ is defined inductively and denoted $\cap_{i\in\mathbb{N}} A_i$.

(iii) The set difference of A and B is denoted $A - B$ and is given by
$$A - B = \{x \in A, x \notin B\}.$$

(iii) We say that A is a subset of B and write $A \subset B$ if every element of A is contained in B. If A is contained in and may be equal to B we write $A \subseteq B$.

Throughout these notes $\emptyset$ will denote the empty set. That is, the set containing no elements.

Definition 1.2. Two sets A, B are said to be disjoint if $A \cap B = \emptyset$.

Unions complements and differences satisfy certain laws. The proof of the next result is a simple exercise.

Proposition 1.3. *Let A, B, C be sets. The following relations hold.*

(i) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(ii) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(iii) $(A \cup B) - C = (A - C) \cup (B - C)$.

(iv) $(A \cap B) - C = (A - C) \cap (B - C)$.

The most important rules for sets are deMorgan's laws.

Definition 1.4. Let $A \subset X$. Then $A^c = \{x \in X : x \notin A\}$. We call A^c the complement of A. We define $X^c = \emptyset$.

Theorem 1.5 (deMorgan)**.** *Let $A_i, i \in \mathbb{N}$ be a collection of sets. Then*

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c, \quad \left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c.$$

Two other useful relationships are $A - B = A \cap B^c$ and $A \subseteq B$ if and only if $B^c \subseteq A^c$.

1.1. Countability.

It is often important to distinguish between different kinds of infinite sets. For example, the rational numbers and the real numbers are both infinite sets, but there is a sense in which the real numbers is a bigger set than the rational numbers. To make this precise we introduce the notion of countability.

Definition 1.6. A set X is countable if there is a one to one function $f : X \to \mathbb{N}$. A countable set is also said to be denumerable. A set which is not countable is said to be uncountable. If f is also onto then we say that X is countably infinite.

Some authors prefer to use the term countable only when the set is countably infinite. A finite set might then be termed finitely countable. However this distinction is unimportant. There are equivalent formulations which are useful in establishing the countability of certain sets.

Theorem 1.7. *Let A be an infinite set. The following are equivalent.*

(i) *A is countable.*

(ii) *There exists a subset B of $\mathbb{N}$ and a function $f : B \to A$ which is onto.*

(iii) *There exists a function $g : A \to \mathbb{N}$ that is one to one.*

Proof. These are all straightforward. For example (iii) follows from the fact that there is a one to one and onto function $f : A \to \mathbb{N}$, so f is invertible. The others are exercises. □

An important fact about countable sets follows.

Theorem 1.8. *Let $X_i, i = 1, 2, 3, ...$ be countable sets. Then the union $X = \bigcup_{i=1}^{\infty} X_i$ is also countable.*

Proof. We let $X_i = \{x_1^i, x_2^i, x_3^i, ...\}$. Let $B = \{2^k 3^n : k, n \in \mathbb{N}\}$. Now define $f : B \to A$ by $f(2^k 3^n) = x_k^n$. Then f maps B onto A, so A is countable by Theorem 1.7. □

The proof of the next result is also an easy consequence of Theorem 1.7 and is an exercise.

Theorem 1.9. *Suppose that $X_i, i = 1, ..., n$, $n < \infty$. are countable sets. Then $X_1 \times \cdots \times X_n$ is countable.*

Example 1.1. The set $\{a, b, c, d\}$ is countable. For example we might have $f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$.

Example 1.2. The natural numbers are countable. Just take $f(n) = n$.

Example 1.3 (Cantor). The rational numbers $\mathbb{Q}$ are countable. Clearly $\mathbb{Q} = \mathbb{Q}^+ \bigcup \mathbb{Q}^-$, where the superscripts denote the negative and non-negative rationals respectively. So it is enough to show that the positive rationals are countable. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ by $f(m, n) = m/n$. Clearly f is onto, so by Theorem 1.7 the rationals are countable.

Example 1.4 (Cantor). The real numbers $\mathbb{R}$ are uncountable. A very nice proof of this result uses the Baire Category Theorem and will be presented later. Cantor presented at least two proofs of this result. The second is the most famous and is known as his diagonal argument. Suppose that we restrict attention to the interval $[0, 1]$ and represent every number in $[0, 1]$ in binary, that is as a possibly infinite sequence of zeroes and ones. We then make a list of all the elements in some order. So for example if we create a list of sequences from the binary expansion of the numbers in $[0, 1]$, the list might look like this:

$$\begin{aligned}
s_1 &= (0, 0, 0, 0, 0, 0, 0, ...)\\
s_2 &= (0, 1, 1, 1, 0, 0, 1, ...)\\
s_3 &= (1, 1, 1, 1, 0, 0, 0, ...)\\
s_4 &= (0, 0, 1, 1, 0, 1, 0, ...)\\
s_5 &= (0, 1, 1, 0, 0, 1, 1, ...)\\
&\vdots
\end{aligned}$$

We claim that no possible list can contain every possible sequence of zeroes and ones. To show this we construct and element s_0 which is not in the given list. We do so by looking down the diagonal of the array of numbers given above. That is, we look at the element s_{ii} and choose element number i of s_0 to not equal s_{ii}. So from the list here we would define

$$s_0 = (1, 0, 0, 0, 1, ...)$$

Notice the first element of s_1 is 0, so we choose the first element of s_0 to be 1. The second element of s_2 is 1, so the second element of s_0 is 0. The third element of s_3 is 1, so the third element of s_0 is 0 and so on.

The sequence s_0 is not in the above list. Suppose otherwise. Then there is an integer N such that S_N is in the above list and $s_0 = s_N$. In particular the Nth term of the sequence s_0 is the Nth term of the sequence s_N. But this is a contradiction, because we constructed s_0 by choosing $s_{0N} \neq s_{NN}$. So s_0 is not in the above list. This is true for any possible countable list. So no countable list of sequences of zeroes and ones can contain every sequence of zeroes and ones. Hence the interval $[0,1]$ is not countable and hence $\mathbb{R}$ is not countable.

1.1.1. *Sets and Real numbers.* Analysis makes use of an axiom and properties of the real numbers. We start with the humble triangle inequality, which is easily the most important inequality in mathematics.

Lemma 1.10. *For any real numbers a, b, $|a+b| \leq |a|+|b|$, where the absolute value is defined by $|x| = \sqrt{x^2}$ for $x \in \mathbb{R}$.*

Proof. This is elementary.

$$\begin{aligned} |a+b|^2 &= (a+b)^2 = a^2 + 2ab + b^2 \\ &\leq |a|^2 + 2|ab| + |b|^2 \\ &= (|a|+|b|)^2. \end{aligned}$$

We used the fact that $ab \leq |ab|$. Now take square roots. □

A rigorous treatment of analysis must begin with the axiom which is the foundation of the subject. If we consider sets of real numbers, then we can ask various questions of them. For example, are they bounded?

Definition 1.11. A finite number u is an upper bound for a set $A \subseteq \mathbb{R}$ if for every $x \in A,\ x \leq u$. Similarly, l is a lower bound for A if for every $x \in A,\ x \geq l$.

Now suppose that $A \subset \mathbb{R}$ is non-empty and that there is an upper bound? We can ask whether or not there is a least upper bound?

Definition 1.12. If $\bar{u}$ is an upper bound of a set $A \subseteq \mathbb{R}$ with the property that $\bar{u} \leq u$, for all other upper bounds u, then $\bar{u}$ is called the least upper bound or supremum of A. We write $\bar{u} = \sup A$. Similarly, a lower bound $\bar{l}$ of a set $A \subset \mathbb{R}$ is the greatest lower bound or infimum, if for every lower bound of A we have $\bar{l} \geq l$. We write $\bar{l} = \inf A$.

Consideration of elementary examples would suggest that every non empty set, bounded above, does indeed have a least upper bound. Indeed it is impossible to write down a counter example. Many examples are straightforward. Take the set $[0,1)$. The least upper bound is obviously 1. This is easy, but it turns out that there is no way that one can prove that every nonempty set of real numbers which is bounded above has a least upper bound. Instead we make it an axiom.

Axiom 1: The Least Upper Bound Axiom: Every non empty set of real numbers which is bounded above has a least upper bound.

Example 1.5. This example is important and uses ideas that we will develop below. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Let $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ be a partition of $[0,1]$. Let $m_i = \max_{x \in [x_{i-1}, x_i]} f(x)$.

Define $L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$. Then by our axiom,

$$I = \sup_{\mathcal{P}}\{L(f,\mathcal{P})\} \tag{1.1}$$

exists. The point is that we could not otherwise prove the existence of this supremum.

Example 1.6. Does the set of rational numbers in $[0,\pi]$ have a least upper bound? Suppose that there is a rational number $0 < \epsilon < \pi$, then by the continuum property of the real numbers, there is a rational number δ, with $\epsilon < \delta < \pi$. So this set has no least upper bound.

From this axiom all of analysis is derived. We start by proving that non-empty sets bounded below have greatest lower bounds.

Theorem 1.13. *A non-empty set of real numbers bounded below has a greatest lower bound.*

Proof. Suppose that A is nonempty and bounded below. Now consider the set $-A = \{-x : x \in A\}$. This set is nonempty and bounded above: If l is a lower bound of A, then $-l$ is an upper bound of $-A$. To see this, notice that if $x \in A$, then $l \leq x$. So $-l \geq -x$. Hence $-l$ is an upper bound. By **Axiom 1**, $-A$ has a least upper bound $\bar{u}$. Then it follows that $\bar{l} = -\bar{u}$ is the greatest lower bound for A. □

An important result we use extensively follows.

Theorem 1.14. *Assume that* $\sup A$ *exists, where* $A \subset \mathbb{R}$. *Then for every* $\epsilon > 0$ *we can find an* x *such that*

$$\sup A - \epsilon < x \leq \sup A.$$

Proof. Suppose that for every $x \in A$, $x \leq \sup A - \epsilon$. Then $\sup A - \epsilon$ is an upper bound for A, less than $\sup A$, which is a contradiction. So there must be some x in A with $x > \sup A - \epsilon$. Clearly $x \leq \sup A$. □

The Archimedean Property This is an obvious property, which nevertheless is fundamental: Given any two positive real numbers x, y, we can find a natural number n such that $nx > y$. Equivalently, there is no largest natural number.

1.2. **Sequences.** We now introduce the concept of a sequence.

Definition 1.15. A sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$. We usually write $f(n) = x_n$ and denote the sequence by $\{x_n\}_{n=1}^{\infty}$ or just $\{x_n\}$.

Sequences are what analysis is made of. Many practical problems have solutions which are given by constructing sequences which "converge" to a solution. That is, which gets closer and closer to the solution as n increases. We can formally define convergence as follows.

Definition 1.16. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be convergent with limit x, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$. We write $x_n \to x$. A sequence which does not converge is said to diverge.

It is easy to establish some basic facts about convergent sequences. From here on we will see just how essential the triangle inequality is to analysis. The subject could not exist without it.

Theorem 1.17. *The limit of a convergent sequence is unique.*

Proof. Suppose that $x_n \to x$ and $x_n \to y$. Then for all n

$$|x - y| = |x - x_n + x_n - y| \leq |x_n - x| + |x_n - y|.$$

But $|x_n - x| \to 0$ and $|x_n - y| \to 0$, so $|x - y| = 0$. □

Theorem 1.18. *Every convergent sequence is bounded.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x and choose N such that $n \geq N$ implies $|x_n - x| < 1$. Now let

$$M = \max\{|x_1|, ..., |x_{N-1}|, 1 + |x|\}.$$

Clearly if $1 \leq n \leq N - 1$ then $|x_n| \leq M$. Conversely, if $n \geq N$, Then

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x| \leq M.$$

So for all n, $|x_n| \leq M$, and hence the sequence is bounded. □

Convergent sequences behave as you would expect under addition, multiplication and division.

Theorem 1.19. *Let a, b be constants and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences, with limits x, y respectively. Then*

(1) $ax_n \to ax$.

(2) $ax_n + by_n \to ax + by$

(3) $x_n y_n \to xy$

(4) *If y_n is never zero and $y \neq 0$, $x_n/y_n \to x/y$.*

Proof. These are easy to prove. For example, let $\epsilon > 0$. Choose M such that $n \geq N$ implies $|x_n - x| < \epsilon/(2|a|)$ and K such that $n \geq K$ implies $|y_n - y| < \epsilon(2|b|)$. Then let $N = \max\{M, K\}$. Then for $n \geq N$,

$$\begin{aligned} |ax_n + by_n - ax - by| &\leq |a||x_n - x| + |b||y_n - y| \\ &< |a|\epsilon/(2|a|) + |b|\epsilon/(2|b|) = \epsilon. \end{aligned}$$

Proofs of the other results are exercises. □

We also have the essential result that increasing bounded sequences are convergent.

Theorem 1.20. *Every monotone increasing sequence which is bounded above has a limit. Every monotone decreasing sequence bounded below has a limit.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded, increasing sequence. Then for all n, $x_{n+1} \geq x_n$. Consider the set $A = \{x_1, x_2, x_3, ...\}$. This set is non-empty and bounded above, so it has a least upper bound, which we denote by x. Now pick $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $x_N > x - \epsilon$. Since x is the least upper bound we can do this, otherwise it would not be the least upper bound. Since $\{x_n\}_{n=1}^{\infty}$ is increasing we have for $n \geq N$, $|x_n - x| = x - x_n < \epsilon$. Hence $x_n \to x$. The case of a decreasing sequence is similar. □

Subsequences play an essential role in analysis.

Definition 1.21. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. A subsequence $\{x_{n_K}\}_{K=1}^{\infty}$ is a sequence contained in $\{x_n\}_{n=1}^{\infty}$, where n_K is an increasing sequence of integers. So $n_K \to \infty$ as $K \to \infty$.

The most important results about sequences on the real line stem from the Bolzano-Weierstrass Theorem. To prove this, we need some preliminaries.

Theorem 1.22. *Every sequence has a monotone subsequence.*

Proof. We sketch the proof. We suppose that the sequence is not constant after some term x_N. If $x_n = a$ for all $n \geq N$, then the result is trivial. So suppose this is not the case. The basic idea is to construct the sequence. We pick the first element, say $y_1 = x_n$, then we move along the sequence till we find another element x_{n_1} which is larger then x_n and then take $y_2 = x_{n_1}$. Now move along the sequence till we come to a larger element, make that the third element of the subsequence. Continuing we construct a monotone increasing sequence. Similarly for the case of a monotone decreasing sequence. □

Theorem 1.23 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. A bounded sequence has a bounded monotone subsequence. Bounded monotone sequences are convergent. So every bounded sequence has a convergent subsequence. □

The reason why this result is so important is that we often need to deal with sequences of real numbers on bounded intervals and in many proofs we pick a convergent subsequence to work with. Texts on elementary real analysis will deal with this and we will see some examples of this in practice.

Definition 1.24. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that $n, m \geq M$ implies $|x_n - x_m| < \epsilon$.

Every Cauchy sequence is convergent. This fact underpins a lot of what follows. Cauchy sequences and convergent sequences are basically the same. The point is that Cauchy's criterion gives us a different way of determining convergence, which is particularly useful when we do not have the limit available to us. First we show an easy result.

Proposition 1.25. *Every convergent sequence is a Cauchy sequence.*

Proof. We pick $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon/2$. Then we have for $n, m \geq N$

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \epsilon/2 + \epsilon/2 = \epsilon,$$

so a convergent sequence is Cauchy. □

Proving the converse is more difficult. We have to show that a Cauchy sequence is bounded. The proof of this result is much the same as the proof that a convergent sequence is bounded. Then we establish the following result:

Proposition 1.26. *If a Cauchy sequence has a convergent subsequence with limit x, then the Cauchy sequence converges to x.*

Proof. To see this, suppose that $\{x_n\}_{n=1}^{\infty}$ is Cauchy and that there is a subsequence $\{x_{n_K}\}_{K=1}^{\infty}$ which converges to x. So that $\lim_{K\to\infty} x_{n_K} = x$. We then choose N large enough to make $|x_n - x_m| < \epsilon/2$ for all $n, m \geq N$ and pick K large enough to make $n_K > N$. Then by the triangle inequality

$$\begin{aligned}|x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

So $x_n \to x$. □

The next step is easy.

Theorem 1.27. *Every Cauchy sequence is convergent.*

Proof. Every Cauchy sequence is bounded. By the Bolzano-Weierstrass Theorem, it follows that every Cauchy sequence has a convergent subsequence. Consequently, every Cauchy sequence converges. □

Cauchy sequences are important because they allow us to establish convergence without knowing what the limit is. In most cases, we cannot compute the limit exactly, so we cannot prove convergence by establishing that $|x_n - x| \to 0$ since x is unknown. We can however often prove that $|x_n - x_m| \to 0$ as $n, m \to \infty$.

It is important to understand that a sequence with the property that $|x_{n+k} - x_n| \to 0$, as $n \to \infty$, for fixed k, is not necessarily Cauchy. We insist that $|x_n - x_m| \to 0$ as both $n, m \to \infty$. For example, the harmonic sequence

$$x_n = \sum_{k=1}^{n} \frac{1}{k},$$

diverges. The proof of this result is quite ancient and is often attributed to Nicolas Oresme (born between 1320-25, died 1382). However it may well have been established in India even earlier. It is based on the observation that

$$\begin{aligned}\sum_{k=N}^{2N} \frac{1}{k} &= \frac{1}{N} + \cdots + \frac{1}{2N} \\ &\geq N \times \frac{1}{2N} = \frac{1}{2}\end{aligned}$$

Similarly $\sum_{k=2N+1}^{4N} \geq \frac{1}{2}$, $\sum_{k=4N+1}^{8N} \geq \frac{1}{2}$ etc. So that

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

and so $x_n \to \infty$. However $|x_{n+1} - x_n| = \frac{1}{n} \to 0$ as $n \to \infty$.

The use of convergent subsequences is one of the most common techniques in analysis. Notice that we have also used the triangle inequality extensively. When we come to extend analysis to other types of spaces, we will want our measure of length to satisfy the triangle inequality. This is not hard to do, but obtaining convergent subsequences will be more difficult. We will discuss this further when we turn to Hilbert and Banach spaces.

1.3. **Series.** As the example of the harmonic sequence shows, we can handle series by treating them as sequences.

Definition 1.28. A series $S = \sum_{n=1}^{\infty} a_n$ is said to be convergent with limit S if the sequence of partial sums $\{S_N\}_{N=1}^{\infty}$ with $S_N = \sum_{n=1}^{N} a_n$ is convergent with limit S. If the series is not convergent, we say it diverges.

The next result is obvious.

Theorem 1.29. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent with sums S and T respectively, then $\sum_{n=1}^{\infty}(a_n + b_n) = S + T$. Further $\sum_{n=1}^{\infty} ca_n = cS$ for all $c \in \mathbb{R}$.*

Proof. This follows from previously established properties of sequences applied to $S_N = \sum_{n=1}^{N} a_n$ and $T_N = \sum_{n=1}^{N} b_n$. □

Note it is not true that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge then $\sum_{n=1}^{\infty} a_n b_n$ is convergent. We require absolute convergence for this.

Definition 1.30. A series is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, the series is said to be conditionally convergent.

Example 1.7. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. However the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. Hence this series is conditionally convergent.

There are many simple but useful properties possessed by convergent series.

Lemma 1.31. *If $\sum_{n=1}^{\infty} a_n$ is convergent, then $lim_{n\to\infty} a_n = 0$.*

Proof. We have $S_n = \sum_{k=1}^{n} a_k$ and $\{S_n\}_{n=1}^{\infty}$ is convergent with limit S. Then $S_{n+1} - S_n \to 0$ but$S_{n+1} - S_n = a_n$. □

Another useful fact is the following. The proof is an exercise.

Proposition 1.32. *Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be convergent. Then as, $N, M \to \infty$, $\sum_{n=M+1}^{N} a_n \to 0$.*

Let us see what is needed to guarantee that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proposition 1.33. *If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.*

Proof. Consider the sequence $S_N = \sum_{n=1}^{N} a_n b_n$. Now $\sum_{n=1}^{\infty} a_n$ is convergent, hence the sequence $a_n \to 0$ and so is bounded. Suppose that $|a_n| \leq K$. Then if $N > M$

$$|S_N - S_M| = |\sum_{n=M+1}^{N} a_n b_n| \leq \sum_{n=M+1}^{N} |a_n b_n|$$
$$\leq K \sum_{n=M+1}^{N} |b_n| \to 0,$$

as $N, M \to \infty$ since $\sum_{n=1}^{\infty} |b_n|$ is convergent. Thus $\{S_N\}_{N=1}^{\infty}$ is Cauchy and hence it converges. □

There are various tests for convergence. Most rely on the comparison test.

Theorem 1.34 (Comparison Test)**.** *Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms. If there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent. Conversely if $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.*

Proof. Let $T = \sum_{k=1}^{\infty} b_k$ and $S_n = \sum_{k=1}^{n} a_k$. Since the a_n are positive S_N is increasing. We show that it is bounded above.

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_{N-1} + a_N + \cdots + a_n \\ &\leq a_1 + \cdots + a_{N-1} + b_N + \cdots + b_n \\ &= b_1 + \cdots + b_{N-1} + b_N + \cdots + b_n \\ &\quad + (a_1 - b_1) + \cdots + (a_{N-1} - b_{N-1}) \\ &= T_n + \sum_{k=1}^{N-1} (a_k - b_k) \\ &\leq T + \sum_{k=1}^{N-1} (a_k - b_k). \end{aligned}$$

So $\{S_n\}_{n=1}^{\infty}$ is increasing and bounded above and hence converges.

The proof of the second part is similar. □

A variant of this is the limit comparison test.

Theorem 1.35 (The Limit Comparison Test). *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be series of strictly positive terms. Suppose that*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l \neq 0.$$

Then either both series converge or both series diverge.

Proof. We can assume that $a_n \to 0$ and $b_n \to 0$, since the series will diverge otherwise. Since $a_n/b_n \to l$, the sequence $\{a_n/b_n\}$ is bounded by some number K. From which it follows that $a_n \leq K b_n$. So that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges by the comparison test. Conversely, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Now we can find $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > \frac{1}{2} l b_n$, so that if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} b_n$. The divergence of $\sum_{n=1}^{\infty} b_n$ implies the divergence of $\sum_{n=1}^{\infty} a_n$. □

Example 1.8. Consider $S_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Then

$$\begin{aligned} S_n &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \\ &< 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{n(n-1)} \\ &= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

So this series is convergent. (It actually equals $\pi^2/6$). Now consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n^3 + n + 3}$. We apply the limit comparison test and compute

$$\lim_{n\to\infty}\left(\frac{n+1}{2n^3+n+3}\right)/\left(\frac{1}{n^2}\right)=1/2\neq 0.$$

So the second series also converges.

A powerful convergence test follows.

Theorem 1.36 (The Ratio Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive terms. Let $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=L$. Then the series converges if $L<1$ and diverges if $L>1$. If $L=1$ then the test is inconclusive.*

Proof. Suppose that $L<1$. Pick $r\in\mathbb{R}$ such that $L<r<1$. We can choose $n\in\mathbb{N}$ such that $n\geq N$ implies

$$\left|\frac{a_{n+1}}{a_n}-L\right|<r-L$$

or

$$-(r-L)<\frac{a_{n+1}}{a_n}-L<r-L.$$

So $a_{n+1}<ra_n$. Also $a_n<ra_{n-1}<r^2a_{n-2}<r^3a_{n-3}$ etc. Indeed $a_n<r^ka_{n-k}$. Now let $k=n-N$. Then

$$a_n\leq r^{n-N}a_N=\left(\frac{a_N}{r^N}\right)r^n.$$

The series $\sum_{n=1}^{\infty}\left(\frac{a_N}{r^N}\right)r^n$ is a geometric series with common ratio $r<1$ and so converges. By the comparison test $\sum_{n=1}^{\infty}a_n$ also converges.

For the case $L>1$ the proof is similar, with the final inequalities reversed and $r>1$, giving a divergent geometric series. Finally, for the series $\sum_{n=1}^{\infty}\frac{1}{n}$, $L=1$ and the series diverges. For the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, $L=1$ and the series converges. So the ratio test is inconclusive if $L=1$. □

Remark 1.37. We can apply the ratio test to series of nonpositive terms. We instead consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L$ and the conclusions are the same as in the given result.

To state the nth root test, we introduce the idea of a limsup.

Definition 1.38. If $\{x_n\}$ is a bounded sequence, then the largest subsequential limit $\bar{l}$ is

$$\bar{l}=\limsup x_n \tag{1.2}$$

and the smallest subsequential limit s is

$$s=\liminf x_n. \tag{1.3}$$

Some authors also write $\lim_{n\to\infty}\sup x_n0$ and $\lim_{n\to\infty}\inf x_n$.

Example 1.9. If $x_n=(-1)^n$, then $\limsup x_n=1$ and $\liminf x_n=-1$.

The convergence of a sequence can be given in terms of its lim sup and lim inf . The proof of the next result is an exercise.

Proposition 1.39. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *converges if and only if*

$$\limsup x_n = \liminf x_n.$$

We now give yet another convergence test.

Theorem 1.40 (The nth root test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be a series and suppose that*

$$\limsup |a_n|^{1/n} = L. \tag{1.4}$$

If $L < 1$ *the series converges. If* $L > 1$ *the series diverges. If* $L = 1$ *the series may converge or diverge.*

Proof. This is another application of the comparison test. If $L < 1$, then there is an r such that $L < r < 1$ and for n large enough $|a_n| < r^n$. Convergence follows from the comparison test with a geometric series. The proof of the second case is similar. Finally, we can exhibit series which converge when $L = 1$ and diverge when $L = 1$. (This is an exercise). □

There are a number of other, lesser known tests for convergence which can be very useful. We present one next.

Theorem 1.41 (Cauchy Condensation Test)**.** *Suppose that the sequence* a_n *is positive and non-increasing. Then the series* $\sum_{n=1}^{\infty} a_n$ *converges, if and only if the series* $\sum_{n=0}^{\infty} 2^n a_{2^n}$ *converges. Moreover we have the estimate*

$$\sum_{n=1}^{\infty} a_n \le \sum_{n=0}^{\infty} 2^n a_{2^n} \le 2 \sum_{n=1}^{\infty} a_n.$$

Proof. Since the sequence is non-decreasing, we have $a_2 + a_3 \le 2a_2$, $a_4 + a_5 + a_6 + a_7 \le 4a_4$ etc. So that

$$\sum_{n=1}^{\infty} a_n \le a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

Thus if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges, then so does $\sum_{n=1}^{\infty} a_n$. Similarly $a_1+a_2 \le 2a_2$, $a_2 + a_4 + a_4 + a_4 \le 2a_2 + 2a_3$, etc. So that

$$\begin{aligned}\sum_{n=0}^{\infty} 2^n a_{2^n} &= a_1 + 2a_2 + 4a_4 + \cdots \le 2a_1 + 2a_2 + 2a_3 + \cdots \\ &= 2\sum_{n=1}^{\infty} a_n.\end{aligned}$$

So by the comparison test the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges if $\sum_{n=1}^{\infty} a_n$ converges. The estimate follows from the above. □

Example 1.10. We consider the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Now

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} \frac{1}{2^{np-n}}.$$

This is a geometric series that will converge for $np - n > 1$ and diverge otherwise. Hence the original series converges for $p > 1$ and diverges for $p \leq 1$.

Finally we mention a test for alternating series.

Theorem 1.42. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms with $a_n \to 0$. Then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.*

Proof. We will show that the sequence of partial sums is Cauchy. Let $\epsilon > 0$ and note that since $a_n \to 0$ we can find an $N \in \mathbb{N}$ such that $a_n < \epsilon$ for all $n \geq N$. Next observe that since a_n is monotone decreasing, $a_n - a_{n+1} \geq 0$. Consequently

$$a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + (-1)^{n+1} a_n \leq a_{m+1}.$$

Now if $S_n = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$, then pick $n > m \geq N$.

$$\begin{aligned} |S_n - S_m| &= |(a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n) - (a_1 - a_2 + a_3 \\ &\quad - \cdots + (-1)^{m+1} a_m)| \\ &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + (-1)^{n+1} a_n| \\ &\leq |a_{m+1}| < \epsilon. \end{aligned}$$

So $\{S_n\}_{n=1}^{\infty}$ is Cauchy and so the series converges. □

1.4. **Continuous Functions.** Once we have the basic facts about sequences, we are able to introduce the concept of continuity. First we must extend the definition of a limit to functions.

Definition 1.43. We define limit points and limits of functions as follows.

(1) A point x is a limit point of a set $X \subseteq \mathbb{R}$ if there is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$. If there is no such sequence, then x is an isolated point.

(2) Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and x_0 a limit point of X. Then L is the limit of f as $x \to x_0$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x \in X$, $|x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$.

Limits of functions satisfy the usual arithmetic properties.

Theorem 1.44. *Let $f, g : X \to \mathbb{R}$ be functions and c a constant. If x_0 is a limit point of X and $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$,* *then*

$$\lim_{x\to x_0} cf(x) = cL \tag{1.5}$$

$$\lim_{x\to x_0} (f(x)+g(x)) = L+M \tag{1.6}$$

$$\lim_{x\to x_0} f(x)g(x) = LM \tag{1.7}$$

$$\lim_{x\to x_0} f(x)/g(x) = L/M, \tag{1.8}$$

provided $M \neq 0$ *and* g *is nonzero.*

Proofs of these results are exercises with the triangle inequality and are left to the reader. We can define right and left limits for functions.

Definition 1.45. Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We say that

$$\lim_{x\to a^+} f(x) = L,$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$. Similarly we say

$$\lim_{x\to a^-} f(x) = L,$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$.

An easy result follows.

Proposition 1.46. *Let* $f : X \to \mathbb{R}$*, where* $X \subseteq \mathbb{R}$*. Then* $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

The proof is an exercise. Finally we define the limit at infinity.

Definition 1.47. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there exists an $M > 0$ such that $x \geq M$ implies $|f(x) - L| < \epsilon$. Similarly we say that $\lim_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $M < 0$ such that $x \leq M$ implies $|f(x) - L| < \epsilon$.

Having established the essentials about limits of functions, we introduce the crucial idea of continuity.

Definition 1.48. A function $f : X \to \mathbb{R}$ is said to be continuous at x if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ which converges to x, we have

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This can be recast in the following form.

Definition 1.49. A function $f : X \to \mathbb{R}$ is continuous at $x \in X$ if for any $\epsilon > 0$, we can find a $\delta_x > 0$ such that $|x - y| < \delta_x$ implies $|f(x) - f(y)| < \epsilon$.

We write δ_x to emphasise the dependence on the point x. So for each x we will require a different δ. If a function is continuous at every point in its domain, we say that it is continuous. The two definitions are clearly equivalent.

Theorem 1.50. *The two definitions of continuity stated above are equivalent.*

Proof. First suppose that f satisfies Definition 1.49. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X with limit x. Pick $\epsilon > 0$ and $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Since $x_n \to x$ we may find an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \delta$. Then $|f(x_n) - f(x)| < \epsilon$, but this means that $f(x_n) \to f(x)$, so f is continuous according to Definition 1.48.

Suppose that f does not satisfy Definition 1.49. Then we can find $\epsilon >$ such that for every $\delta > 0$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| \epsilon$. Now choose a sequence $\{x_n\}_{n=1}^{\infty}$ in X with limit $x \in X$. Then given $\delta > 0$ we may find an $N \in \mathbb{N}$ such that $|x_n - x| < \delta$, but $|f(x_n) - f(x)| \geq \epsilon$. So $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(x)$ and thus f is not continuous by Definition 1.48. □

There are other notions of continuity. We will see a third definition in terms of open sets and encounter the concept of absolute continuity later. We can also consider functions which are right and left continuous.

Definition 1.51. We say that f is right continuous at x_0 if $\lim_{x \to x_0^+} f(x)$ exists. If $\lim_{x \to x_0^-} f(x)$ exists, then we say that f is left continuous.

The most important form of continuity for the Riemann integral is uniform continuity.

Definition 1.52. A function $f : X \to \mathbb{R}$ is said to be uniformly continuous if given $\epsilon > 0$ we can find a $\delta > 0$ such that whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

The point here is that unlike ordinary continuity, δ does not depend on x or y. Only on how far apart they are. Uniform continuity implies continuity, but the converse is false. Most functions on the real line are not uniformly continuous, but they are on compact intervals. In order to prove this we introduce an equivalent idea.

Definition 1.53. A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is sequentially uniformly continuous if given $x_n, y_n \in X$, $y_n - x_n \to 0$ implies $f(y_n) - f(x_n) \to 0$.

The proof of the following is straightforward and we omit it.

Theorem 1.54. *A function $f : X \to \mathbb{R}$ is sequentially uniformly continuous if and only if it is uniformly continuous.*

Now we will prove a major result.

Theorem 1.55. *A continuous function on a closed bounded interval* $[a, b]$ *is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. It therefore cannot be sequentially uniformly continuous. Choose $r \geq 0$ such that for every $\delta > 0$ there exists $x, y \in [a, b]$ such that $|x-y| < \delta$ and $|f(x)-f(y)| > r$.

For each $N \in \mathbb{N}$, choose $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| \geq r.$$

By the Bolzano-Weierstrass Theorem, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose that $x_{n_K} \to x$. Since $\{x_{n_K} - y_{n_K}\}_{K=1}^{\infty}$ is a subsequence of $\{x_n - y_n\}_{n=1}^{\infty}$ and $x_n - y_n \to 0$, so $x_{n_K} - y_{n_K} \to 0$. So we have

$$y_{n_K} = x_{n_K} - (x_{n_K} - y_{n_K}) \to x - 0 = x.$$

But f is continuous on $[a, b]$ and hence at x. So $f(x_{n_K}) \to f(x)$. and $f(y_{n_K}) \to f(x)$ and so $f(x_{n_K}) - f(y_{n_K}) \to 0$. But we have assumed that

$$|f(x_{n_K}) - f(y_{n_K})| \geq r > 0, \tag{1.9}$$

for all $K > 0$. We have a contradiction. So f is sequentially uniformly continuous and hence uniformly continuous. □

The fact that continuous functions on closed and bounded intervals are uniformly continuous is essential to many other results. For example, the proof of Riemann's theorem that every continuous function is Riemann integrable requires it. So does the proof of the Fundamental Theorem of Calculus.

Another widely used type of continuity is Lipschitz continuity.

Definition 1.56. A function $f : X \to \mathbb{R}$ is Lipschitz continuous if there exists a constant M such that

$$\text{for every } x, y \in X,\ |f(x) - f(y)| \leq M|x - y|.$$

Lipschitz continuous functions are obviously continuous and in fact uniformly continuous.

We now turn to another of the big results about continuous functions. This is about maxima and minima.

Theorem 1.57. *A continuous function on a closed, bounded interval* $[a, b]$ *is bounded. Moreover it attains its maximum and minimum values on* $[a, b]$.

Proof. Suppose that f is unbounded. Then given $n \in \mathbb{N}$, n is not a bound for f and thus there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. However, we know that $[a, b]$ is closed and bounded, and so the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose that $x_{n_K} \to x$ as $K \to \infty$. By continuity of f, $f(x_{n_K}) \to f(x)$. But this is impossible, since $f(x_{n_K}) > n_K$ for each K and $n_K \to \infty$, so the sequence $\{f(x_{n_K})\}_{K=1}^{\infty}$ is not convergent, and hence f is not continuous at x. This is a contradiction and we therefore conclude that f is bounded.

Now suppose that $M = \sup_{x\in[a,b]} f(x)$. For each $n \in \mathbb{N}$ choose $x_n \in [a, b]$ such that $f(x_n) > M - 1/n$. Then $f(x_n) \to M$. $\{x_n\}_{n=1}^{\infty}$ is contained in $[a, b]$, so is bounded and hence has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose $x_{n_K} \to c \in [a, b]$. By continuity, $f(x_{n_K}) \to f(c)$. But the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is convergent, so the sequence $\{f(x_{n_K})\}_{K=1}^{\infty}$ has the same limit. Thus $f(c) = M$, so f reaches its maximum. The case for the minimum is similar. □

We also need to mention the intermediate value property. This is the result which tells us that we can solve certain equations.

Theorem 1.58. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $f(a)f(b) < 0$. Then there is a $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Without loss of generality, we suppose that $f(a) < 0, f(b) > 0$. Let $A = \{x \in [a, b] : f(x) < 0\}$. Then $a \in A$ and so A is nonempty and bounded above. It therefore has a least upper bound, which we we call c. Choose x_n such that $c - 1/n < x_n \leq c$. Then $f(x_n) < 0$. By continuity, $f(c) = \lim_{n\to\infty} f(x_n) \leq 0$. Now take $y_n = c+(b-c)/n$. Then $y_n \to c$ and by continuity $f(c) = \lim_{n\to\infty} f(y_n) \geq 0$. Hence $f(c) = 0$. The case $f(a) > 0$ and $f(b) < 0$ is similar. □

Corollary 1.59. *Let f be continuous on $[a, b]$. Suppose that $f(a) \neq f(b)$ and that M lies between $f(a)$ and $f(b)$. Then there is a $c \in [a, b]$ such that $f(c) = M$.*

Proof. Apply Theorem 1.58 to the function $g(x) = f(x) - M$. □

Definition 1.60. A function is said to be monotone increasing if for each $x \geq y$ we have $f(x) \geq f(y)$. We say that f is monotone decreasing if $f(y) \leq f(x)$.

An important question that arose at the end of the nineteenth century was which functions are differentiable? Weierstrass had constructed a nowhere differentiable function, an event that was a considerable shock to mathematicians. Lebesgue proved that every monotone function is differentiable 'almost everywhere'. As a first step we can show that monotone functions are continuous except possibly on a countable set of points.

Theorem 1.61. *Suppose that f is monotone on (a, b). Then f is continuous except possibly on a countable set of points in (a, b).*

Proof. Without loss of generality we can assume that f is increasing. If f is decreasing we can just multiply by minus one to obtain an increasing function. We can also assume that (a, b) is bounded. Otherwise we can write it as a countable union of open, bounded subintervals and the discontinuities of f will be a countable union of the discontinuities on each subinterval. Now let $x_0 \in (a, b)$ and then by the least upper bound axiom

$$f(x_0^-) = \sup\{f(x) : a < x < x_0\}, \tag{1.10}$$

and

$$f(x_0^+) = \inf\{f(x) : x_0 < x < b\}, \tag{1.11}$$

both exit and $f(x_0^-) \leq f(x_0^+)$. The only way that f can be discontinuous at x_0 is if there is a jump at x_0. We define the jump interval at x_0 by $J(x_0) = \{y \in (f(x_0^-), f(x_0^+))\}$. Clearly $J(x_0) \subseteq (a, b)$ and so it is bounded. The jump intervals for f are also obviously disjoint. So for every $n \in N$, there are only a finite number of jump intervals of length greater then $1/n$. Hence the set of points of discontinuity of f is a countable union of finite sets and is therefore countable.

□

1.5. **The Derivative.** The derivative is one of the two major tools of calculus. It is the limit of the Newton quotient.

Definition 1.62. A function $f : X \to \mathbb{R}$, where X is open, is said to be differentiable at x if

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1.12}$$

exists. We say that $f'(x)$ is the derivative of f at x. We also write $\frac{df}{dx}$ for f'.

An equivalent formulation is

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{1.13}$$

Derivatives are defined on open sets. So one talks about a function being differentiable on an open interval (a, b) rather than on $[a, b]$, because the limit in the definition is not necessarily defined at the end points of an interval. The basic rules of differentiation are well known.

Theorem 1.63. *Let c be constant and f, g be differentiable at x_0. Then*

$$(cf)'(x_0) = cf'(x_0) \tag{1.14}$$

$$(f+g)'(x_0) = f'(x_0) + g'(x_0) \tag{1.15}$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \tag{1.16}$$

Proof. Once more this an exercise manipulating limits. For example, the product rule is proved as follows.

$$\begin{aligned}
(fg)'(x_0) &= \lim_{x\to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \left[\frac{f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\right] \\
&= \lim_{x\to x_0} f(x)\frac{g(x)-g(x_0)}{x-x_0} + \lim_{x\to x_0} g(x_0)\frac{f(x)-f(x_0)}{x-x_0} \\
&= f(x_0)g'(x_0)+f'(x_0)g(x_0).
\end{aligned}$$

□

The next result is easy to prove and will be used in the proof of the chain rule.

Theorem 1.64. *If f is differentiable at a point x, then it is continuous at x.*

Proof. We can write

$$f(x) = (x-x_0)\left(\frac{f(x)-f(x_0)}{x-x_0}\right) + f(x_0) \tag{1.17}$$

Since f is differentiable at x_0, we have

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left((x-x_0)\left(\frac{f(x)-f(x_0)}{x-x_0}\right)+f(x_0)\right) \\
&= \lim_{x\to x_0}(x-x_0)\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+f(x_0) \\
&= 0\times f'(x_0)+f(x_0) = f(x_0).
\end{aligned}$$

So f is continuous at x_0. □

Again the converse of this result is false. The function $f(x) = |x|$ is continuous at zero, but is not differentiable there. Indeed Weierstrass proved that there are functions which are continuous everywhere, but differentiable nowhere. We will see Weierstrass' nowhere differentiable function later.

The most important result is the chain rule.

Theorem 1.65 (The Chain Rule)**.** *Suppose that g is differentiable at x and f is differentiable at $y = g(x)$. Then*

$$(f\circ g)'(x) = f'(y)g'(x). \tag{1.18}$$

Proof. Write $k = g(x+h) - g(x)$. Since g is differentiable at x, it is continuous there and so as $h \to 0$, $k \to 0$. Now

$$\begin{aligned}\frac{f(g(x+h)) - f(g(x))}{h} &= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \\ &= \frac{f(y+k) - f(y)}{k} \frac{g(x+h) - g(x)}{h}.\end{aligned}$$

Suppose that at no value of h does $k = 0$. Then taking the limit as $h \to 0$ gives the result. To take care of the case $k = 0$ we let

$$F(k) = \begin{cases} \frac{f(y+k)-f(y)}{k} & k \neq 0 \\ f'(y) & k = 0. \end{cases} \tag{1.19}$$

By differentiability of f, as $k \to 0$ $F(k) \to f'(y)$ and so F is continuous at 0. Thus as $h \to 0$, $F(k) \to f'(y)$. So for $k \neq 0$

$$\frac{f(g(x+h)) - f(g(x))}{h} = F(k) \frac{g(x+k) - g(x)}{h}. \tag{1.20}$$

This also holds when $k = 0$ since both sides will be zero. Consequently

$$\frac{f(g(x+h)) - f(g(x))}{h} \to f'(y)g'(x) \tag{1.21}$$

as $h \to 0$. □

Example 1.11. Let us compute the derivative of a reciprocal. We have $f(x) = 1/g(x) = h(g(x))$, where $h(u) = 1/u$. Hence

$$\frac{d}{dx} f(x) = g'(x)h'(u) = -\frac{g'(x)}{(g(x))^2}.$$

Example 1.12. The quotient rule is obtained by combining the chain rule and the product rule:

$$\begin{aligned}\frac{d}{dx} \frac{f(x)}{g(x)} &= \frac{f'(x)}{g(x)} + f(x) \frac{d}{dx} \frac{1}{g(x)} \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

The first application of differentiation that we see is usually to the problem of obtaining maxima and minima.

Definition 1.66. A function $f : X \to \mathbb{R}$ has a local maximum at $c \in X$ if there is a subset $Y \subseteq X$ such that $c \in Y$ and $f(c) > f(x)$ for all $x \in Y$. A point c is a local minimum for f if there is a subset $Y \subseteq X$ such that $c \in Y$ and $f(c) < f(x)$ for all $x \in Y$. If f has a local maximum at c, then c is called a maximimiser. If f has a local minimum at c, Then c is called a minimiser. In general c is called an extreme point.

Theorem 1.67. *Let I be an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be differentiable at $c \in I$. If f attains a local maximum or minimum at c, then $f'(c) = 0$.*

Proof. There are two cases to consider, which turn out to be very similar. So we only prove the case for a local maximum. The proof proceeds by contradiction, so we assume that c is a point where f attains a local maximum and that $f'(c) > 0$. Choose $\delta > 0$ such that for $x \in I$ and $0 < |x - c| < \delta$ we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c).$$

Pick an $x > c$ with $|x - c| < \delta$. Then we have

$$-f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < f'(c).$$

Which implies

$$\frac{f(x) - f(c)}{x - c} > 0$$

and hence $f(x) > f(c)$, which is a contradiction. Thus $f'(c) \leq 0$.

Suppose then that $f'(c) < 0$. Pick a $\delta > 0$ such that for $x \in I$ and $0 < |x - c| < \delta$ we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c).$$

Pick an $x < c$ with $|x - c| < \delta$. Then

$$f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < -f'(c).$$

Which implies

$$\frac{f(x) - f(c)}{x - c} < 0,$$

and hence $f(x) > f(c)$, since $x - c < 0$, which is a contradiction once more. Thus $f'(c) = 0$. The proof for a local minimum is essentially the same. □

A useful corollary of this is called Rolle's Theorem.

Theorem 1.68 (Rolle's Theorem)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$ then then there is a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Continuous functions attain their maximum and minimum values on closed bounded intervals. If $c \in (a, b)$ is an extreme point, then $f'(c) = 0$. Suppose that both the maximum and minimum values occur at $[a, b]$. Then since $f(a) = f(b)$, it follows that f is constant and so $f'(x) = 0$ for all $x \in (a, b)$. □

The main applications of Rolle's Theorem are to prove the Mean Value Theorem and Taylor's Theorem, which are two of the most useful results in analysis.

Theorem 1.69 (Mean Value Theorem)**.** *Let $[a,b]$ be a closed and bounded interval on $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ a continuous function which is differentiable on (a,b). Then there is a point $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. The proof is an application of Rolle's Theorem. We consider the function

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Then $g(a) = g(b) = 0$ and

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem there is a $c \in (a,b)$ with $g'(c) = 0$, which proves the result. □

The MVT is one of the most powerful results in calculus. Let us consider some simple applications. Later we will see it used to prove a result about the behaviour of limits for sequences of derivatives. One can also use it to prove the Fundamental Theorem of Calculus. It is quite ubiquitous.

Corollary 1.70. *If $[a,b]$ is a closed and bounded interval in $\mathbb{R}$ and f is continuous on $[a,b]$ and differentiable on (a,b), then f is Lipschitz continuous on $[a,b]$.*

Proof. For any $x, y \in (a,b)$ the MVT gives, $|f(x)-f(y)| \leq |f'(c)||x-y|$ for some $c \in (x,y)$. □

The following result is well known from high school calculus, but usually is not given a rigourous proof.

Corollary 1.71. *If f is continuous on $[a,b]$ and differentiable on (a,b), and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.*

Proof. For any $x, y \in (a,b)$, $f(x) - f(y) = f'(c)(x - y) = 0$. Hence $f(x) = f(y)$ for all x, y and so f is constant on (a,b). By continuity it is also constant on $[a,b]$. □

Let us use this to prove uniqueness for the solution of a differential equation.

Proposition 1.72. *The equation $y' = ky, y(0) = y_0$ has a unique solution.*

Proof. We let $y(x) = y_0 e^{kx}$. Then this is clearly a solution of the differential equation. Now suppose that f is any solution of the equation. Consider $h(x) = f(x)e^{-kx}$. Then

$$h'(x) = f'(x)e^{-kx} - ke^{-kx}f(x) = e^{-kx}(f'(x) - kf(x)) = 0.$$

Thus h is constant. Hence $f(x) = Ce^{kx}$. The condition that $f(0) = y_0$ proves the result. □

There is a more general version of the MVT. It is due to Cauchy and is often called the Cauchy Mean Value Theorem.

Theorem 1.73 (Generalised Mean Value Theorem)**.** *Suppose that f and g are continuous functions on $[a, b]$, which are differentiable on (a, b) and suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{1.22}$$

Proof. This again relies upon Rolle's Theorem. First, observe that if $g(b)-g(a) = 0$, then the Mean Value Theorem tells us that there exists a point $c \in (a, b)$ such that $g'(c) = 0$. However we have assumed that g' is nonzero, so $g(b) - g(a) \neq 0$. Next introduce the function

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - f(b)g(a) = h(b). \end{aligned}$$

Rolle's Theorem then tells us that there is a $c \in (a, b)$ such that $h'(c) = 0$. Which means that

$$f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0. \tag{1.23}$$

Rearranging gives the result. □

As an application of this result we prove L'Hôpital's rule.

Theorem 1.74. *Suppose that f and g are differentiable on (a, b) and that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose further that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then,*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}, \tag{1.24}$$

provided the right side exists.

Proof. Suppose that

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L.$$

Then given $\epsilon > 0$ we can find $\delta > 0$ such that if $c \in (a, a+\delta)$ then

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon.$$

However, by the generalised MVT, if $x \in (a, a+\delta)$ then

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f(x)-f(a)}{g(x)-g(a)} - L\right| < \epsilon.$$

□

The extension of this result to the case when

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$$

can also be established using the generalised MVT.

Remark 1.75. L'Hôpital's rule was actually discovered by the Swiss mathematician Johann Bernoulli, who taught Euler and worked for L'Hôpital. L'Hôpital published the rule in his textbook on calculus, and it became known by his name.

1.5.1. *Inverse Functions.* We first state our definitions.

Definition 1.76. A function $f : X \to Y$ is said to be one to one if for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. We also say that such an f is a bijection. If $f : X \to Y$ is one to one then it has an inverse function $f^{-1} : Y \to X$ which satisfies

$$f(f^{-1}(f)) = f^{-1}(f(x)) = x$$

for all $x \in X$.

Suppose that $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing (or decreasing). Then f is clearly one to one, and hence it has an inverse. If f is continuous, then the inverse function will also be continuous.

Theorem 1.77. *Suppose that $f : X \subseteq \mathbb{R} \to Y$ is a strictly increasing (or decreasing) continuous function. Then the inverse function f^{-1} exists and is continuous and increasing (or decreasing) on $f(X)$.*

Proof. We only deal with the case when f is increasing. We show that f^{-1} is increasing. Assume not. Then we can find $y_1, y_2 \in Y$ with $y_2 > y_1$ and $f^{-1}(y_2) < f^{-1}(y_1)$. But f is increasing, so

$$f(f^{-1}(y_2)) < f(f^{-1}(y_1)),$$

so that $y_2 < y_1$ which is a contradiction.

To prove continuity, take $y_0 \in f(X)$. Then there exists $x_0 \in X$ with $f(x_0) = y_0$. We suppose that y_0 is not an endpoint, so x_0 is not an endpoint and we may find $\epsilon_0 > 0$ such that the interval

$$(f^{-1}(y_0) - \epsilon_0, f^{-1}(y_0) + \epsilon_0) \subset X.$$

Pick $\epsilon < \epsilon_0$. Then there exist $y_1, y_2 \in f(X)$ such that $f^{-1}(y_1) = f^{-1}(y_0) - \epsilon$ and $f^{-1}(y_2) = f^{-1}(y_0) + \epsilon$. Because f is increasing $y_1 < y_0 < y_2$ and the inverse is increasing so for all $y \in (y_1, y_2)$ we have the inequality

$$f^{-1}(y_0) - \epsilon < f^{-1}(y) < f^{-1}(y_0) + \epsilon.$$

Consequently, if $\delta = \min\{y_2 - y_0, y_0 - y_1\}$, then

$$|f^{-1}(y_0) - f^{-1}(y)| < \epsilon$$

whenever $|y_0 - y| < \delta$. So f^{-1} is continuous at y_0.

We can also prove that if y_0 is a left (or right) endpoint, then f^{-1} is left (or right) continuous at y_0. □

The most important result about inverse functions relates the derivative of f and that of f^{-1}.

Theorem 1.78 (The Inverse Function Theorem)**.** *Suppose that f is differentiable and one to one on an open interval I. If $f'(a) \neq 0$, $a \in I$, then f^{-1} exists and is differentiable at $f(a)$ and*

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since f' is nonzero on I, it follows that f is either increasing or decreasing on I, and hence f is invertible. The inverse is continuous. Since f is decreasing or increasing, for $x \neq a$ it follows that $f(x) \neq f(a)$. Now

$$\begin{aligned}\lim_{y \to f(a)} \frac{f^{-1}(y) - f^{-1}(f(a))}{y - f(a)} &= \lim_{f(x) \to f(a)} \frac{f^{-1}(f(x)) - f^{-1}(f(a))}{f(x) - f(a)} \\ &= \lim_{x \to a} \left(\frac{x - a}{f(x) - f(a)}\right)^{-1} \\ &= \frac{1}{f'(a)}.\end{aligned}$$

□

1.5.2. *Convex Functions.* An interesting and important class of functions for which we can establish some very general results about differentiability are convex functions. We begin with the definition.

Definition 1.79. A function $f : I \to \mathbb{R}$ is said to be convex on an open interval I if for all $x \in I$ and $a > 0, b > 0$ with $a + b = 1$, we have

$$f(ax + by) \leq af(a) + bf(y).$$

If

$$f(ax + by) \geq af(a) + bf(y)$$

then f is said to be concave.

An equivalent formulation of convexity is that for all $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{1.25}$$

Convex functions are automatically continuous. To prove this we have a preliminary result.

Proposition 1.80. *If f is convex on an open interval $I \subset \mathbb{R}$, then the left and right hand derivatives, defined respectively by,*

$$f'(x^+) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h},$$

and

$$f'(x^-) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h},$$

both exist for each $x \in I$. Moreover, if $x, y \in (a,b)$ and $y > x$, then

$$f'(x^-) \leq f'(x^+) \leq \frac{f(y) - f(x)}{y - x} \leq f'(y^-) \leq f'(y^+). \tag{1.26}$$

Proof. We let $0 < h_1 < h_2$, then observe that

$$\frac{f(x+h_1) - f(x_1)}{h_1} \leq \frac{f(x+h_2) - f(x_1)}{h_2}.$$

Hence

$$F(h) = \frac{f(x+h) - f(x_1)}{h}$$

is an increasing function on some interval $(0, \delta)$ and hence $\lim_{h \to 0^+} F(h)$ exists. Similarly for the second limit. The inequality follows from (1.25) and is an easy exercise. □

Note, this result does not mean that f is differentiable at x. We have not established equality of the limits and in fact, this may not hold. However, it is a remarkable fact that convex functions are differentiable except possibly on a countable set of points. We will prove this below.

An application of the mean value theorem allows us to establish the following test for convexity.

Theorem 1.81. *Suppose that f is differentiable on an open interval I. Then f is convex on I if and only if f' increases on I.*

Proof. Suppose that f' is increasing on I and pick three points $x_1 < x_2 < x_3 \in I$. Then by the mean value theorem there exists points $a, b \in I$ with $b > a$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(a) \tag{1.27}$$

and

$$\frac{f(x_3)-f(x_2)}{x_3-x_2}=f'(b). \tag{1.28}$$

Now f' is increasing, hence $f'(a)\geq f'(b)$ and thus f is convex.

Conversely, suppose that f is convex. Then for points $x_1<x_2<x_3<x_4$ we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(x_3)-f(x_2)}{x_3-x_2}\leq\frac{f(x_4)-f(x_3)}{x_4-x_3}. \tag{1.29}$$

Letting $x_2\to x_1^+$ and $x_2\to x_4^-$ shows that $f'(x_3)\leq f'(x_4)$ and so f is increasing. □

For twice differentiable functions we have a simple result.

Theorem 1.82. *Let f be twice differentiable on an open interval I. Then f is convex on I if and only if $f''(x)\geq 0$ for all $x\in I$.*

Now we prove continuity for convex functions.

Theorem 1.83. *Suppose that f is convex on an open interval I. Then f is continuous on I.*

Proof. We let $x\in I$. Then

$$\lim_{h\to 0^+}\left(f(x+h)-f(x)\right)=\lim_{h\to 0^+}\left(\frac{f(x+h)-f(x)}{h}\right)h=0$$

and

$$\lim_{h\to 0^-}\left(f(x+h)-f(x)\right)=\lim_{h\to 0^-}\left(\frac{f(x+h)-f(x)}{h}\right)h=0.$$

Thus both limits exist and are equal, so f is continuous at x. □

Actually, convex functions are not just continuous.

Proposition 1.84. *Let f be a convex function on (a,b). Then f is Lipschitz continuous on each closed bounded subinterval $[c,d]$ of (a,b).*

Proof. This follows from the inequality

$$f'(c^+)\leq f'(u^+)\leq\frac{f(v)-f(u)}{v-u}\leq f'(v^-)\leq f'(d^-), \tag{1.30}$$

valid for $c\leq u\leq v\leq d$. So that for all $u,v,\in[c,d]$ with $M=\max\{|f'(c^+),f'(d^-)|\}$ we have

$$|f(u)-f(v)|\leq M|u-v|.$$

□

Indeed we can show something even stronger.

Theorem 1.85. *A convex function f on an interval (a,b) is differentiable except at most on a countable set of points. Moreover, the derivative is an increasing function.*

Proof. We already know that the left and right derivatives at a point exist. They are also increasing, and so they are continuous except at most on a countable set of points, which we denote by $\mathcal{D}$. Take a point $x \in (a, b) - \mathcal{D}$ and let $x_n \to x^+$ and apply the inequality (1.30). This gives

$$f'(x^-) \leq f'(x^+) \leq f'(x^-)$$

so that $f'(x^+) = f'(x^-)$ and hence f is differentiable at x. That f is increasing also follows from (1.30). □

Lebesgue proved a deeper result about differentiability. Namely that any monotone function is differentiable almost everywhere. We will discuss this later.

We can define higher derivatives in the obvious way. So

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right),$$

or $f''(x) = (f')'(x)$ etc.

Definition 1.86. A function $f : X \to \mathbb{R}$ for which the nth derivative $f^{(n)}$ exists for all $n \in \mathbb{N}$ is said to be infinitely differentiable, or smooth. We write $f \in C^\infty(X)$. (Pronounced C infinity on X). If f is n times differentiable for finite n we write $f \in C^n(X)$.

1.6. **Power Series.** A power series about a point x_0 is an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

By the ratio test such a series will converge if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = L < 1.$$

Upon rewriting this becomes

$$|x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1. \tag{1.31}$$

We can think of this as determining the values of x for which the series converges.

Definition 1.87. Suppose that for the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$|x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \tag{1.32}$$

for all $|x - x_0| < R$. We call R the radius of convergence of the power series.

Note a power series with radius of convergence R may converge or diverge when $|x - x_0| = R$. One has to check convergence at the end points individually.

Example 1.13. The series $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ is convergent for all $|x| < 1$. Hence the radius of convergence is 1.

For simplicity we will take $x_0 = 0$ in what follows. All results can be transferred to the more general case by making the replacement $x \to x - x_0$

Power series have very nice properties. In particular they converge absolutely within their radius of convergence.

Theorem 1.88. *Let $\sum_{n=o}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.*

Proof. Let $t \in (-R, R)$, then $\sum_{n=0}^{\infty} a_n t^n$ converges and the sequence $a_n t^n \to 0$ and is thus bounded. Let M be a bound. Now pick x with $|x| < |t|$, then

$$|a_n x^n| = |a_n t^n| \left|\frac{x}{t}\right|^n \leq M r^n$$

where $r = |x/t| < 1$. But $\sum_{n=0}^{\infty} M r^n$ is a convergent geometric series, and so $\sum_{n=0}^{\infty} |a_n x^n|$ converges by the comparison test. The other result is similar. □

Power series actually converge uniformly, a result we prove later. An important fact is that we can differentiate power series term by term and this does not change the radius of convergence.

Theorem 1.89. *Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R. Then the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.*

Proof. Suppose that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence $R_d < R$. Choose r, s so that $R_d < r < s < R$. Clearly $\sum_{n=0}^{\infty} a_n s^n$ converges which shows that $a_n s^n \to 0$ and so is bounded by a constant M. Then

$$\begin{aligned} |n a_n r^{n-1}| &= n |a_n| s^{n-1} \left(\frac{r}{s}\right)^{n-1} \\ &\leq \frac{M}{s} n \left(\frac{r}{s}\right)^{n-1}. \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{(M/s)(n+1)(r/s)^n}{(M/s)n(r/s)^{n-1}} = \frac{r}{s} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{M}{s} n \left(\frac{r}{s}\right)^{n-1}$ is convergent by the ratio test. Thus $\sum_{n=1}^{\infty} n a_n r^{n-1}$ is absolutely convergent, which is a contradiction since $r > R_d$. Hence $R \leq R_d$. Similarly we show that $R_d > R$ leads to a contradiction. (Exercise). Hence $R = R_d$. □

From this we can establish an important corollary.

Theorem 1.90. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

To prove this result we actually need some more information about the convergence of series. The key is that the series for f and f' both converge uniformly. We will discuss uniform convergence later.

The most commonly encountered power series are functions given by Taylor series expansions.

Definition 1.91. Let f be smooth in a neighbourhood X of a point a. We let the Taylor series for f at a be given by

$$T_f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \cdots .$$

If the series is convergent for all $x \in X$ and $|T_f(x) - f(x)| = 0$ for all $x \in X$, we say that f is analytic at a. If we truncate the Taylor expansion after n terms, the resulting expression is known as the nth Taylor polynomial.

Even if the Taylor series does not converge, smooth functions can be approximated by Taylor polynomials.

Theorem 1.92 (Taylor's Theorem)**.** *Let I be an open interval in $\mathbb{R}$, $n \in \mathbb{N}$ and $f \in C^{n+1}(I)$. Let $a \in I$ and $x \in I$, with $x \neq a$. Then there is a point ξ between a and x such that*

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{n!}(x-a)^n.$$

Proof. The proof uses Rolle's Theorem and is conceptually similar to the proof of the MVT. We define a function

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!} f'(t)(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n. \tag{1.33}$$

Plainly $F(x) = 0$. Since $f \in C^{(n+1)}(I)$ we see that F is differentiable. Now

$$\begin{aligned} F'(t) &= -f'(t) - f''(t)(x-t) + f'(t) - \frac{f'''(t)}{2!}(x-t)^2 + 2\frac{f''(t)}{2!}(x-t) \\ &\quad - \cdots - \frac{f^{(n+1)}t}{n!}(x-t)^n + n\frac{f^{(n)}(t)}{n!}(x-t)^{n-1} \\ &= -\frac{f^{(n+1)}}{n!}(x-t)^n. \end{aligned}$$

Next we introduce the function

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a).$$

Obviously $G(a) = 0$ and $G(x) = F(x) = 0$. Then

$$\begin{aligned} G'(t) &= F'(t) \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)\frac{(x-t)^n}{(x-a)^{n+1}}F(a). \end{aligned}$$

By Rolle's Theorem there is a point ξ between x and a such that $G'(\xi) = 0$. That is

$$\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n = (n+1)\frac{(x-\xi)^n}{(x-a)^{n+1}}F(a).$$

Rearranging we get

$$F(a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

If we substitute this into (1.33) we have Taylor's Theorem.

□

The other major tool in analysis is the integral. Although the Fundamental Theorem of Calculus was first stated by Newton and Leibnitz, the first rigorous theory of integration was developed by Cauchy, and extended by Riemann. Let us briefly summarise Riemann's theory.

1.7. **The Riemann Integral.** We take an interval $[a, b]$ and partition it as

$$\mathcal{P} = \{x_0, x_1, ..., x_n\},$$

where $x_0 = a$, $x_0 < x_1 < \cdots < x_n$ and $x_n = b$.

Now let f be a bounded function on $[a, b]$ then define

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i)\},$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i)\}.$$

We then form the upper and lower Riemann sums

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \tag{1.34}$$

and

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}). \tag{1.35}$$

The least upper bound axiom establishes that the upper and lower integrals

$$\overline{\int_a^b} f = \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\} \tag{1.36}$$

and

$$\underline{\int_a^b} f = \sup\{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\} \tag{1.37}$$

both exist. We then say that f is Riemann integrable on $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. The Riemann integral is then equal to the upper (or lower) integral.

It is easy to prove the following results.

Proposition 1.93. *The Riemann integral has the following properties.*

(1) *If c is a constant, $\int_a^b c dx = c(b - a)$.*

(2) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

The most important results about the Riemann integral are as follows.

Theorem 1.94 (Riemann's Criterion)**.** *Let f be a bounded function on the closed interval $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if, given any $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.*

From this one establishes the first major result.

Theorem 1.95. *Every continuous function on a closed bounded interval $[a, b]$ is Riemann integrable.*

Proof. The function f is continuous on $[a, b]$ and so is bounded. Let $\epsilon > 0$. Since f is continuous it is uniformly continuous and so we can choose $\delta > 0$ such that $x, y \in [a, b]$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$. Now choose $N \in \mathbb{N}$ such that $N > (b - a)/\delta$. For each $i = 0, 1, ...N$, let $x_i = a + (b - a)i/N$. Then $\mathcal{P} = \{x_0, x_1, ...x_N\}$ is a partition of $[a, b]$, with $|x_i - x_{i-1}| < \delta$. By continuity, f attains its maximum and minimum values on each closed subinterval $[x_{i-1}, x_i]$. Now let

$$f(c_i) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \tag{1.38}$$

$$f(d_i) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}. \tag{1.39}$$

Obviously $|d_i - c_i| < \delta$ and $f(d_i) \geq f(c_i)$. By uniform continuity

$$f(d_i) - f(c_i) < \frac{\epsilon}{(b-a)}.$$

So we have

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{i=1}^{N} f(d_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N} (f(d_i) - f(c_i))(x_i - x_{i-1}) \\ &< \sum_{i=1}^{N} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^{N}(x_i - x_{i-1}) = \epsilon. \end{aligned}$$

Thus by Riemann's criterion, f is integrable on $[a, b]$.

□

1.7.1. *Calculating Integrals By Riemann Sums.* It is possible to explicitly compute a surprisingly large class of integrals by evaluating Riemann sums. For monotone functions, the construction of upper and lower sums is straightforward. One simply picks sample points at the ends of each subinterval. We restrict our attention to $[0, 1]$. We can extend to the interval $[a, b]$ by a linear change of variable.

Example 1.14. We integrate $f(x) = x^2$ on $[0, 1]$ Since f is increasing we can take $\mathcal{P} = \{0, 1/n, 2/n, ..., n/n\}$ and note that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1). \tag{1.40}$$

Now we observe that

$$\begin{aligned} m_i(f,\mathcal{P}) &= \inf\{x^2 : x \in [\frac{i-1}{n}, \frac{i}{n})\} \\ &= \frac{(i-1)^2}{n^2} \\ M_i(f,\mathcal{P}) &= \sup\{x^2 : x \in [\frac{i-1}{n}, \frac{i}{n})\} \\ &= \frac{i^2}{n^2}. \end{aligned}$$

Then

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}\left(\frac{i}{n} - \frac{(i-1)}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2.$$

Also

$$U(f,\mathcal{P}) = \sum_{i=1}^{n} \frac{i^2}{n^2}\left(\frac{i}{n} - \frac{(i-1)}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

Using (1.40) we get

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{1}{6}n(n+1)(2n+1)\frac{1}{n^3} - \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{n}.$$

By Riemann's Criterion, f is Riemann integrable if for any $\epsilon > 0$ we can find a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$. Clearly we can do this by taking $n > 1/\epsilon$. So f is Riemann integrable. Further

$$\begin{aligned}\int_0^1 f(x)dx &= \sup\{L(f,\mathcal{P}), \mathcal{P} \text{ a partition of } [0,1]\} \\ &= \sup_{n\geq 1}\left\{\frac{(n-1)n(2n-1)}{6n^3}\right\} \\ &= \sup_{n\geq 1}\left\{\frac{1}{6n^2} - \frac{1}{2n} + \frac{1}{3}\right\} = \frac{1}{3}.\end{aligned}$$

Example 1.15. Let $a \neq 0$ and consider $f(x) = e^{ax}$ on $[0,1]$. The function is monotone and we take the same partition as in the previous example. Then

$$m_k(f,\mathcal{P}) = \inf\left\{e^{ax} : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)\right\} = e^{(k-1)a/n} \tag{1.41}$$

$$M_k(f,\mathcal{P}) = \sup\left\{e^{ax} : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)\right\} = e^{ka/n} \tag{1.42}$$

Then

$$\begin{aligned}L(f,\mathcal{P}) &= \sum_{k=1}^{n} m_k(f,\mathcal{P})(x_k - x_{k-1}) \\ &= \frac{1}{n}(1 + e^{a/n} + \cdots + e^{(n-1)a/n})\end{aligned}$$

and

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=1}^{n} M_k(f,\mathcal{P})(x_k - x_{k-1}) \\ &= \frac{1}{n}(e^{a/n} + e^{2a/n} + \cdots + e^{an/n}). \end{aligned}$$

So

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{e^a - 1}{n}.$$

This can be made smaller than ϵ by picking $n > \epsilon/(e^a - 1)$. Thus by Riemann's Criterion, f is Riemann integrable on $[0,1]$. We can explicitly evaluate the upper and lower sums by noticing that they are sums of geometric progressions with common ratio e^a. Hence

$$\begin{aligned} L(f,\mathcal{P}) &= \frac{1}{n}(1 + e^{a/n} + \cdots + e^{(n-1)a/n}) \\ &= \frac{1}{n}\frac{(1 - e^a)}{(1 - e^{a/n})}. \end{aligned}$$

So we have

$$\begin{aligned} \int_0^1 e^{ax}dx &= \sup_n \left\{ \frac{1}{n}\frac{(1 - e^a)}{(1 - e^{a/n})} \right\} \\ &= \lim_{u\to 0} \frac{u(1 - e^a)}{1 - e^{au}} \\ &= \frac{1}{a}(e^a - 1) \end{aligned}$$

where we put $u = 1/n$ and used L'Hôpital's rule to evaluate the limit.

We can actually prove that bounded monotone functions are Riemann integrable.

Theorem 1.96. *Suppose that $f : [0,1] \to \mathbb{R}$ is monotone increasing and $f(1)$ is bounded. Then f is Riemann integrable on $[0,1]$.*

Proof. With the previous partition of $[0,1]$ we have

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{1}{n}(f(1) - f(0)). \tag{1.43}$$

Since f is monotone increasing, $f(0)$ must be finite and $f(1)$ is also finite, we can make this smaller than any $\epsilon > 0$ by suitable choice of n. So f is Riemann integrable. □

It is possible to evaluate many integrals by means of Riemann sums- in particular, we can integrate any polynomial- but it is clearly a laborious procedure. Fortunately we have a far more powerful means of doing integration. The key is the following result, which is at the heart of modern science.

Theorem 1.97 (Fundamental Theorem of Calculus)**.** *If f is a continuous function on $[a, b]$, then for all $x \in [a, b]$*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Proof. We define the function $F(x) = \int_a^x f(t)dt$. Since f is continuous, it is bounded. Thus there is an $M > 0$ such that $|f(t)| \leq M$ for all $t \in [a, b]$. Then

$$\begin{aligned} |F(x) - F(y)| &= \left|\int_y^x f(t)dt\right| \\ &\leq \int_y^x |f(t)|dt \\ &\leq M|x - y|. \end{aligned}$$

Consequently, F is Lipschitz continuous on $[a, b]$ and hence continuous. Now

$$\begin{aligned} \frac{F(x) - F(y)}{x - y} - f(y) &= \frac{1}{x - y}\left(F(x) - F(y) - (x - y)f(y)\right) \\ &= \frac{1}{x - y}\int_y^x (f(t) - f(y))dt. \end{aligned}$$

By uniform continuity of f, given $\epsilon > 0$, we may find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. We choose such an ϵ and δ to obtain

$$\begin{aligned} \left|\frac{F(x) - F(y)}{x - y} - f(y)\right| &\leq \frac{1}{|x - y|}\int_y^x |f(t) - f(y)|dt \\ &< \frac{1}{|x - y|}\epsilon(x - y) = \epsilon \end{aligned}$$

as $x > y$. Thus F is differentiable and $F' = f$. □

In other words, integration is essentially the inverse of differentiation. From this we can establish the well known second form of the fundamental theorem.

Corollary 1.98 (The Fundamental Theorem of Calculus II)**.** *Let f be a Riemann integrable function on $[a, b]$. Then if $F' = f$ on (a, b) the integral is given by*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1.44}$$

Proof. Suppose that $G(x) = \int_a^x f(t)dt$ and $F'(x) = f(x)$. It follows that $G - F$ is a constant, since $G' = f$. Hence $G(b) - F(b) = G(a) - F(a)$. But $G(a) = 0$. Hence $G(b) = \int_a^b f(x)dx = F(b) - F(a)$. □

There is a mean value theorem for the Riemann integral which is often useful.

Theorem 1.99 (Mean Value Theorem for Integrals)**.** *Suppose that f and g are continuous on $[a, b]$ and $g(x) \geq 0$, for all $x \in [a, b]$. Then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx. \tag{1.45}$$

Proof. By continuity f is bounded. Suppose that for all $t \in [a, b]$ $m \leq f(t) \leq M$. Then

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$$

Let $F(t) = f(t) \int_a^b g(t)dt$. By the Intermediate Value Theorem, there is a $c \in [a, b]$ such that

$$F(c) = f(c) \int_a^b g(t)dt = \int_a^b f(x)g(x)dx.$$

□

Notice that if $g = 1$ and $F' = f$ then we have the existence of a $c \in [a, b]$ such that

$$\int_a^b f(x)dx = F(b) - F(a) = F'(c)(b - a) \tag{1.46}$$

which is the mean value theorem. Actually the mean value theorem can be used to prove the fundamental theorem of calculus. This is an exercise.

1.7.2. *Integration Rules.* Integration is intrinsically more difficult than differentiation. Useful rules for evaluating integrals exist however. Integration by parts is simply the product rule of differentiation backwards. Specifically

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives the integration by parts rule

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx. \tag{1.47}$$

The most important technique for evaluating integrals is the use of substitutions. This is the chain rule in reverse. The chain rule says that $(f \circ g)'(x) = f'(g(x))g'(x)$. Thus letting $u = g(x)$ gives

$$\int_a^b f'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du. \tag{1.48}$$

We can use integration by parts to show how Taylor's Theorem follows from the Fundamental Theorem of Calculus. Assume that f is continuously differentiable $n + 1$ times. We know that

$$f(x) - f(a) = \int_a^x f'(t)dt. \tag{1.49}$$

We are going to integrate by parts. Notice however that instead of using the obvious anti-derivative of 1, we are going to use $t - x$, which is also an anti-derivative of 1. So that

$$\begin{aligned} f(x) - f(a) &= [(t-x)f'(t)]_a^x - \int_a^x (t-x)f'(t)dt \\ &= (x-a)f'(a) + \int_a^x (x-t)f'(t)dt \\ &= (x-a)f'(a) + \frac{(x-a)^2}{2}f''(x) + \frac{1}{2}\int_a^x (x-t)^2 f''(t)dt. \end{aligned}$$

Repeating this n times gives

$$\begin{aligned} f(x) = f(a) &+ (x-a)f'(a) + \frac{(x-a)^2}{2}f''(x) + \cdots \\ &+ \frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n)}(t)dt. \end{aligned}$$

This gives us the useful form for the remainder in the Taylor series expansion

$$R_n(a,x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n)}(t)dt.$$

Using the mean value theorem for integrals we can show that this is the same as the derivative form we found earlier.

1.7.3. *Improper Riemann Integrals.* It is often the case that we wish to consider an integral of a function over a set where the function is discontinuous.

Definition 1.100. Let $f : [a,b] \to \mathbb{R}$ be continuous on $(a,b]$, but f is discontinuous at a. Then the improper Riemann integral of f over $[a,b]$ is define by

$$\int_a^b f(x)dx = \lim_{X \to a} \int_X^b f(x)dx, \tag{1.50}$$

provided the limit exists. Similarly, if the discontinuity is at $x = b$ then

$$\int_a^b f(x)dx = \lim_{X \to b} \int_a^X f(x)dx, \tag{1.51}$$

provided the limit exists.

Example 1.16. Consider $f(x) = 1/\sqrt{x}$ on $[0,1]$. Then f is continuous on $(0,1]$ with a discontinuity at 0. Thus the improper Riemann integral of f over $[0,1]$ is

$$\begin{aligned}\int_0^1 f(x)dx &= \lim_{X\to 0}\int_X^1 \frac{dx}{\sqrt{x}}\\ &= \lim_{X\to 0} 2\sqrt{x}]_X^1\\ &= \lim_{X\to 0}(2\sqrt{1}-\sqrt{X}) = 2.\end{aligned}$$

For integrals on unbounded domains we can use the same idea.

Definition 1.101. The improper Riemann integral of f over $\mathbb{R}$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{0} f(x)dx + \lim_{T\to\infty}\int_0^T f(x)dx, \tag{1.52}$$

provided the limits exist.

One has to be careful to distinguish between Definition 1.101 and the Cauchy Principal value.

Definition 1.102. The quantity

$$\text{pv}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx,$$

is known as the Cauchy Principal value of the integral, provided that the limit exists.

Example 1.17. The improper Riemann integral $\int_{-\infty}^{\infty} xdx$ does not exist, but

$$\begin{aligned}\text{pv}\int_{-\infty}^{\infty} xdx &= \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R}\\ &= \frac{1}{2}(R^2-R^2) = 0.\end{aligned}$$

1.8. **Sequences of Functions.** The Riemann integral is a powerful tool, but it has limitations. The most important relates to the question of swapping integrals and limits. To see what this involves, let us introduce the notion of convergence of a sequence of functions.

Definition 1.103. We say that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function f on a set $X \subseteq \mathbb{R}$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

Given such a sequence, we would like to be able to conclude that

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

A common example is where we have a function defined as an infinite sum

$$f = \sum_{n=1}^{\infty} f_n,$$

and we would like to determine $\int f$ by term by term integration. So that we would like

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

Unfortunately this is not in general true. Consider the sequence

$$f_n(x) = nxe^{-nx^2}. \tag{1.53}$$

Then on $[0, 1]$, $f_n \to 0$ pointwise as $n \to \infty$. Hence

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = 0.$$

However

$$\int_0^1 nxe^{-nx^2} dx = \frac{1 - e^{-n}}{2}.$$

Hence

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2} \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

In order to safely swap limits and Riemann integrals, we need uniform convergence.

Definition 1.104. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ on a set $X \subseteq \mathbb{R}$ converges uniformly to f on X if for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$.

The first result is a trivial exercise.

Lemma 1.105. *If $f_n \to f$ uniformly on X, then $f_n \to f$ pointwise.*

The converse of this result is false. Pointwise convergent sequences usually do not converge uniformly. The example (1.53) converges pointwise but not uniformly. There is a result due to Egoroff which we will see later, that tells us that a sequence of functions converging pointwise on a closed and bounded interval $[a, b]$, converges uniformly on $[a, b] - E$, where E is a small set. However this is not enough to swap limits and Riemann integrals.

Uniformly convergent sequences have nice properties. An important one is that they preserve continuity.

Theorem 1.106. *If $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $X \subseteq \mathbb{R}$, with $f_n \to f$ then f is continuous on X.*

Proof. Since f_k is continuous at $x \in X$, given $\epsilon > 0$, we may choose $\delta > 0$ such that for all y satisfying $0 < |x - y| < \delta$ we have

$$|f_k(x) - f_k(y)| < \epsilon/3.$$

By uniform convergence, we may choose $N \in \mathbb{N}$ such that $k \geq N$ implies

$$|f(x) - f_k(x)| < \epsilon/3,$$

for all $x \in X$. Consequently, given $x \in X$, then for all $y \in X$ satisfying $0 < |x - y| < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus f is continuous at x. □

This result is not true for pointwise convergence. For example, the pointwise convergent double sequence $f_{n,j}(x) = (\cos(n!\pi x))^{2j}$ does not have a continuous limit on $[0, 1]$. In fact it converges to the Dirichlet function

$$\mathbb{D}(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ 0 & x \notin \mathbb{Q} \cap [0, 1]. \end{cases} \tag{1.54}$$

To see this, observe that if x is rational, then $x = p/q$ for some integers p, q. Then for $k > q$ it follows that $\pi k!x = N\pi$ for some integer N. Now $\cos^{2j}(N\pi) = 1$ for all j. Thus if x is rational, $\lim_{j,k\to\infty} g_{k,j}(x) = 1$. Suppose however that x is irrational. Then $\pi k!x$ is never an integer multiple of π and so $-1 < \cos(\pi k!x) < 1$. Now if $|r| < 1$, $r^{2j} \to 0$ as $j \to \infty$. So for x irrational, $\lim_{j,k\to\infty} g_{k,j}(x) = 0$. Hence the limit of this sequence of functions is a function that is 1 if x is rational and 0 if x is irrational. This function is not Riemann integrable and it is not even continuous, despite the fact that every function in the sequence is not only continuous, but is analytic.

Uniform convergence is preserved under addition.

Proposition 1.107. *If $f_n \to f$ and $g_n \to g$ uniformly, then $af_n + bg_n \to af + bg$ uniformly, where a, b are constants.*

Proof. Suppose that $a, b \neq 0$. $f_n, g_n : X \subseteq \mathbb{R} \to \mathbb{R}$. Then given $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$\sup_{x\in X} |f_n(x) - f(x)| < \epsilon/(2|a|)$$

and there exists $N_2 \in \mathbb{N}$ such that

$$\sup_{x\in X} |g_n(x) - g(x)| < \epsilon/(2|b|).$$

Let $N = \max N_1, N_2$. Then for $n \geq N$

$$\begin{aligned}\sup_{x\in X} |af_n(x) + bg_n(x) - af(x) - bg(x)| &\leq |a| \sup_{x\in X} |f_n(x) - f(x)| \\ &\quad + |b| \sup_{x\in X} |g_n(x) - g(x)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

□

However, uniform convergence is *not* preserved under pointwise multiplication. That is, if $f_n \to f$ uniformly on $X \subseteq \mathbb{R}$ and $g_n \to g$ uniformly on X, it is not in general true that $f_n g_n \to fg$ uniformly on X. The best we can say is the following.

Theorem 1.108. *Suppose that $f_n \to f$ uniformly on the closed and bounded interval $[a, b]$ and $g_n \to g$ uniformly on $[a, b]$. Then $f_n g_n \to fg$ uniformly on $[a, b]$.*

Proof. Since $f_n \to f$ uniformly, it converges pointwise on $[a, b]$ and hence each sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded, for all $x \in [a, b]$. Consequently f is also bounded. Similarly for $\{g_n\}_{n=1}^{\infty}$. Let

$$A = \sup_{x\in[a,b]} |f(x)|, \quad B = \sup_{x\in[a,b],n\geq 1} |g_n(x)|.$$

Choose an $\epsilon > 0$. We can find $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|f_n(x) - f(x)| < \epsilon/(2B)$ and an $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies that $|g_n(x) - g(x)| < \epsilon/(2A)$. Then take $N = \max\{N_1, N_2\}$ and for $n \geq N$

$$\begin{aligned}\sup_{x\in[a,b]} |f_n(x)g_n(x) - f(x)g(x)| &= \sup_{x\in[a,b]} |f_n(x)g_n(x) - f(x)g_n(x) \\ &\quad + f(x)g_n(x) - f(x)g(x)| \\ &= \sup_{x\in[a,b]} |g_n(x)||f_n(x) - f(x)| \\ &\quad + \sup_{x\in[a,b]} |f(x)||g_n(x) - g(x)| \\ &< \epsilon/(2A) + \epsilon/(2B) = \epsilon.\end{aligned}$$

□

There are various tests for uniform convergence. For series we have the following powerful result.

Theorem 1.109 (Weierstrass M-Test)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on X such that $|f_n(x)| \leq M_n$ all $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.*

Proof. Let $S_N(x) = \sum_{n=1}^{\infty} f_n(x)$ and suppose that $|f_n(x)| \le M_n$. Then for all $N \ge M$

$$\begin{aligned} |S_N(x) - S_M(x)| &= |\sum_{n=M+1}^{N} f_n(x)| \\ &\le \sum_{n=M+1}^{N} |f_n(x)| \\ &\le \sum_{n=M+1}^{N} M_n \to 0, \end{aligned}$$

as $N, M \to \infty$. So the series S_N converges independently of x and hence is uniformly convergent. □

Example 1.18. The M test is generally easy to use. To illustrate, consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+1}. \tag{1.55}$$

Letting $f_n(x) = \dfrac{\cos(nx)}{n^2+1}$, we immediately see that

$$|f_n(x)| \le \frac{1}{n^2+1}, \tag{1.56}$$

and by the comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \infty$. Hence the series (1.55) is uniformly convergent and so f is a continuous function.

As an application we prove a result about power series.

Theorem 1.110. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $0 < r < R$. Then the series converges uniformly on $[-r, r]$.*

Proof. Let $\sum_{n=0}^{\infty} a_n x^n$ be convergent for $|x| < R$. Then it is absolutely convergent. Pick $x = x_0 > r$ and $x_0 < R$ and we have $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, hence $a_n x_0^n \to 0$. So there is an $M > 0$ such that $|a_n x_0^n| \le M$. Then

$$\begin{aligned} |a_n x^n| &\le |a_n| r^n \\ &= |a_n x_0^n| \left|\frac{r}{x_0}\right|^n \\ &\le M \left|\frac{r}{x_0}\right|^n. \end{aligned}$$

Now $\sum_{n=0}^{\infty} M \left|\frac{r}{x_0}\right|^n$ converges and hence the power series converges uniformly by the Weierstrass M test. □

It is important to note that this theorem does not say that a power series converges uniformly on $(-R,R)$. Indeed the series $\sum_{n=0}^{\infty} x^n$ converges on $(-1,1)$ but the convergence is not uniform. It does converge uniformly on $[-r,r]$ for $r<1$. The point is we cannot necessarily extend the uniform convergence to the entire interval of convergence.

Another useful test is due to Dini.

Theorem 1.111 (Dini)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[a,b]$ which converges monotonically to a continuous function f on $[a,b]$. Then $f_n \to f$ uniformly on $[a,b]$.*

Proof. We can with no loss of generality suppose that $f=0$ and that $f_n(x)$ decreases monotonically to 0 for all $x \in [a,b]$. If this is not the case then we can consider the functions $g_n = \pm(f_n - f)$ depending on whether f_n increases or decreases. The sequence $\{g_n\}$ will then decrease monotonically to 0.

Now set

$$M_n = \sup\{f_n(x) : x \in [x,b]\}.$$

Since f_n decreases to 0, it follows that M_n is decreasing. We claim that $M_n \to 0$. This will be sufficient to establish that the convergence is uniform, since then given $\epsilon > 0$ we will be able to find N such that for all $n \geq N$ $M_n < \epsilon$ and so for $n \geq N$ we will have

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon. \tag{1.57}$$

We proceed by contraction. Suppose otherwise. Then we can find $\delta > 0$ such that for every n, $M_n > \delta$. Consequently for every n there is a point x_n such that $f(x_n) > \delta$. The sequence $\{x_n\} \in [a,b]$ is bounded, so by the Bolzano-Weierstrass Theorem it contains a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Let $x_{n_k} \to \alpha$ as $k \to \infty$. By assumption, $f_n(\alpha) \to 0$ as $n \to \infty$. So there exists $N > 0$ such that for $n \geq N$, we have $|f_n(\alpha)| < \delta$. But each f_n is continuous, so that we can find $\epsilon > 0$ such that $|x-\alpha| < \epsilon$ implies $|f_n(x)| < \delta$ for all $n \geq N$. But this is a contradiction, because we can choose N such that for all $n_k \geq N$, $|x_{n_k} - \alpha| < \epsilon$ and $f_n(x_{n_k}) > \delta$. Thus our assumption must be false and hence $M_n \to 0$. □

1.8.1. *The Weierstrass Approximation Theorem.* One of the most important results on uniform approximation of functions is due to Weierstrass. This says that any continuous function on a closed and bounded interval $[a,b]$ can be approximated uniformly by a polynomial. Equivalently, there is a sequence of polynomials converging uniformly to f. This can be proved in a number of ways. We can use Fourier series to establish the result, but Bernstein actually constructed a sequence of polynomials which approximates any continuous function uniformly. To present Bernstein's proof, we require a preliminary lemma.

Lemma 1.112. *For each fixed x, the following identities hold.*

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 \tag{1.58}$$

$$\sum_{k=1}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = nx \tag{1.59}$$

$$\sum_{k=1}^{n} k^2\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 + nx. \tag{1.60}$$

Proof. For the first identity, observe that $1^n = (x+(1-x))^n$ and apply the Binomial Theorem to both sides. For the second,

$$\begin{aligned} k\binom{n}{k} = \frac{kn!}{k!(n-k)!} &= \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} \\ &= n\binom{n-1}{k-1}. \end{aligned}$$

So that

$$\begin{aligned} \sum_{k=1}^{n} k\binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= n\sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} (1-x)^{n-1-j} \\ &= nx(x+(1-x))^{n-1} = nx. \end{aligned}$$

For the final identity, notice that $k^2 = k(k-1)+k$, so that

$$\begin{aligned} \sum_{k=1}^{n} k^2\binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=1}^{n} (k(k-1)+k)\binom{n}{k} x^k (1-x)^{n-k} \\ &= nx + \sum_{k=1}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} \end{aligned}$$

Now

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$$

so that

$$\begin{aligned} \sum_{k=1}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=1}^{n} n(n-1)\binom{n-2}{k-2} x^k (1-x)^{n-k} \\ &= n(n-1)x^2(x+1-x)^{n-2} \\ &= n(n-1)x^2. \end{aligned}$$

This completes the proof. □

Now we come to Weierstrass' Theorem.

Theorem 1.113 (Weierstrass Approximation Theorem)**.** *Let f be a continuous function on a closed and bounded interval $[a, b]$. Then given any $\epsilon > 0$ there is a polynomial P with the property that*

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Proof. This is Bernstein's proof. For simplicity we can restrict attention to the interval $[0, 1]$, since $[0, 1]$ can be mapped to $[a, b]$ by the function $\varphi(t) = a(1 - t) + bt$, where $0 \leq t \leq 1$. It is not hard to show that if $P_n \to g$ uniformly on $[0, 1]$, then

$$P_n(\varphi(t)) \to g\left(\frac{t-a}{b-a}\right)$$

uniformly on $[a, b]$.

Now let f be continuous on $[0, 1]$ and define the polynomials

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{1.61}$$

These are the Bernstein polynomials for f. We claim that $P_n \to f$ uniformly on $[0, 1]$. That is, given $\epsilon > 0$ we want to find an $n \in \mathbb{N}$ such that $n \geq N$ implies $\sup_{x\in[0,1]} |P_n(x) - f(x)| < \epsilon$. Notice that

$$\begin{aligned} P_n(x) - f(x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \\ &\quad - f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n}\left[f\left(\frac{k}{n}\right) - f(x)\right]\binom{n}{k}x^k(1-x)^{n-k}. \qquad (1.62) \end{aligned}$$

We want to use the uniform continuity of f to make this small. So let $\epsilon > 0$ and pick $\delta > 0$ such that $|x - y| < \epsilon/2$ implies

$$|f(x) - f(y)| < \epsilon/2.$$

We therefore want to consider values of k/n such that $|x - k/n| < \delta$. Since k, n are integers we need to use the integer part function $[x] =$ greatest integer $\leq x$. We split the sum as

$$\begin{aligned} P_n(x) - f(x) = &\sum_{[x-k/n]<\delta}\left[f\left(\frac{k}{n}\right) - f(x)\right]\binom{n}{k}x^k(1-x)^{n-k} \\ &+ \sum_{[x-k/n]\geq\delta}\left[f\left(\frac{k}{n}\right) - f(x)\right]\binom{n}{k}x^k(1-x)^{n-k}. \end{aligned}$$

Using the fact that $\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=1$ and the continuity of f we get

$$\left|\sum_{[x-k/n]<\delta}\left[f\left(\frac{k}{n}\right)-f(x)\right]\binom{n}{k}x^k(1-x)^{n-k}\right|<\frac{\epsilon}{2},$$

since $|k/n-x|<\delta$. For the second sum

$$\left|\sum_{[x-k/n]\geq\delta}\left[f\left(\frac{k}{n}\right)-f(x)\right]\binom{n}{k}x^k(1-x)^{n-k}\right|$$
$$\leq 2\sup_{x\in[0,1]}|f(x)|\left|\sum_{[x-k/n]\geq\delta}\binom{n}{k}x^k(1-x)^{n-k}\right|$$

Now $|x-k/n|\geq\delta$, so that $(x-k/n)^2/\delta^2\geq 1$. We can then produce the estimate

$$\begin{aligned}\sum_{[x-k/n]\geq\delta}\binom{n}{k}x^k(1-x)^{n-k}&\leq\frac{1}{\delta^2}\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k}\\&=\frac{1}{\delta^2}\sum_{k=0}^{n}\left(x^2-\frac{2xk}{n}+\frac{x^2}{k^2}\right)\binom{n}{k}x^k(1-x)^{n-k}\\&=\frac{1}{\delta^2}\left(x^2-\frac{2x}{n}nx+\frac{1}{n^2}(n(n-1)x^2+nx)\right)\\&=\frac{x(1-x)}{n\delta^2}\leq\frac{1}{4n\delta^2},\end{aligned}$$

since $x(1-x)\leq 1/4$ if $x\in[0,1]$. We thus arrive at

$$\left|\sum_{[x-k/n]\geq\delta}\left[f\left(\frac{k}{n}\right)-f(x)\right]\binom{n}{k}x^k(1-x)^{n-k}\right|\leq\frac{1}{4n\delta^2}2A, \qquad (1.63)$$

where $A=\sup_{x\in[0,1]}|f(x)|$. Hence we need to choose $n\geq 2\epsilon A/\delta^2$. This will guarantee that $\sup_{x\in[0,1]}|P_n(x)-f(x)|<\epsilon$. □

Weierstrass' Theorem was extended by Marshall Stone to algebras of functions on more abstract spaces and the result is known as the Stone-Weierstrass Theorem. It plays a central role in much modern analysis.

Remark 1.114. A word of caution. Weierstrass' Theorem does not mean that the Taylor series of a function f will converge to f uniformly. The Taylor series may not exist, since we only assume continuity, not differentiability for f. Even when f is infinitely differentiable the Taylor series may still not converge.

Uniform convergence allows us to reverse the order of a limit and a Riemann integral.

Theorem 1.115. *If* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of Riemann integrable functions converging uniformly to* f *on* $[a,b]$*, then* f *is Riemann integrable on* $[a,b]$ *and*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. If the functions f_n are continuous then the proof is easy. By uniform convergence we can choose N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon/(b-a).$$

As f is continuous by Theorem 1.106, we have for $n \geq N$

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leq \int_a^b |f_n(x) - f(x)|\, dx \tag{1.64}$$

$$< \int_a^b \frac{\epsilon}{(b-a)}dx = \epsilon. \tag{1.65}$$

If the functions $\{f_n\}$ are not assumed to be continuous, then we have to prove the limit is integrable. Each f_n is bounded, so the limit f is bounded. Pick $\epsilon > 0$ and by uniform convergence we can choose $N \in \mathbb{N}$ such that for all $x \in [a,b]$ $n \geq N$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}$. Since f_N is integrable, by Riemann's criterion we can choose a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) < \frac{\epsilon}{3}.$$

Now $\sup_{x\in[a,b]}|f_N(x) - f(x)| < \frac{\epsilon}{3(b-a)}$, so we have

$$\begin{aligned}
U(f,\mathcal{P}) - L(f,\mathcal{P}) &= U(f + f - f_N,\mathcal{P}) - L(f + f_N - f,\mathcal{P})\\
&\leq U(f_N,\mathcal{P}) + U(f - f_N,\mathcal{P}) - L(f_N,\mathcal{P})\\
&\quad - L(f - f_N,\mathcal{P})\\
&= U(f_N,\mathcal{P}) - L(f_N,\mathcal{P}) + U(f - f_N,\mathcal{P})\\
&\quad - L(f - f_N,\mathcal{P})\\
&< \frac{\epsilon}{3} + 2\frac{\epsilon}{3(b-a)}(b-a) = \epsilon.
\end{aligned}$$

The inequalities used above can be verified by direct calculation. So f is Riemann integrable. The rest of the proof is as in the continuous case. □

This swapping of limits does not work for pointwise convergence with the Riemann integral. The limit function may not even be Riemann integrable, as is the case with the double sequence $f_{n,k}$ above. Unfortunately when we have sequences of functions, they often do not converge uniformly. This leads to the question of how can we modify the integral in such a way as to be able to swap limits and integrals, even when we do not have uniform convergence? This problem led to a new theory of integration, which we will consider, beginning in section three.

Uniform convergence is equivalent to a sequence being uniformly Cauchy.

Definition 1.116. A sequence $f_n : X \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly Cauchy if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\sup_{x \in I} |f_n(x) - f_m(x)| < \epsilon.$$

The next result connects uniformly Cauchy and uniformly convergent sequences.

Theorem 1.117. *Every uniformly convergent sequence of functions is uniformly Cauchy.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly convergent sequence of functions defined on $X \subseteq \mathbb{R}$ and suppose it has limit f. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $x \in X, n \geq N$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Then if $x \in X, m, n \geq N$ implies

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x)| \\ &\leq |f_m(x) - f(x)| + |f(x) - f_n(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy. □

The converse of this result is also true, but the proof is a little harder.

Theorem 1.118. *Every uniformly Cauchy sequence of functions is uniformly convergent.*

Proof. There are two parts. First we have to define the limit and then we have to prove that the convergence is uniform. The first part proceeds as follows.

Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly Cauchy sequence on $X \subseteq \mathbb{R}$. Let $x_0 \in X$. Then $\{f_n(x_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent. Let us denote the limit by $f(x_0)$. This process defines a limit function for each $x \in X$. By construction, $f_n \to f$ pointwise.

Now we prove that $f_n \to f$ uniformly. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $x \in X$, $n, m \geq N$ we have

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2}.$$

Now $f_m \to f$ pointwise. So for all $x \in X$, $n \geq N$

$$|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f(x)| < \frac{\epsilon}{2}.$$

Thus $f_n \to f$ uniformly. □

Interchanging limits and derivatives is actually harder than interchanging a limit and an integral. Consider the sequence of functions $f_n(x) = \sqrt{x^2 + 1/n^2}$. Then for every n, $f_n'(x)$ exists for all $x \in \mathbb{R}$. However $f_n \to |x|$ which is not differentiable at zero. Yet the convergence is uniform. To see this, observe that

$$\begin{aligned}\sqrt{x^2 + \frac{1}{n^2}} - |x| &= (\sqrt{x^2 + \frac{1}{n^2}} - |x|)\frac{\sqrt{x^2 + \frac{1}{n^2}} + |x|}{\sqrt{x^2 + \frac{1}{n^2}} + |x|} \\ &= \frac{1}{n^2(\sqrt{x^2 + \frac{1}{n^2}} + |x|)} \\ &\leq \frac{1}{n}.\end{aligned}$$

Thus we can bound the difference between the nth term of the sequence and the limit independently of x and so the convergence is uniform. So uniform convergence is not enough to guarantee that the limit function is differentiable.

Even if the limit is differentiable, it does not follow that $f_n' \to f'$.

Example 1.19. Let $f_n(x) = \dfrac{x}{1 + nx^2}$. Now $f_n \to 0$ for all x as $n \to \infty$. But

$$f_n'(x) = \frac{(1 - nx^2)}{(1 + nx^2)^2}$$

and so $f_n'(0) \to 1 \neq 0 = f'(0)$.

We actually require uniform convergence of the derivatives in order to swap differentiation and limits. The relevant result follows.

Theorem 1.119. *Let I be an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on I which converges pointwise to f on I. Let $g : I \to \mathbb{R}$ and let the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converge uniformly to g on I. Then f is differentiable on I and $f'(x) = g(x)$ for all $x \in I$.*

Proof. Let $\epsilon > 0$ and pick an $N_1 \in \mathbb{N}$ such that

$$\sup_{x \in I} |f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

The sequence $\{f_n'\}_{n=1}^{\infty}$ is uniformly Cauchy on I. So we can find $N_2 \in \mathbb{N}$ such that for all $x \in I$, $n, m \geq N_2$

$$|f_m'(x) - f_n'(x)| < \frac{\epsilon}{3}.$$

Now let $N = \max N_1, N_2$. The function f_N is differentiable on I and so at any point $x_0 \in I$ there exists $\delta > 0$ such that for $x \in I$, $0 <$

$|x - x_0| < \delta$ we have
$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N(x_0)\right| < \frac{\epsilon}{3}.$$
Now let $x \in I$, $x \neq x_0$ and choose $M \geq N$. $f_M - f_N$ is differentiable and so by the Mean Value Theorem we can find c between x and x_0 such that
$$\frac{(f_M - f_N)(x) - (f_M - f_N)(x_0)}{x - x_0} = (f_M - f_N)'(c).$$
which is the same as
$$(f_M - f_N)(x) - (f_M - f_N)(x_0) = (f_M - f_N)'(c)(x - x_0).$$
From this we deduce that
$$|f_M(x) - f_M(x_0) - (f_N(x) - f_N(x_0))| = |f'_M(c) - f'_N(c)||x - x_0| \tag{1.66}$$
$$< \frac{\epsilon}{3}|x - x_0|,$$
where $|f'_M(c) - f'_N(c)| < \epsilon/3$ by the fact that the sequence is uniformly Cauchy and $M, N \geq N$ and $c \in I$. Taking limits as $M \to \infty$ in (1.66) we get
$$|f(x) - f(x_0) - (f_N(x) - f_N(x_0))| \leq \frac{\epsilon}{3}|x - x_0|,$$
which leads to the inequality
$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right| \leq \frac{\epsilon}{3}.$$
Finally, we put this altogether and let $x \in I$, $0 < |x - x_0| < \delta$, then adding and subtracting appropriate terms, and using the triangle inequality, we can write
$$\begin{aligned}\left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| &\leq \left|\frac{f(x) - f(x_0)}{x - x_0} - \left(\frac{f_N(x) - f_N(x_0)}{x - x_0}\right)\right| \\ &\quad + \left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0)\right| \\ &\quad + |f'_N(x_0) - g(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$
Hence f is differentiable at x_0 and $f'(x_0) = g(x_0)$.

□

For many problems, the methods we have developed are sufficient to provide a solution. However analysis does not stop at this point. There are areas where more sophisticated techniques are needed. For example, do we really need uniform convergence to swap a limit and an integral, or can we do better? The answer is that yes we can, but it will require us to completely redefine what we mean by integration. Much of the remainder of these notes will flow from giving a better answer to the question: What is the best way to define the integral?

2. The Riemann-Stieltjes Integral

2.1. **Basic Concepts.** In this section we will consider a modification of the integral due to Thomas Stieltjes. This integral has many uses so we will develop it in some detail, before turning to the more important Lebesgue integral.

Recall that for a function f, we can form a Riemann sum by taking

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}). \tag{2.1}$$

Here $x_i^* \in [x_{i-1}, x_i)$. While studying problems involving the calculation of moments, Thomas Stieltjes generalised the Riemann integral by considering a sum

$$\sum_{i=1}^{n} f(x_i^*)[\phi(x_i) - \phi(x_{i-1})] = \sum_{\mathcal{P}} f\Delta\phi, \tag{2.2}$$

for some bounded function ϕ. Taking $\phi(x) = x$ we obtain a Riemann sum. However Stieltjes claimed without proof that if we let

$$|\mathcal{P}| = \max_i\{|x_i - x_{i-1}|\} \to 0, \tag{2.3}$$

then $\sum_{\mathcal{P}} f\Delta\phi$ converges to a limit. This turns out to be true under suitable assumptions on f and ϕ and this limit is a new kind of integral.

Definition 2.1 (Riemann-Stieltjes Integral)**.** Suppose that we have two bounded functions f and ϕ on $[a, b]$ and that there is a number A such that for each $\epsilon > 0$ there is a $\delta > 0$ for which

$$\left|\sum_{k=1}^{n} f(c_k)\left(\phi(x_k) - \phi(x_{k-1})\right) - A\right| < \epsilon, \tag{2.4}$$

where $x_{k-1} \leq c_k \leq x_k$ and $\max_k |x_k - x_{k-1}| < \delta$. Then we say that f is Riemann-Stieltjes integrable and we write

$$RS\int_a^b f(x)d\phi(x) = A, \tag{2.5}$$

or usually just

$$\int_a^b f(x)d\phi(x) = A. \tag{2.6}$$

We call this the Riemann-Stieltjes integral or the RS integral. If $\phi(x) = x$, the standard Riemann integral is to be understood.

We easily see that for ϕ a constant, the integral is zero. Conversely, if $f(x) = k$ then

$$\int_a^b k d\phi(x) = k[\phi(b) - \phi(a)]. \tag{2.7}$$

If ϕ is a step function with discontinuities at $\hat{x}_1, ..., \hat{x}_n$ then the RS integral has a simple expression.

Theorem 2.2. *Suppose that f is continuous on $[a, b]$ and that ϕ is a step function with*

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{[x_{k-1}, x_k)}(x),$$

where $\chi_A(x) = 1$ if $x \in A$ and zero otherwise. Then the RS integral of f with respect to ϕ exists and

$$RS \int_a^b f(x) d\phi(x) = \sum_{k=1}^{n} f(\hat{x}_k)[\phi(\hat{x}_k^+) - \phi(\hat{x}_k^-)]. \tag{2.8}$$

Here $\phi(a^+) - \phi(a^-)$ means $\phi(a^+) - \phi(a)$ and $\phi(b^+) - \phi(b^-)$ means $\phi(b) - \phi(b^-)$.

The essential feature to understand is that $\phi(\hat{x}_k^+) - \phi(\hat{x}_k^-)$ is the change in ϕ across any jump point $\hat{x}_k$. So if there is a jump at $x = 2$ and to the left $\phi(2^-) = 5$ and to the right $\phi(2^+) = 9$, then the change is 4. This result is best understood by an example.

Example 2.1. Let $f(x) = x^2$ and $\phi(x) = [x]$, the greatest integer $\leq x$. Then the jumps occur at $\hat{x}_k = 1, 2, 3...$ and the jump size is always 1. So $\phi(\hat{x}_k^+) - \phi(\hat{x}_k^-) = 1$ for every k.

$$\int_0^n x^2 d([x]) = \sum_{k=1}^{n} k^2 = \frac{n}{6}(n + 1)(2n + 1). \tag{2.9}$$

Note: The RS integral can exist only if f and ϕ have no common points of discontinuity. If f and ϕ both jump at x_k, then there is a problem. Do we take $f(x^-)$ or $f(x^+)$ in the RS Sum? Taking $f(x^-)[\phi(x^+) - \phi(x^-)]$ will lead to one value for the integral and taking $f(x^+)[\phi(x^+) - \phi(x^-)]$ will give a different value for the integral. Since functions may have many different jump points, there are many different possible integrals. So no consistent choice is possible.

2.2. **Evaluating the RS Integral.** The next result is the most important. This tells us how to actually evaluate a RS integral when ϕ is differentiable.

Theorem 2.3. *Suppose that f is continuous and ϕ is differentiable on (a, b), with ϕ' being Riemann integrable on $[a, b]$. Then the RS integral of f with respect to ϕ exists and*

$$RS \int_a^b f(x) d\phi(x) = R \int_a^b f(x) \phi'(x) dx \tag{2.10}$$

where the right side is a regular Riemann integral.

Proof. It is clear that $f\phi'$ is Riemann integrable and

$$\int_{x_{k-1}}^{x_k} \phi'(x)dx = \phi(x_k) - \phi(x_{k-1}). \tag{2.11}$$

So

$$\begin{aligned}\sum_{\mathcal{P}} f\Delta\phi - R\int_a^b f(x)\phi'(x)dx &= \sum_{k=1}^n f(c_k)(\phi(x_k) - \phi(x_{k-1})) \\ &\quad - R\int_a^b f(x)\phi'(x)dx \\ &= \sum_{k=1}^n R\int_{x_{k-1}}^{x_k} (f(c_k) - f(x))\phi'(x)dx.\end{aligned}$$

Now f is uniformly continuous on $[a,b]$. Choose $\delta > 0$ such that $|x_k - x_{k-1}| < \delta$ implies

$$|f(x_k) - f(x_{k-1})| < \frac{\epsilon}{n(1 + |\phi(b) - \phi(a)|)}.$$

Then

$$\begin{aligned}\left|\sum_{k=1}^n R\int_{x_{k-1}}^{x_k} (f(c_k) - f(x))\phi'(x)dx\right| &< \sum_{k=1}^n R\int_{x_{k-1}}^{x_k} |f(c_k) - f(x)||\phi'(x)|dx \\ &< \sum_{k=1}^n \frac{\epsilon|\phi(b) - \phi(a)|}{n(1 + |\phi(b) - \phi(a)|)} \\ &< \epsilon.\end{aligned}$$

□

Example 2.2. Using the previous result we can easily compute most RS integrals. For example

$$\int_{-1}^1 x^2 d(x^2) = \int_{-1}^1 x^2(2xdx) = 2\int_{-1}^1 x^3 dx = 0.$$

There is also an integration by parts formula.

Theorem 2.4. *Suppose that f, ϕ are bounded functions with no common discontinuities on the interval $[a,b]$ and that the RS integral $\int_a^b f d\phi$ exists. Then the RS integral $\int_a^b \phi df$ exists and*

$$\int_a^b \phi(x)df(x) = f(b)\phi(b) - f(a)\phi(a) - \int_a^b f(x)d\phi(x).$$

Proof. Let $\mathcal{P}$ be a partition of $[a,b]$ whose subintervals have length less than $\delta/2$. If $c_0 = a$ and $c_{n+1} = b$, then $\{c_0, c_1, ..., c_{n+1}\}$ is a partition of $[a,b]$ for $c_k \le x_k \le c_{k+1}$ and $c_k - c_{k-1} < \delta$. We have

$$\left|\sum_{\mathcal{P}} f\Delta\phi - \int_a^b f d\phi\right| < \epsilon$$

for this partition. Then

$$\left|\sum_{k=1}^{n}\phi(c_k)[f(x_k)-f(x_{k-1})]-\left(f(b)\phi(b)-f(a)\phi(a)-\int_a^b f(x)d\phi(x)\right)\right|$$

$$=\left|\sum_{k=1}^{n}\phi(c_k)[f(x_k)-f(x_{k-1})]-(f(x_n)\phi(c_{n+1})-f(x_0)\phi(c_0))\right.$$

$$\left.+\int_a^b f(x)d\phi(x)\right|$$

$$=\left|\sum_{k=0}^{n}\phi(c_k)f(x_k)-\sum_{k=1}^{n+1}\phi(c_k)f(x_{k-1})+\int_a^b f(x)d\phi(x)\right|$$

$$=\left|\int_a^b f(x)d\phi(x)-\sum_{k=0}^{n}f(x_k)[\phi(c_{k+1})-\phi(c_k)]\right|<\epsilon.$$

□

There is a fundamental theorem of calculus for the RS integral.

Theorem 2.5. *If f is continuous on $[a,b]$ and ϕ is monotone increasing on $[a,b]$, then $\int_a^b f(x)d\phi(x)$ exists. If $F(x)=\int_a^x f(t)d\phi(t)$ then*

(1) *F is continuous at any point where ϕ is continuous*

(2) *F is differentiable at any point where ϕ is differentiable and $F'(x)=f(x)\phi'(x)$.*

Proof. Proof of the existence of the integral is left as an exercise. First note the elementary point that since ϕ is monotone increasing, we can assume $\phi(b)-\phi(a)>0$. If $\phi(b)=\phi(a)$ then ϕ is constant and hence the RS integral is zero.

To show that F is continuous at any point where ϕ is, we use the fact that a continuous function on a closed, bounded interval is bounded. So we can assume that $\max_{a\le x\le b}|f(x)|=B$. We wish to show that given $\epsilon>0$ we can find $\delta>0$, such that for a given x, $|x-y|<\delta$ implies that $|F(x)-F(y)|<\epsilon$. By linearity of the integral

$$\begin{aligned}|F(x)-F(y)|&\le\left|\int_y^x f(t)d\phi(t)\right|\\&\le B\left|\int_y^x d\phi(t)\right|\\&\le B|\phi(x)-\phi(y)|,\end{aligned}$$

and so if δ is chosen so that $|x-y|<\delta$, then $|\phi(x)-\phi(y)|<\frac{\epsilon}{B}$ and the desired inequality for F follows. Hence F is continuous wherever ϕ is.

Next, we suppose that ϕ is differentiable at $x \in (a,b)$. By continuity of f, on the interval $[a,b]$, if $h>0$ and sufficiently small

$$\min_{t\in[x,x+h]} f(t)[\phi(x+h)-\phi(x)] \leq \int_x^{x+h} f(t)d\phi(t) \leq \max_{t\in[x,x+h]} f(t)[\phi(x+h)-\phi(x)].$$

Since F is continuous, we can apply the intermediate value theorem to conclude that there is a $c \in [x,x+h]$ such that

$$\int_x^{x+h} f(t)d\phi(t) = f(c)[\phi(x+h)-\phi(x)]. \tag{2.12}$$

We then write

$$\begin{aligned}\left|\frac{F(x+h)-F(x)}{h} - f(x)\phi'(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)d\phi(t) - f(x)\phi'(x)\right| \\ &= \left|f(c)\frac{\phi(x+h)-\phi(x)}{h} - f(x)\phi'(x)\right| \\ &\leq \left|f(c)\left(\frac{\phi(x+h)-\phi(x)}{h} - \phi'(x)\right)\right| \\ &\quad + |\phi'(x)[f(x)-f(c)]|.\end{aligned}$$

Since ϕ is differentiable, the first term goes to zero as $h \to 0$. Similarly, as $h \to 0$, $|f(c)-f(x)| \to 0$ since $c \in [x,x+h]$.

□

2.3. **Euler-MacLaurin Summation.** As an application of the RS integral, we prove a famous result due to Euler and MacLaurin. A very common problem in mathematics and its applications is to sum a series of numbers given by some sequence. Thus practical methods of summing series had to be developed. Probably the best known is that of Euler and MacLaurin.

Theorem 2.6. *Suppose that f and f' are continuous on $[0,\infty)$. Then*

$$\sum_{k=1}^N f(k) = \int_1^N f(x)dx + \frac{1}{2}(f(1)+f(N)) + \int_1^N (x-[x]-1/2)f'(x)dx.$$

Proof. From our earlier considerations, we see that $\int_1^N f(x)d([x]) = \sum_{k=2}^N f(k)$ since the first jump occurs at 2. Integration by parts for the RS integral gives

$$\begin{aligned}\int_1^N f(x)d([x]) &= Nf(N) - f(1) - \int_1^N [x]df(x) \\ &= Nf(N) - f(1) - \int_1^N [x]f'(x)dx\end{aligned}$$

So

$$\int_1^N (x-[x]-1/2)f'(x)dx = \int_1^N (x-1/2)f'(x) - \int_1^N [x]f'(x)dx$$
$$= (N-1/2)f(N) - 1/2f(1) - \int_1^N f(x)dx$$
$$- Nf(N) + f(1) + \int_1^N f(x)d([x])$$
$$= -1/2(f(N)+f(1)) - \int_1^N f(x)dx$$
$$+ \sum_{k=1}^N f(k),$$

since

$$\int_1^N f(x)d([x]) = \sum_{k=2}^N f(k) = \sum_{k=1}^N f(k) - f(1).$$

The result follows. □

To use Euler-Maclaurin summation we have to evaluate the integral

$$\int_1^N (x-[x]-1/2)f'(x)dx.$$

The best way to do this is by means of Fourier series. (We will discuss Fourier series later in the course, but now we only assume basic knowledge from the subject 35231). We assume that f is sufficiently smooth on $[1,N]$. Since $x-[x]-1/2$ is periodic, with period 1, we can represent it as a Fourier series. The Fourier series

$$P_1(x) = -\sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{2n\pi} \tag{2.13}$$

represents $x-[x]-1/2$ on $[1,N]$. Now if we integrate P_1 we obtain

$$P_2(x) = \sum_{n=1}^{\infty} \frac{2\cos(2n\pi x)}{(2n\pi)^2} + A \tag{2.14}$$

The constant A we can take to be zero, since we will be evaluating a definite integral by parts. Integrating repeatedly we are lead to the odd and even indefinite integrals of P_1.

$$P_{2k+1}(x) = (-1)^{k+1}\sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{(2n\pi)^{2k+1}}$$
$$P_{2k}(x) = (-1)^{k+1}\sum_{n=1}^{\infty} \frac{2\cos(2n\pi x)}{(2n\pi)^{2k}}.$$

We notice $P_{2k+1}(N) = 0$ for any integer N and

$$P_{2k}(N) = 2(-1)^{k+1}\frac{1}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = 2(-1)^{k+1}\frac{1}{(2\pi)^{2k}}\zeta(2k),$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function. Euler proved that

$$\zeta(2k) = (-1)^{k+1}\pi^{2k}B_{2k}\frac{2^{2k-1}}{(2k)!}, \tag{2.15}$$

where B_n is the nth Bernoulli number, defined by $B_n = B_n(0)$ and the Bernoulli polynomials satisfy

$$\sum_{n=0}^{\infty}B_n(x)t^n = \frac{te^{xt}}{e^t - 1}.$$

These can be computed and we have $\zeta(2) = \pi^2/6, \zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$ etc.

Now

$$\begin{aligned}\int_1^N (x - [x] - 1/2)f'(x)dx &= \int_1^N P_1(x)f'(x)dx \\ &= P_2(x)f'(x)]_1^N - \int_1^N P_2(x)f''(x)dx \\ &= \frac{1}{12}[f'(N) - f'(1)] - P_3(x)f''(x)]_1^N \\ &\quad + \int_1^N P_3(x)f'''(x)dx \\ &= \frac{1}{12}[f'(N) - f'(1)] + P_4(x)f'''(x)]_1^N \\ &\quad - \int_1^N P_4(x)f^{(iv)}(x)dx \\ &= \frac{1}{12}[f'(N) - f'(1)] - \frac{1}{720}[f'''(N) - f'''(1)] \\ &\quad + \cdots\end{aligned}$$

We are thus lead to the approximation

$$\sum_{k=1}^{N}f(k) = \int_1^N f(x)dx + \frac{1}{2}(f(1) + f(N)) + \frac{1}{12}[f'(N) - f'(1)] - \\ \frac{1}{720}[f'''(N) - f'''(1)] + \frac{1}{30240}[f^{(v)}(N) - f^{(v)}(1)] + \cdots .$$

The error in this approximation when we truncate at the $2n + 1$st derivative can be shown to go to zero as $n \to \infty$. We thus have a very practical way of summing series.

Convergence theorems for the RS integral are similar to those for the Riemann integral. If we have uniform convergence, then swapping limits and integrals is valid.

Theorem 2.7. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions converging uniformly to f on $[a,b]$ and ϕ is monotone increasing, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)d\phi(x) = \int_a^b f(x)d\phi(x). \tag{2.16}$$

Proof. Recall that if a sequence of continuous functions converges uniformly, then the limit is continuous. Since $f_n \to f$ uniformly, f is continuous. Next observe that ϕ is monotone, so $A = \int_a^b d\phi(x)$ exists and A is finite. Now pick an $\epsilon > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon/|A|$. Then $\int_a^b f(x)d\phi(x)$ exists and so for $n \geq N$

$$\begin{aligned}\left|\int_a^b f_n(x)d\phi(x) - \int_a^b f(x)d\phi(x)\right| &\leq \max_{x\in[a,b]}|f_n(x) - f(x)|\left|\int_a^b d\phi(x)\right| \\ &< \frac{\epsilon}{|A|}\left|\int_a^b d\phi(x)\right| \\ &< \epsilon.\end{aligned}$$

So

$$\lim_{k\to\infty}\int_a^b f_k(x)d\phi(x) = \int_a^b f(x)d\phi(x). \tag{2.17}$$

□

Example 2.3. From the previous result

$$\begin{aligned}\lim_{k\to\infty}\int_0^{\pi/2}(1 - x/n)^n d(\sin x) &= \int_0^{\pi/2} e^{-x}d(\sin x) \\ &= \int_0^{\pi/2} e^{-x}\cos x dx \\ &= 1/2(1 + e^{-\pi/2}).\end{aligned}$$

3. Measure Theory

3.1. **Definition of a Measure.** We saw a different type of integral in the previous section. In this section we begin the task of introducing the most important integral for modern analysis, namely the Lebesgue integral. This will require the development of measure theory. Basically a measure is a way of assigning a size to a set. More precisely:

Definition 3.1. A measure is a set function with the properties that

(i) $m(\emptyset) = 0$, where $\emptyset$ is the empty set.

(ii) If $\{A_n\}$ are pairwise disjoint then $m\left(\bigcup_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} m(A_n)$.

The concept of a measure was introduced by Borel in 1898, though it builds on earlier work of Jordan. In his PhD thesis in 1900 Lebesgue extended the idea and used it to construct a new type of integral, which has since gone on to become the standard integral used in analysis.

The point of a measure is to assign a size to a set. For finite discrete sets, the measure might be the number of elements in the set. For subsets of the real numbers, such as intervals, the measure might be the length. In this course we are interested in subsets of the real numbers. Besides the two basic properties, a wish list for a measure on $\mathbb{R}$ might consist of the following:

(i) Every set of real numbers is measurable, no matter how complicated.

(ii) If $A \subseteq B$ then $m(A) \leq m(B)$.

(iii) The measure of a single point $\{x\}$ is zero.

(iv) $m([a, b]) = b - a$.

(v) $m(A + x) = A$, where $A + x = \{y + x, y \in A\}$.

Unfortunately it turns out that no such function exists. We will see later that there are sets of real numbers so weird that they do not have a measure. At the moment however, we are unaware of this problem. So we proceed as follows. First we define something called an *outer measure*. We recall that the length of an interval $I = (a, b)$ is given by $l(I) = b - a$. Note that closed, and half open intervals have the same length as the corresponding open intervals. Now we make a definition.

Definition 3.2. Let A be a set of real numbers. The Lebesgue outer measure of A is

$$m^*(A) = \inf\left\{\sum_k l(I_k) : A \subseteq \bigcup_k I_k\right\}, \tag{3.1}$$

and the I_k are open intervals.

The idea is to cover our set A with the smallest collection of intervals possible and take the sum of the lengths of the intervals to be the outer measure. We can similarly define the inner measure

Definition 3.3. Let A be a set of real numbers. The Lebesgue inner measure of A is

$$m_*(A) = \sup\left\{\sum_k l(I_k) : \bigcup_k I_k \subseteq A\right\} \tag{3.2}$$

and the I_k are open intervals.

Here we are filling up our set A with the biggest collection of intervals possible, then taking the sum of the lengths of the intervals to be the inner measure.

3.2. **Measurable Sets.** For arbitrary sets it will turn out that the Lebesgue outer measure is not in general additive. The best we can conclude in the general case is that Lebesgue outer measure is subadditive. The proof of the next result is a tutorial exercise.

Lemma 3.4. *For any collection of disjoint sets $A_1, A_2, \ldots$ the Lebesgue outer measure satisfies the inequality*

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n). \tag{3.3}$$

In order to get additivity, we will have to restrict our attention to certain types of sets. There are actually two different approaches to this problem, which turn out to be completely equivalent. The original idea of Lebesgue is to consider sets which have the same inner and outer measure. Then in 1912 Caratheodory introduced a criterion for measurability which has since become the standard approach.

If E^c denotes the compliment on a set $E \subseteq X$, that is

$$E^c = \{x \in X, x \notin E\}, \;\; X^c = \emptyset,$$

then given another set A, any set E can be decomposed into two disjoint sets:

$$E = (E \cap A) \cup (E \cap A^c).$$

This suggests that the outer measure should satisfy

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c).$$

This forms the basis of our definition of measurability.

Definition 3.5 (Caratheodory's Measurability Condition)**.** A set $A \subset \mathbb{R}$ is Lebesgue measurable if and only if

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c),$$

for every subset E of $\mathbb{R}$.

We already know that outer measure is always sub-additive. So that

$$m^*(E) \le m^*(E \cap A) + m^*(E \cap A^c)$$

is always true. To prove measurability for a given set we therefore need to establish that the reverse inequality is true.

This definition of measurability turns out to be enough to guarantee additivity for countable unions of disjoint sets. Lebesgue's definition of measurability was alluded to above.

Definition 3.6. A set $A \subset \mathbb{R}$ is measurable if $m^*(A) = m_*(A)$. That is, if the inner and outer measures are equal.

A result which we will not prove connects these two definitions.

Theorem 3.7. *Definitions 3.5 and 3.6 are equivalent. That is, if a set is measurable according to one definition, it is measurable according to the other.*

We will work mainly with Caratheodory's condition for measurability, but the second is also very useful. First we want to know what sets are measurable.

Theorem 3.8. *The empty set $\emptyset$ and $\mathbb{R}$ are Lebesgue measurable. If A is measurable, then so is A^c.*

Proof. If A is measurable, then for any set E,

$$\begin{aligned} m^*(E) &= m^*(E \cap A) + m^*(E \cap A^c) \\ &= m^*(E \cap A^c) + m^*(E \cap A) \\ &= m^*(E \cap Y) + m^*(E \cap Y^c), \end{aligned}$$

where $Y = A^c$. So A^c is measurable. Now $\mathbb{R}^c = \emptyset$ and it is clear that $m^*(\emptyset) = 0 = m_*(\emptyset)$, so $\emptyset$ is measurable and hence $\mathbb{R}$ is measurable. □

Theorem 3.9. *Intervals are Lebesgue measurable.*

Proof. If I is an interval then $m^*(I) = l(I) = m_*(I)$ and so I is measurable. □

Using these basic facts we can produce many other measurable sets if we can show that unions and intersections of measurable sets are also measurable. This is our next result.

Theorem 3.10. *If A and B are measurable then $A \cup B$ and $A \cap B$ are also measurable.*

Proof. A is measurable and so for any subset E of $\mathbb{R}$ we can write

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap B \cap A^c). \end{aligned} \tag{3.4}$$

Now we also know that B is measurable. So we can write

$$\begin{aligned} m^*(E) &= m^*(E \cap A) + m^*(E \cap A^c) \\ &= m^*(E \cap A) + [m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) \cap B^c)] \\ &= [m^*(E \cap A) + m^*(E \cap B \cap A^c)] + m^*(E \cap (A \cup B)^c) \\ &= m^*(E \cap (A \cup B)) + m^*(E \cap (E \cup B)^c). \end{aligned}$$

We used the deMorgan law $(A \cup B)^c = A^c \cap B^c$ and (3.4) in the last step. Thus $A \cup B$ is measurable. Now A^c and B^c are also measurable, hence $A^c \cup B^c$ is measurable and thus $A \cap B = (A^c \cup B^c)^c$ is measurable. □

It is clear by induction that for any finite collection of measurable sets $A_1, ..., A_N$, the union $\bigcup_{k=1}^{N} A_k$ is measurable.

It turns out that we can extend the previous result to countable unions of measurable sets. In order to establish this, we need a preliminary result, the proof of which is another exercise in induction.

Theorem 3.11. *Let $A_1, ..., A_N$ be pairwise disjoint measurable sets. If E is a subset of $\mathbb{R}$, then*

$$m^*\left(E \cap \left(\bigcup_{k=1}^{N} A_k\right)\right) = \sum_{k=1}^{N} m^*(E \cap A_k). \tag{3.5}$$

Now we come to the major result.

Theorem 3.12. *Suppose that $\{A_n\}_{n=1}^{\infty}$ are pairwise disjoint measurable sets. Then $\bigcup_{n=1}^{\infty} A_n$ is measurable and*

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{k=1}^{\infty} m^*(A_n). \tag{3.6}$$

Proof. Let $B_N = \bigcup_{k=1}^{N} A_k$ and let $B = \bigcup_{k=1}^{\infty} A_k$. We know that B_N is measurable, so for any set $E \subset \mathbb{R}$

$$\begin{aligned} m^*(E) &= m^*(E \cap B_N) + m^*(E \cap B_N^c) \\ &= \sum_{n=1}^{N} m^*(E \cap A_n) + m^*(E \cap B_N^c) \end{aligned}$$

by (3.5). Now $B^c \subset B_N^c$ so we may take the limit as $N \to \infty$ to get

$$\begin{aligned} m^*(E) &\geq \sum_{n=1}^{\infty} m^*(E \cap A_n) + m^*(E \cap B^c) \\ &\geq m^*(E \cap B) + m^*(E \cap B^c), \end{aligned}$$

since $E \cap B = \bigcup_{n=1}^{\infty} E \cap A_n$. We know that outer measure is always sub-additive, so $m^*(E) \leq m^*(E \cap B) + m^*(E \cap B^c)$. Because both inequalities hold we have

$$m^*(E) = m^*(E \cap B) + m^*(E \cap B^c). \tag{3.7}$$

Hence B is measurable. Now let $E = B$ in the relation

$$m^*(E) = \sum_{n=1}^{\infty} m^*(E \cap A_n) + m^*(E \cap B^c). \tag{3.8}$$

This gives

$$\begin{aligned} m^*(B) &= \sum_{n=1}^{\infty} m^*(B \cap A_n) + m^*(B \cap B^c) \\ &= \sum_{n=1}^{\infty} m^*(A_n) + m^*(\emptyset) \end{aligned}$$

and the result follows since $m^*(\emptyset) = 0$.

□

So we see that the Caratheodory condition gives us what we need. If we restrict our attention to sets satisfying this condition, then we get countable additivity. In the previous result, for measurability, we do not need the sets to be disjoint. The next result is an exercise.

Theorem 3.13. *Let $\{A_n\}_{n=1}^{\infty}$ be any collection of measurable sets. Then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are measurable.*

Henceforth we restrict our attention to sets which are measurable according to Caratheodory's condition, (or equivalently Lebesgue's definition).

Definition 3.14. Let A be a Lebesgue measurable set. Then the Lebesgue measure of A is $m(A) = m^*(A)$. Equivalently $m(A) = m_*(A)$.

The moral of all this is it if we restrict our attention to sets satisfying Caratheodory's condition, then the outer measure is a measure. Are there sets which do not satisfy Caratheodory's condition? The answer we will see is yes.

Lebesgue measure has many useful properties. We conclude this section by grouping some of them together. Some we have already proved, other are exercises or can be found in the standard references.

Theorem 3.15 (Properties of Lebesgue Measure)**.** *The following facts hold for Lebesgue measure.*

(i) *Complements, countable unions and countable intersections of measurable sets are measurable.*

(ii) *Any interval is measurable and its measure is its length.*

(iii) *Suppose that $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are sequences of measurable sets such that $A \supset A_1 \supset A_2 \supset A_3 \supset \cdots$ and $B_1 \subset B_2 \subset B_3 \subset$*

$\cdots$, and $m(A) < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

and

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n).$$

(iv) *Translation invariance. If A is measurable, then $m(A+h) = m(A)$, where $A+h = \{x+h, x \in A\}$.*

(v) *Open and closed subsets of $\mathbb{R}$ are measurable.*

(vi) *If A is measurable, then for any $\epsilon > 0$ there exists a closed set B and an open set C such that $B \subset A \subset C$ and $m(C \cap B^c) < \epsilon$. In particular $m(B) \geq m(A) - \epsilon$ and $m(C) \leq m(A) + \epsilon$. If $m(A)$ is finite, then B can be taken to be bounded.*

4. Measure Theory II

We have established that if we restrict Lebesgue outer measure to the sets satisfying the Caratheodory condition, then the desired properties of a measure hold for the Lebesgue outer measure applied to these sets. The Lebesgue measurable sets on $\mathbb{R}$ form a special kind of set.

Definition 4.1. Let X be a set. A σ-algebra on X is a set S of subsets of X such that

(i) $\emptyset \in S$

(ii) $A \in S$ implies $A^c \in S$

(iii) $A_n \in S$ for all $n \in \mathbb{N}$ implies $\cup_{n=1}^{\infty} A_n \in S$.

The next result follows immediately from our previous considerations.

Theorem 4.2. *The Lebesgue measurable sets form a σ-alegbra.*

In the theory of integration we work with measure spaces.

Definition 4.3. We will define the following spaces.

(i) A measurable space is an ordered pair (X, S) where X is a set and S is a σ-algebra on X.

(ii) A measure space is a triple (X, S, μ) where X, S are as in (i) and μ is a measure on S.

Example 4.1. If $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets, m is Lebesgue measure then $(\mathbb{R}, \mathcal{L}, m)$ is a measure space.

One of the most important σ algebras is the one generated by open sets.

Definition 4.4. We define open and closed sets as follows.

(i) Let $A \subset \mathbb{R}$. We say that A is open if given any $x_0 \in A$ may find $\epsilon > 0$ such that the ball
$$B_\epsilon(x_0) = \{x \in \mathbb{R} : |x_0 - x| < \epsilon\} \subset A.$$

(ii) A point $x \in A$ is said to be a limit point of A if there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$.

(iii) A set A is closed if it contains all its limit points.

Example 4.2. The interval $[0, 1]$ is closed, whereas $(0, 1)$ is open. Given $x \in (0, 1)$ we can choose $\epsilon = \frac{1}{2}\min(x, 1 - x)$ and then it is clear that $B_\epsilon(x) \subset (0, 1)$. Obviously 0,1 are limit points of $(0, 1)$, so $[0, 1]$ is closed.

A fundamental result is that

Theorem 4.5. *A subset of* $\mathbb{R}$ *is open if and only if its compliment is closed.*

Open sets are fundamental to analysis. Now suppose that we consider the set of all open sets in $\mathbb{R}$. Then we form the collection

$$\mathfrak{B} = \{A \subset \mathbb{R} : A = (\cup_{n=1}^{\infty} A_n) \cup (\cup_{n=1}^{\infty} B_n^c)\}$$

where the A_n, B_n are open. In other words, $\mathfrak{B}$ is formed by taking countable unions and compliments of open sets.

Definition 4.6. The set $\mathfrak{B}$ is called the Borel σ-algebra. Elements of $\mathfrak{B}$ are called Borel sets.

It is easy to show the following.

Theorem 4.7. *Every Borel set is Lebesgue measurable.*

The converse of this result is false. Not every Lebesgue measurable set is a Borel set, though this is much harder to prove. It involves actually constructing a Lebesgue set that is not a Borel set.

Sets whose measure is zero are commonly encountered. It is clear that every countable set has Lebesgue measure zero. (Why?) The converse is once again false. There are uncountable sets with measure zero.

Theorem 4.8 (Cantor)**.** *There exists a subset* C *of* $[0, 1]$ *which is measurable, uncountable and* $m(C) = 0$.

Note: Cantor did not state this result in terms of measures. He showed that the total length of the sets removed from $[0, 1]$ to make the Cantor set is one. This obviously implies the result as we have stated it.

We have said that not every subset of $\mathbb{R}$ is measurable. That is, that not every subset of the reals satisfies Caratheodory's condition. In 1905 Giuseppe Vitali constructed a non-measurable set by invoking the axiom of choice. Zermelo introduced the axiom of choice in order to prove a result that had eluded Cantor when he established set theory in the 19th century. The result Zermelo was looking to establish was the so called 'well ordering principle' that any set of real numbers can be well ordered. Cantor thought this to be obvious. Many mathematicians today think it is obviously false! It actually turned out that the axiom of choice -given any collection of non-empty sets it is possible to choose an element from each one- is *logically equivalent* to the well ordering principle. There is another result called Zorn's lemma which is also logically equivalent to the axiom of choice. This led one mathematician to observe 'the axiom of choice is obviously true. The well ordering principle is obviously false and Zorn's lemma, who knows?'

The point is that the axiom of choice is controversial and many do not like what it implies. Yet the proof of the Hahn-Banach theorem, Tychonoff's Theorem,[1] the existence of a basis for a vector space and many other major results rely on it.

Here is Vitali's example. It should be clear where the axiom of choice is used.

Theorem 4.9. *There exists a nonmeasurable subset of* $\mathbb{R}$.

Proof. Define the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$. Now partition $\mathbb{R}$ into disjoint non-empty sets E_α with $x, y \in E_\alpha$, for some α if and only if $x \sim y$. Here $\alpha \in I$, for some index set I.

Now $x - [x] \sim x$ and $x - [x] \in [0, 1]$. Here $[x]$ is the largest integer less than or equal to x. Now construct a set A by choosing one element from $E_\alpha \cap [0, 1]$ for each α. Let $\{x_n\}$ be any enumeration of the rationals in $(-1, 1)$. Define the sequence of sets $A_n = A + x_n$. Clearly $A \subset (0, 1)$. The sets A_n are pairwise disjoint. To prove this, observe that if $x \in A_m \cap A_n$ then for some points x', x'' in A, $x = x' + x_m = x'' + x_n$, so $x' - x'' = x_m - x_n \in \mathbb{Q}$. So $x' \sim x''$, so $x_m = x_n$ and $n = m$. Now

$$(0, 1) \subset \bigcup_{n=1}^{\infty} A_n \subset (-1, 2). \tag{4.1}$$

In fact, by construction, for each $x \in (0, 1)$, there is a unique $x' \in A \subset [0, 1)$ such that $x - x' \in \mathbb{Q}$. Then $x - x' = x_n$ for some n and so $x \in A_n$. The other conclusion is obvious since $A \subset [0, 1]$ and $|x_n| < 1$ for all n.

We now prove that A is not measurable. Suppose that it is. Then from (4.1) and elementary properties of Lebesgue measure,

$$1 = m((0, 1)) \leq m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n) \leq 3 = m((-1, 2)). \tag{4.2}$$

But $m(A_n) = m(A + x_n) = m(A)$. So we have $1 \leq m(A) + m(A) + m(A) + \cdots \leq 3$. This is impossible. Either $m(A) = 0$, or $m(A) > 0$ and both violate the previous inequality. Consequently, $m(A)$ does not exist and A is not a measurable set. □

The existence of non-measurable sets is a problem that many mathematicians want to banish, since it leads to paradoxes. It also violates our intuition. Imagine that we have a non-measurable set and we throw darts at it, where the darts are of course line segments. The probability of landing in the set should be the measure of the set. Yet the measure does not exist. So does that mean that the probability does not exist? Our intuition seems to suggest a paradox. These sorts of questions have lead many mathematicians to conclude that something is not right here. There is an even stranger consequence however.

The *Banach-Tarski paradox* says that it is possible to decompose the unit sphere into five segments, then rotate and translate (but not

[1]This is a famous result in topology: The product of any collection of compact topological spaces is compact. See any standard textbook on toplogy for the details.

dilate) the segments into two spheres each identical to the original.[2] Hausdorff earlier showed a similar result is true in the plane. The point is that we cut the sphere into non-measurable sets.

In the construction of a non-measurable set, we used the Axiom of Choice, and some would like to abandon this, but there is considerable argument about this. Solovay showed in 1970 that non-measurable sets exist only if we accept the axiom of choice in the case where we can make a choice from an uncountable collection of sets. If we only allow choice for countable collections, then all sets are measurable. However the axiom of choice in general form is so useful, that most mathematicians just accept that there really are non-measurable sets. We cannot actually turn one sphere into two. The theorem tells us that the sets which are needed to transform one sphere into two exist. It gives no method for actually constructing them.

The general view seems to be that the axiom of choice is fine, as long as we understand that the predictions of mathematics do not always have to correspond to our physical reality. Mathematics exists in and of itself. The fact that it is useful for understanding the world should not blind us to the fact that we can easily imagine things which are mathematically possible, but physically impossible.

4.1. **Measurable Functions.** In order to construct the integral, we need to consider the types of functions which will be used. We start with a definition.

Definition 4.10. Let $f : X \to Y$. Then

$$f^{-1}(A) = \{x \in X : f(x) \in A \subseteq Y\}.$$

We call $f^{-1}(A)$ the inverse image of A under f.

We give an alternative definition of continuity using this.

Definition 4.11. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is continuous if $f^{-1}(A)$ is open, whenever A is open.

It is not hard to show that this is equivalent to the usual definition of continuity in terms of sequences: f is continuous at x if $x_n \to x$ implies $f(x_n) \to f(x)$. The new definition of continuity is easy to adapt to other settings, since we can define open sets by fiat. (This is the subject of topology)

Next we introduce the characteristic function.

Definition 4.12. The characteristic function of a set E is

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases} \tag{4.3}$$

[2]This led students at UNSW in 1987 to propose developing the Banach potato which could be infinitely reproduced from a single speciman and used to solve world hunger. Sadly no funding was forthcoming.

A simple function is one of the form $f = \sum_{i=1}^{n} a_i \chi_{E_i}$.

In order to define the Lebesgue integral, we require the sets E_i to be measurable. This leads to the definition of a measurable function.

Definition 4.13. A real valued function f is measurable if $f^{-1}(A)$ is measurable whenever A is open.

Note we do not require A to be measurable in this definition. Open sets are measurable, but requiring measurability of A would lead to an unnecessarily restrictive class of functions.

It is immediate that continuous functions are measurable.

Theorem 4.14. *Every continuous function is measurable.*

Proof. If f is continuous then $f^{-1}(A)$ is open whenever A is open. But open sets are measurable. So $f^{-1}(A)$ is measurable whenever A is open. □

Various operations on functions preserve measurability. To establish the necessary results we require a more convenient characterisation of measurability. The proof is an exercise.

Proposition 4.15. *f is measurable if any of the following hold:*

(i) *$f^{-1}(A)$ is measurable whenever A is closed.*

(ii) *$f^{-1}(A)$ is measurable whenever A is an interval.*

(iii) *$f^{-1}(A)$ is measurable whenever A is an open interval*

(iv) *$f^{-1}(A)$ is measurable whenever A is an interval of the form (a, ∞) or $(-\infty, a)$.*

Now we introduce operations on two functions.

Definition 4.16. If f and g are real valued functions then

$$(f \wedge g)(x) = \min(f(x), g(x)) \tag{4.4}$$

$$(f \vee g)(x) = \max(f(x), g(x)). \tag{4.5}$$

Proposition 4.17. *If c is real and f, g are measurable, then cf, $f+g$, fg, $|f|$, $f \wedge g$ and $f \vee g$ are all measurable.*

Proof. We introduce the notation $\{f > a\} = \{x \in \mathbb{R} : f(x) > a\} = f^{-1}((a, \infty))$. Since f is measurable $\{f > a\}$ is measurable. If $c > 0$ then $\{cf > a\} = \{f > a/c\}$ and similarly for $c \leq 0$ so cf is measurable. Next $\{|f| > a\} = \{f > a\} \cup \{f < -a\}$ which is the union of measurable sets, so $|f|$ is measurable. Then $\{f \wedge g > a\} = \{f > a\} \cap \{g > a\}$ and $\{f \vee g > a\} = \{f > a\} \cup \{g > a\}$ and these are measurable.

If $f(x) > r$ and $g(x) > a - r$ then $f(x) + g(x) > a$. Conversely, if $(f+g)(x) > a$, then there is a rational r such that $f(x) > r > a - g(x)$. So that we can write

$$\{f + g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > r\} \cap \{g > a - r\}$$

and this is measurable.

Next $\{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}$, so f^2 is measurable. And finally $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$ and so fg is measurable. $\square$

In the next section we turn to the construction of the Lebesgue integral.

5. The Lebesgue Integral

We saw in the previous section that measurability is preserved when we perform certain operations on functions. It will be important to consider the question of what happens when we take limits of sequences of measurable functions. It turns out that pointwise limits are measurable.

Theorem 5.1. *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions converging pointwise to f. Then f is measurable.*

Proof. Suppose that $f(x) > a$. If $f(x) = b$, then $b - a > 0$ and we can find m such that $2/m < (b - a)$. So there exists $m \in \mathbb{N}$ such that $f(x) > a + 2/m$. Now $f_n(x) \to f(x)$ so given m we can find $N \in \mathbb{N}$ so that for all $n \geq N$, $|f_n(x) - f(x)| < 1/(m)$. Which implies $f_n(x) > f(x) - 1/(m)$. Hence there is an N such that $f_n(x) > a + 1/m$ for all $n \geq N$. So

$$\{f > a\} \subset \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n > a + 1/m\}. \tag{5.1}$$

The right is formed from countable unions and intersections of measurable sets and so is measurable. Now if x belongs to the set on the right, then for some m and some N we have $f_n(x) > a + 1/m$ for all $n \geq N$. So in the limit $f(x) \geq a + 1/m$. Therefore

$$\bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n > a + 1/m\} \subset \{f > a\}. \tag{5.2}$$

So the two sets are equal and hence f is measurable. □

It is convenient to work on the extended real numbers, which we denote by $\mathbb{R}^*$. This is simply the real numbers with $\pm\infty$ included. For the most part nothing changes when we work on the extended reals, though we have to be careful about sums and products, since quantities like $\infty - \infty$ is not defined. Measurability can be recast as: f is measurable if $f^{-1}(A)$ is measurable for every set $A = (a, \infty]$.

Sequences of measurable functions have other nice properties.

Theorem 5.2. *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Let*

$$g(x) = \inf_n \{f_n(x)\}, \ h(x) = \sup_n \{f_n(x)\}. \tag{5.3}$$

Then g, h are measurable.

Proof. For any a

$$\{g < a\} = \bigcup_{n=1}^{\infty} \{f_n < a\}, \ \{h > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}.$$

These are countable unions of measurable sets and hence measurable. □

An interesting question is how close pointwise convergence is to uniform convergence? Egoroff's Theorem gives and answer. Before stating the result we need a definition.

Definition 5.3. Suppose that $f = g$ for all $x \notin E$ where $m(E) = 0$. Then we say that $f = g$ almost everywhere, or $f = g$ a.e. In general a property of a function that holds except on a set of measure zero is said to hold almost everywhere.

Theorem 5.4 (Egoroff). *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that converges to a real valued function f almost everywhere on the interval $[a, b]$. Then for any $\delta > 0$ there is a measurable subset E of $[a, b]$ such that $m(E) < \delta$ and the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on $[a, b] - E$.*

The proof of this is an exercise, and quite a hard one. The point of the theorem is that we can make convergence uniform on "most" of the interval $[a, b]$. Think about convergence. We have a sequence $f_n \to f$ on $[a, b]$. This means that we can make $|f_n(x) - f(x)| < \epsilon$ by choosing n large enough. The problem is that we will need in general a different n for each x. If the convergence is uniform, then we can use the same n for every x. What Egoroff's Theorem tells us is that the same n will do for all $x \in [a, b] - E$. However the smaller ϵ is, the larger n will have to be to compensate. If the convergence is not uniform on the whole of $[a, b]$, then as we let $\epsilon \to 0$, $n \to \infty$. Estimates of how n grows as E shrinks exist, but we will not discuss this issue. We will use Egoroff's Theorem later.

Simple functions lie at the heart of the theory of integration. Recall that a simple function is one of the form

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where the sets A_i are measurable and pairwise disjoint. We first introduce the integral of a simple function and then extend it to more general functions. The key fact is that we can approximate measurable functions by simple functions as the next result shows.

Theorem 5.5. *For any non-negative measurable function f defined on a measurable set E, there exists a sequence of simple functions $\{\phi_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \phi_n = f$ on E. If f is bounded on E, then $\lim_{n\to\infty} \phi_n = f$ uniformly on E. If f is non-negative, the sequence $\{\phi_n\}_{n=1}^{\infty}$ may be constructed so that it is monotonically increasing.*

Proof. For each $n \geq 1$ and for each $x \in E$ let

$$\phi_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \\ n & \text{if } f(x) \geq n. \end{cases} \tag{5.4}$$

Then the ϕ_n are non-negative functions and $\phi_{n+1}(x) \geq \phi_n(x)$. At a point where $f(x) < \infty$ we have

$$0 \leq f(x) - \phi_n(x) < \frac{1}{2^n}$$

if $f(x) < n$. At a point where $f(x) = \infty$ we have $\phi_n(x) = n$. This shows that for any point x, $\phi_n(x) \to f(x)$ as $n \to \infty$. Since

$$|f(x) - \phi_n(x)| \leq \frac{1}{2^n} \tag{5.5}$$

the convergence is independent of x and so is uniform. □

We define the Lebesgue integral for simple functions first.

Definition 5.6. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ be a simple function. Then the Lebesgue integral of f is

$$\int f = \sum_{i=1}^n a_i m(A_i),$$

where m is Lebesgue measure. If $\int f < \infty$ we say that f is an integrable simple function, or ISF.

Example 5.1. Let $f = \chi_{\mathbb{Q}}$. Then $\int f = m(\mathbb{Q}) = 0$.

Proposition 5.7. *Let f and g be ISFs. The following properties hold.*

(i) *If c is a constant then cf is an ISF and $\int cf = c \int f$.*

(ii) *$f + g$ is an ISF and $\int (f+g) = \int f + \int g$.*

(iii) $|\int f| \leq \int |f|$.

(iv) *If $f \leq g$, then $\int f \leq \int g$.*

(v) *If $f_a(x) = f(x-a)$ then f_a is an ISF and $\int f_a = \int f$.*

Proof. We will only prove additivity. The other properties are exercises. Suppose that $f = a\chi_A$ and $g = b\chi_B$. Then

$$f + g = a\chi_{(A-(A\cap B))} + b\chi_{(B-(B\cap A))} + (a+b)\chi_{A\cap B}.$$

Hence

$$\begin{aligned}\int (f+g) &= am(A-(A\cap B)) + bm(B-(B\cap A)) + (a+b)m(A\cap B)\\ &= a\,(m(A-(A\cap B)) + m(A\cap B)) + b(m(B-(B\cap A))\\ &\quad + m(A\cap B)) = am(A) + bm(B)\\ &= \int f + \int g.\end{aligned}$$

We made use of the decomposition $(A-(A\cap B))\bigcup(A\cap B) = A$, which implies that $m(A) = m(A-(A\cap B)) + m(A\cap B)$. Similarly for $m(B)$.

Now let $f = \sum_{i=1}^n a_i\chi_{A_i}$ where the A_is are disjoint and $g = b_1\chi_{B_1}$. Then

$$f+g = \sum_{i=1}^n \left(a_i\chi_{(A_i-(A_i\cap B))} + (a_i+b_1)\chi_{A_i\cap B_1} + b_1\chi_{(B_1-\cup A_i)}\right).$$

From this we have

$$\begin{aligned}\int (f+g) &= \sum_{i=1}^n [a_i m(A_i-(A_i\cap B_1)) + (a_i+b_1)m(A_i\cap B_1)\\ &\qquad + bm(B_1-\cup A_i)]\\ &= \sum_{i=1}^n a_i[m(A_i-(A_i\cap B_1)) + m(A_i\cap B_1)]\\ &\quad + b_1\sum_{i=1}^n [m(A_i\cap B_1) + m(B_1-\cup A_i)]\\ &= \sum_{i=1}^n a_i m(A_i) + b_1 m(B_1)\\ &= \int f + \int g.\end{aligned}$$

Observe that $f+g$ is itself a simple function, which we call h. Now let $k = b_2\chi_{B_2}$ consider $f+g+k = h+k$. Then by our previous argument

$$\begin{aligned}\int (f+g+k) = \int (h+k) = \int h + \int k = \int f + \int g + \int k\\ = \sum_i^n a_i m(A_i) + \sum_{j=1}^2 b_j m(B_j).\end{aligned}$$

By induction it follows that if $g = \sum_{j=1}^{m} b_j \chi_{B_j}$ with the B_js disjoint, then

$$\int (f+g) = \sum_{i=1}^{n} a_i m(A_i) + \sum_{j=1}^{m} b_j m(B_j)$$
$$= \int f + \int g,$$

so the integral is linear.

□

To define the Lebesgue integral in general, we consider the positive case first.

Definition 5.8. If $f : \mathbb{R} \to [0, \infty)$ is a measurable function, then

$$\int f = \sup \left\{ \int \phi : 0 \le \phi \le f, \phi \text{ is an ISF} \right\}. \tag{5.6}$$

Note that this allows $\int f = \infty$. We will have to exclude this possibility when we extend the integral to arbitrary functions.

Theorem 5.9. *If f and g are non-negative and measurable, then*

(i) $\int af = a \int f$

(ii) $\int (f+g) = \int f + \int g$.

(iii) *If $f \le g$, then $\int f \le \int g$.*

The proof requires a lemma.

Lemma 5.10. *Suppose that f is a bounded, measurable function and A is a set with $m(A) < \infty$. For any $\epsilon > 0$, there are ISFs f_1, f_2 such that $f_1 \le f \le f_2$ on A and $f_2 - f_1 < \epsilon$.*

Proof. Choose M so that $|f(x)| \le M$ for all x. Then decompose $[-M, M]$ into n disjoint intervals $I_1, ..., I_n$ each of length less than ϵ. So

$$[-M, M] = \bigcup_{k=1}^{n} I_k, \; I_k \cap I_j = \emptyset, \; k \ne j, I_k = [a_k, b_k).$$

Let $A_k = A \cap f^{-1}(I_k)$ and let $f_1 = \sum_{k=1}^{n} a_k \chi_{A_k}$ and $f_2 = \sum_{k=1}^{n} b_k \chi_{A_k}$. Then the result follows. □

Now we prove Theorem 5.9.

Proof. Parts (i) and (iii) are trivial. So we only prove additivity. Suppose that f and g are non-negative measurable and f_1, g_1 are ISFs such that $0 \le f_1 \le f$ and $0 \le g_1 \le g$. Then $f_1 + g_1 \le f + g$. So

$$\int f_1 + \int g_1 \le \int (f+g).$$

Taking supremums of both sides gives

$$\int f + \int g \leq \int (f+g).$$

Now we prove the reverse inequality.

Suppose that h is an ISF and $0 \leq h \leq f+g$. Let $A = \{x : h(x) > 0\}$. Then $m(A) < \infty$ since h is an ISF, so its support must have finite measure. Now h is bounded, so $h \wedge f$ and $h \wedge g$ are bounded. We choose ISFs f_1 and g_1 such that $0 \leq f_1 \leq h \wedge f \leq f_1 + \epsilon$ and $0 \leq g_1 \leq h \wedge g \leq g_1 + \epsilon$. Now $h \leq f+g$, so

$$h \leq h \wedge f + h \wedge g \leq f_1 + g_1 + 2\epsilon \chi_A.$$

Now

$$\int h \leq \int f_1 + \int g_1 + 2\epsilon m(A) \leq \int f + \int g + 2\epsilon m(A).$$

Taking first the infimum over ϵ and then the supremum over $h \leq f+g$ we find that $\int (f+g) \leq \int f + \int g$ and the theorem is proved. □

We use additivity to extend to the general case.

Definition 5.11. Let f be any real valued or extended real valued function. We define $f^+(x) = f(x)$ if $f(x) > 0$ and $f^+(x) = 0$ if $f(x) < 0$. Conversely, $f^-(x) = -f(x)$ if $f(x) < 0$ and $f^-(x) = 0$ if $f(x) > 0$.

Lemma 5.12. *If f is measurable, then f^+ and f^- are measurable. Any function f can be written $f = f^+ - f^-$. Further $|f| = f^+ + f^-$.*

We extend the Lebesgue integral to arbitrary functions.

Definition 5.13. A measurable function is Lebesgue integrable if $\int |f| < \infty$. If f is Lebesgue integrable, then we define

$$\int f = \int f^+ - \int f^-.$$

This definition requires $\int f^+ < \infty$ and $\int f^- < \infty$, since $\infty - \infty$ is not defined. If we restrict our attention to positive functions then we can allow the Lebesgue integral of a function to be infinite. However for the more general case we must insist on finiteness, because otherwise the integral will not be well defined.

Theorem 5.14. *Suppose that f, g are Lebesgue integrable and a is real. Then $f+g$ is Lebesgue integrable and (i)$\int af = a\int f$, (ii) $\int (f+g) = \int f + \int g$. (iii) $|\int f| \leq \int |f|$. Further if $f \leq g$ then $\int f \leq \int g$. Finally, $\int f_a = \int f$.*

Proof. Observe that $|f+g| \leq |f| + |g|$. So $f+g$ is integrable. Next

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-).$$

Therefore $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. So

$$\int (f+g)^+ + \int f^- + \int g^- = \int (f+g)^- + \int f^+ + \int g^+$$

Thus

$$\int (f+g)^+ - \int (f+g)^- = \int f^+ - \int f^- + \int g^+ - \int g^-.$$

Hence $\int (f+g) = \int f + \int g$.

Next, $f \leq g$ implies $g - f \geq 0$ so $\int g - \int f = \int (g-f) \geq 0$. Property (iii) follows from $-|f| \leq f \leq |f|$. And the final result follows from the corresponding result for simple functions. □

Finally we extend the Lebesgue integral to arbitrary sets.

Definition 5.15. Let f be Lebesgue measurable. Then for $A \subset \mathbb{R}$ we have $\int_A f = \int f\chi_A$.

It is easy to establish the following very useful results.

Theorem 5.16. *If f is measurable and bounded almost everywhere on A, with $m(A) < \infty$, then it is Lebesgue integrable on A. If g, h are Lebesgue integrable, f is Lebesgue measurable and $g \leq f \leq h$, then f is Lebesgue integrable.*

Proposition 5.17. *If $A = B \cup C$, where B, C are disjoint, then $\int_A f = \int_B f + \int_C f$.*

Proof. Since $A = B \cup C$, $B \cap C = \emptyset$, then $\chi_A = \chi_{B\cup C}$. Hence

$$\chi_A(x) = \begin{cases} 1 & x \in B \\ 1 & x \in C \\ 0 & x \notin B \cup C. \end{cases} \tag{5.7}$$

From which we have $\chi_A = \chi_B + \chi_C$. Then

$$\begin{aligned} \int_A f &= \int \chi_A f \\ &= \int (\chi_B + \chi_C) f \\ &= \int \chi_B f + \int \chi_C f \\ &= \int_B f + \int_C f. \end{aligned}$$

□

Before we introduce the convergence theorems which lie at the heart of Lebesgue's theory of integration, we should briefly discuss the behaviour of integrable functions at infinity. It is clearly not true that a function which is integrable must satisfy $\lim_{|x|\to\infty} |f(x)| = 0$. As a simple counter example, consider the function $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$. Then $\int |f| = 0$, since $\mathbb{Q}$ has measure zero. Thus f is integrable, but $\lim_{|x|\to\infty} |f(x)| \neq 0$.

However, suppose that there is an $\epsilon > 0$ and an $X > 0$ such that $|f(x)| \geq \epsilon$ for all $x > X$. It is then clear that

$$\int_{-\infty}^{\infty} |f(x)| dx \geq \int_{X}^{\infty} |f(x)| dx \geq \int_{X}^{\infty} \epsilon dx = \infty.$$

Thus, given $\epsilon > 0$ the set of points for which $|f| > \epsilon$, must have finite measure. That is, if f is integrable, then given $\epsilon > 0$, there must be an X such that $|f(x)| < \epsilon$ for all $x > X$, except possibly for a set of measure zero.

When we evaluate Fourier transforms in what follows, we will typically be dealing with smooth, integrable functions and it will be safe to assume that the function decays to zero at infinity. But for functions which are not continuous, some caution may be needed. Typically we will explicitly state that we are assuming that the functions we work with satisfy $\lim_{|x|\to\infty} |f(x)| = 0$.

6. The Convergence Theorems and Differentiation

One of the great strengths of the Lebesgue integral is that it has stronger convergence properties than the Riemann integral. Recall that in order to interchange the order of integration and a limit, we require uniform convergence, if we use the Riemann integral. This is very restrictive, since establishing uniform convergence can be very difficult, and we often do not have it all. For the Lebesgue integral the situation is much better. We only need to be able to bound the sequence by a constant if we are integrating over a finite set. This make the Lebesgue integral far more suited to problems in analysis than the Riemann integral.

In this section we detail the main results. The most important is the Dominated convergence Theorem. We begin with a Lemma.

Lemma 6.1. *Suppose that g is a non-negative ISF, and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that*

$$g \geq f_1 \geq f_2 \geq f_n \cdots \geq 0 \tag{6.1}$$

and $\lim_{n\to\infty} f_n(x) = 0$ *a.e. Then*

$$\lim_{n\to\infty} \int f_n = 0. \tag{6.2}$$

Proof. Since g is an ISF, the set $A = \{x : g(x) > 0\}$ has finite measure and g is bounded. Say $0 \leq g(x) \leq M$ for all x. Given any $\epsilon > 0$, let $A_n = \{x : f_n(x) > \epsilon\}$. Our assumptions imply that

$$A \supset A_1 \supset A_2 \supset \cdots$$

and $\cap_{n=1}^{\infty} A_n = \emptyset$. Therefore, by Theorem 3.15 $\lim_{n\to\infty} m(A_n) = 0$. Choose N so that $m(A_n) < \epsilon$ for all $n \geq N$. Then $n \geq N$ implies $f_n = 0$ on A^c and $f_n \leq \epsilon$ on $A \cap A_n^c$. Also, $f_n \leq M$. So

$$\begin{aligned} 0 \leq \int f_n &= \int_{A_n} f_n + \int_{A\cap A_n^c} f_n \\ &\leq Mm(A_n) + \epsilon m(A \cap A_n^c) \\ &< \epsilon(M + m(A)), \end{aligned}$$

and the result follows. □

We use this result to prove

Theorem 6.2 (Lebesgue's Dominated Convergence Theorem)**.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, such that for almost all real x, $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise. Suppose further that there exists a Lebesgue integrable function g such that $|f_n(x)| \leq g(x)$ for all x. Then*

$$\lim_{n\to\infty} \int f_n = \int f. \tag{6.3}$$

Proof. We consider three cases.
Case 1

Suppose that $f = 0$ and $\{f_n\}$ is a non-increasing sequence. Given $\epsilon > 0$ choose an ISF with $0 \leq g_1 \leq g$ and $\int g \leq \int g_1 + \epsilon$. Now we can write

$$f_n = f_n \wedge g_1 + [f_n - f_n \wedge g_1] \leq f_n \wedge g_1 + (g - g_1). \tag{6.4}$$

The functions $f_n \wedge g_1$ satisfy the conditions of the previous lemma, since they are dominated by the ISF g_1. Thus for n sufficiently large, $\int f_n \wedge g_1 < \epsilon$, hence

$$\int f_n \leq \int f_n \wedge g_1 + \int (g - g_1) < \epsilon + \epsilon \tag{6.5}$$

and so $\int f_n \to 0$.
Case 2

Let $f = 0$ and $f_n \geq 0$ for all n. Let

$$g = \sup\{f_n, f_{n+1}, f_{n+2}, ...\}.$$

Then the g_n's decrease to 0 and Case 1 applies to the g_n. So we have

$$0 \leq \int f_n \leq \int g_n \to 0. \tag{6.6}$$

Case 3

From the preceding, we establish the general case. Let $g_n = |f_n - f|$. Then g_n is non-negative, measurable and converges to 0 pointwise almost everywhere. Further $g_n \leq |f_n| + |f| \leq 2g$. We thus apply case 2 to the g_n. Hence

$$0 \leq \left| \int f_n - \int f \right| \leq \int |f_n - f| \leq \int g_n \to 0. \tag{6.7}$$

□

This is the most important result in the theory of integration. It is one of the most important tools of modern analysis. Swapping limits and integrals is so common. that much of modern analysis would be impossible without this result. We present an immediate consequence.

Theorem 6.3 (Monotone Convergence theorem)**.** *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a nondecreasing sequence of measurable functions with* $0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots$ *. Let* $f_n \to f$ *pointwise. Then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Clearly f is measurable, since it is the limit of a sequence of measurable functions. Let h be an ISF such that $0 \leq h \leq f$, then the Dominated Convergence Theorem implies that

$$\int h = \lim_{n\to\infty} \int h \wedge f_n \leq \lim_{n\to\infty} \int f_n. \tag{6.8}$$

Taking the supremum over all h's gives $\int f \leq \lim_{n\to\infty} \int f_n$. But it is also clear that $\int f_n \leq f$ and so the reverse inequality also holds. $\square$

Let us consider another way of proving the Dominated Convergence Theorem, or rather a simpler version of it. This makes use of Egoroff's Theorem and illustrates Littlewood's three principles of measure theory.

(1) Measurable sets are *nearly* open sets.

(2) Measurable functions are *nearly* continuous functions.

(3) Pointwise convergence is *nearly* uniform convergence.

We will discuss this further after the next result and its proof. The proof illustrates Littlewood's idea.

Theorem 6.4. *If $\{f_n\}$ is a uniformly bounded sequence of Lebesgue measurable functions converging pointwise almost everywhere to f on $[a, b]$ then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \tag{6.9}$$

Proof. Define f to be zero where $\lim f_n \neq f$. Sets of measure zero do not effect the Lebesgue integral, so it does not matter what the limit is on such a set. Because $|f_n| \leq B$ on $[a, b]$ and limits of measurable functions are measurable, we can conclude that f is measurable and bounded and hence Lebesgue integrable. So $\int_a^b f$ exists.

Let $\epsilon > 0$ be given. By Egoroff's Theorem there is a subset E of $[a, b]$ such that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $[a, b] - E$ and $m(E) < \epsilon$. By the triangle inequality $|f - f_n| \leq 2B$. So

$$\begin{aligned}\left|\int_a^b f_n - \int_a^b f\right| &\leq \int_a^b |f_n - f| \\ &= \int_E |f_n - f| + \int_{[a,b]-E} |f_n - f| \\ &\leq 2Bm(E) + \int_{[a,b]-E} |f_n - f| \\ &< 2B\epsilon + \int_{[a,b]-E} |f_n - f|.\end{aligned}$$

Now on $[a, b] - E$ $|f_n - f| \to 0$ uniformly. So we can choose N such that for $n \geq N, |f_n - f| < \epsilon$ on $[a, b] - E$. So for $n \geq N$ we have

$$\begin{aligned}\left|\int_a^b f_n - \int_a^b f\right| &\leq 2B\epsilon + \epsilon m([a, b] - E) \\ &< 2B\epsilon + \epsilon(b - a).\end{aligned}$$

This holds for all $\epsilon > 0$ so we conclude $|\int_a^b f_n - \int_a^b f| \to 0$. □

The crux of the problem is to be able to swap limits and integrals. We can do this if we have uniform convergence, because we can make $|f_n - f|$ uniformly small over the region of integration. That is, we can choose n large enough so that

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon.$$

The choice of n is independent of x. This is the value of uniform convergence. So we have $\int_a^b |f_n(x) - f(x)| dx \le \epsilon(b - a)$. If we needed a different n to do this for each x, then this argument just doesn't work.

Uniformity is nice. In the proof of the bounded convergence theorem just given, we use the fact that pointwise convergence is nearly uniform convergence, (Egoroff's Theorem says exactly this). Split the region of integration into the part where the convergence is uniform and the part where it is not. The part where we have uniform convergence we don't need to worry about. All we have to do is take care of the small part of the domain where the convergence is not uniform, and the argument above does this.

It is also true that measurable functions are nearly continuous functions. This is the content of the following result.

Theorem 6.5 (Lusin). *Let f be a measurable function on a set E and let $\epsilon > 0$. Then there is a set F such that f is continuous on F and $m(E - F) < \infty$.*

Proof. We sketch the idea for the case $m(E) < \infty$. The case $m(E) = \infty$ can be obtained as a modification of this argument.

We can approximate f by a sequence of simple functions $\{\phi_n\}$. It is possible to prove that given a simple function ϕ_n on a set E there is a continuous function g_n and a closed set $F_n \subseteq E$, such that $\phi_n = g_n$ on F_n and $m(E - F_n) < \epsilon/2^{n+1}$. Egoroff's Theorem tells us that there is a closed set $F_0 \subseteq E$ such that $\phi_n \to f$ uniformly on F_0 and $m(E - F_0) < \epsilon/2$. Let $F = \cap_{n=0}^{\infty} F_n$. Then it is straightforward to show that $m(E - F) < \epsilon$. Now $\phi_n = g_n$ on F_n and since $\phi_n \to f$ uniformly on F, $g_n \to f$ uniformly on F. Now the uniform limit of a sequence of continuous functions is continuous. So f must be continuous on F. □

We now have an explicit statement of what it means to say that measurable functions are nearly continuous. When we deal with measurable functions, we can use continuity arguments on the set where the function is continuous, then worry about what happens on the small (measure less than ϵ) part of the domain where it is not continuous. This approach to measure theoretical problems has proved to be enormously effective over the years.

Another useful result is Fatou's Lemma. This is the strongest statement that can in general be made if we do not have dominated convergence and impose no other conditions.

Theorem 6.6 (Fatou's Lemma). *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative integrable functions, converging pointwise almost everywhere to f on $[a,b]$, then*

$$\int_a^b f \leq \liminf \int_a^b f_n. \tag{6.10}$$

Fatou's lemma is used extensively in applications of the Lebesgue integral. Some proofs of the dominated convergence theorem make use of it. We will not have any further use for it, however.

Next we turn to the relationship between the Lebesgue and Riemann integrals.

Theorem 6.7. *If f is Riemann integrable on $[a,b]$, then f is also Lebesgue integrable on $[a,b]$ and*

$$\mathcal{L}\int_a^b f = \mathcal{R}\int_a^b f. \tag{6.11}$$

Proof. Since f is Riemann integrable on $[a,b]$ it is bounded. Consider the partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of $[a,b[$ and let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i)\}$$

and

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i)\}.$$

Then introduce the simple functions

$$\phi_n = \sum_{i=1}^n m_i \chi_{[x_{i-1},x_i)}$$
$$\psi_n = \sum_{i=1}^n M_i \chi_{[x_{i-1},x_i)},$$

which satisfy $\phi_n \leq f \leq \psi_n$. Since $\phi_n \uparrow f$ and $\psi_n \downarrow f$, then f is measurable and Lebesgue integrable, since it is the limit of sequences of simple functions. Now by the Riemann integrability of f we have that the lower sum

$$L(f,\mathcal{P}) = \sum_{i=1}^n m_i(x_i - x_{i-1})$$
$$= \mathcal{L}\int \phi_n.$$

The sequence $\{\phi_n\}_{n=1}^{\infty}$ is montone increasing, so by the fact that f is Riemann integrable and the monotone convergence theorem

$$\begin{aligned}\mathcal{L}\int_a^b f &= \lim_{n\to\infty}\mathcal{L}\int \phi_n \\ &= \mathcal{L}\int \lim_{n\to\infty}\phi_n \\ &= \mathcal{R}\int_a^b f(x)dx.\end{aligned}$$

□

The result means that we can use standard results about Riemann integrals to evaluate Lebesgue integrals. The converse is of course false. There are many Lebesgue integrable functions that are not Riemann integrable.

However with the Riemann approach, we have the concept of an improper Riemann integral. Specifically, in Riemann's sense, we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{T\to\infty}\int_{-T}^{0} f(x)dx + \lim_{R\to\infty}\int_0^R f(x)dx. \tag{6.12}$$

A consequence of this is that there are functions which are not Lebesgue integrable for which the improper Riemann integral does exist. For example let $f(x) = \dfrac{\sin x}{x}, x \neq 0,\ f(0) = 1$. This function is not Lebesgue integrable on $\mathbb{R}$. But the improper Riemann integral does exist and equals

$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}. \tag{6.13}$$

For f to be Lebesgue integrable, we require $\int |f| < \infty$. Whereas the improper Riemann integral is simply given by

$$\int_0^{\infty}\frac{\sin x}{x}dx = \lim_{R\to\infty}\int_0^R\frac{\sin x}{x}dx. \tag{6.14}$$

Of course the integrals $\int_0^R f(x)dx$ exist in the Lebesgue sense and are equal to the Riemann integral. So it is true that

$$\lim_{R\to\infty} L\int_0^R f(x)dx = \pi/2. \tag{6.15}$$

But the limit is not a Lebesgue integral. We can of course define it to be the *improper Lebesgue integral* of f. That is, if f is Lebesgue integrable on $[0,n]$ for each n and

$$\lim_{n\to\infty}\int f\chi_{[0,n]} = I,\ < \infty. \tag{6.16}$$

Then we can say that the improper Lebesgue integral

$$\text{Imp}\int_{[0,\infty)} f = I.$$

Obviously if $\int_{[0,\infty)} |f| < \infty$, then we must have $\int_{[0,\infty)} f = I$. That is the Lebesgue integral and the improper Lebesgue integral will coincide.

This is perfectly fine. However, we must be careful because in these situations it is not clear that we can use the convergence theorems, or other useful properties of the Lebesgue integral. For example, suppose that the improper Lebesgue integral of f_k on $\mathbb{R}^+$ exists for each $k = 0, 1, 2, ...$ and that $f_k \to f$. Are there conditions under which it is true that

$$\lim_{k\to\infty} \text{Imp}\int_{[0,\infty)} f_k = \text{Imp}\int_{[0,\infty)} f, \ ? \tag{6.17}$$

In fact there are, but we will not discuss them in this subject. One must however be careful to not simply assume that every nice property of the Lebesgue integral can be assumed about the improper version.

Nevertheless, despite this caveat, the Lebesgue approach to the integral is the standard one for modern analysis. And we shall use it for now on. As we shall see, the DCT is a result that is essential to a great deal of modern mathematics.

There is a fundamental theorem of calculus for the Lebesgue integral, which we state without proof. First, a definition.

Definition 6.8. A function f is said to be absolutely continuous on the interval $[a, b]$ if, given any $\epsilon > 0$, we can find a $\delta > 0$ such that for any finite collection of pairwise disjoint intervals $(a_k, b_k) \subset [a, b], k = 1, 2..., n$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$ we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

This is a very strong form of continuity. Even differentiable functions are not necessarily absolutely continuous, though the converse is (essentially) true. Naturally, absolutely continuous functions are also continuous. The following is also true.

Proposition 6.9. *Every absolutely continuous function is uniformly continuous.*

Proof. Just check the definition. □

The Lebesgue integral returns absolutely continuous functions.

Theorem 6.10 (Lebesgue, 1904)**.** *Given that f is Lebesgue integrable on $[a, b]$, define a function F on $[a, b]$ by $F(x) = \int_a^x f(t)dt$. Then F is absolutely continuous on $[a, b]$. Further $F' = f$ almost everywhere on*

$[a,b]$. *Moreover, if* F' *exists and is bounded on* $[a,b]$ *then* F' *is Lebesgue integrable and*

$$\int_a^x F' = F(x) - F(a)$$

for $x \in [a,b]$.

Actually, we can say more. The next result of Lebesgue tells us more than the 1904 result.

Theorem 6.11 (Lebesgue)**.** *Let* $F(x) = \int_a^x f(t)dt$.

(i) *If* f *is bounded on* $[a,b]$ *then* F *is Lipschitz continuous on* $[a,b]$.

(ii) *Given an* $\epsilon > 0$, *there is a* $\delta > 0$ *such that if* E *is any measurable subset of* $[a,b]$ *with* $m(E) < \delta$, *then* $\int_E |f| < \epsilon$.

(iii) *If* $F(x) = 0$ *for all* $x \in [a,b]$ *then* $f =$ *a.e.*

(iv) *If* $f \geq 0$ *then* F *is non-decreasing,*

(v) *If* f *is bounded on* $[a,b]$, *then* $F'(x) = f(x)$ *a.e.*

The proof of this result is an exercise.

One of the fundamental unanswered problems about the Riemann integral was the following. Can one explicitly describe a class of functions $\mathcal{R}$, with the property that every function $f \in \mathcal{R}$ is Riemann integrable, but if $f \notin \mathcal{R}$, then it is not Riemann integrable? Certainly every continuous function is Riemann integrable, but there are functions that are Riemann integrable, but not continuous. Lebesgue gave the following answer to this question.

Theorem 6.12 (Lebesgue)**.** *A bounded function* $f : [a,b] \to \mathbb{R}$ *is Riemann integrable if and only if it is continuous almost everywhere.*

6.1. **Differentiation.** Let us briefly consider a few facts about the derivative. We will keep the proofs to a minimum. The rest would take us too far away from the main thrust of the course.

Definition 6.13. A function f is said to have bounded variation on $[a,b]$ if for every partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ the quantity

$$\text{variation}(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

is bounded.

Example 6.1. We list some examples.

(i) The function $f(x) = x\sin(\pi/x)$ for $x \neq 0$ and $f(0) = 0$ is not of bounded variation. (To see this, take the points $x_k = 2/(2k+1)$ and consider the resulting sum. It is unbounded by comparison with the harmonic series).

(ii) If f is monotone, then f has bounded variation. This follows from the inequality $\sum_{\mathcal{P}} |f(x_k) - f(x_{k-1})| \leq |f(b) - f(a)|$ for any partition $\mathcal{P}$ of $[a, b]$.

(iii) Brownian motion has unbounded variation. (However it has bounded quadratic variation[3]).

(iv) Absolutely continuous functions have bounded variation.

(v) If f is continuously differentiable it has bounded variation.

To see the last one, observe that $F(x) = \int_a^x F'(y)dy + F(a)$, and this is absolutely continuous and hence has bounded variation.

In 1894 Jordan proved an important result about functions of bounded variation.

Theorem 6.14 (Jordan)**.** *Every function of bounded variation can be written as the difference of two monotone functions.*

Proof. Monotone functions are of bounded variation and by the triangle inequality, the difference of two functions of bounded variation has bounded variation. To prove the result we construct the necessary monotone functions. Let f be of bounded variation on $[a, b]$. Define

$$V(x) = \sup \sum_{\mathcal{P}} |f(x_k) - f(x_{k-1})|,$$

where the supremum is taken over all partition of $[a, x]$, $a \leq x \leq b$. Then V is monotone increasing. Obviously $f = V - (V - f)$. So we show that $V - f$ is monotone increasing. Suppose $x < y$. Now the variation of f on $[x, y]$ is no smaller than $|f(x) - f(y)|$, since this is variation for a trivial partition. We thus have $V(y) - V(x) \geq f(y) - f(x)$. Which is the same as $V(x) - f(x) \leq V(y) - f(y)$. So $V - f$ is monotone increasing. □

Absolutely continuous functions have another very important property.

Theorem 6.15. *An absolutely continuous function is differentiable almost everywhere.*

We have already seen that convex functions can be differentiated almost everywhere, in fact except possibly at a countable number of points. We can actually say more. Lebesgue proved the following famous result. The proof, using the so called Dini derivatives is very involved, so we omit it.

[3]the quadratic variation of f is $QV(f) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^2$.

Theorem 6.16 (Lebesgue)**.** *If f is non-decreasing, then it is differentiable almost everywhere.*

Corollary 6.17. *Functions of bounded variation are differentiable almost everywhere.*

Proof. A function of bounded variation is the difference of two monotone functions. ☐

Since we are talking about sets of measure zero, we need to answer an important question. Suppose that $f' = 0$. We know that f is constant. What if $f' = 0$ a.e.?

Theorem 6.18. *If f is absolutely continuous on $[a, b]$ and $f' = 0$ a.e. on $[a, b]$ then f is constant on $[a, b]$.*

Proof. Given $\epsilon > 0$ we can find $\delta > 0$ such that if (a_k, b_k) is a pairwise disjoint collection of subintervals of length $\sum_k (b_k - a_k) < \delta$, we have $\sum_k |f(b_k) - f(c_k)| < \epsilon$. Now consider the set $E = \{x \in (a, c) : f'(x) = 0\}$. Then E is equal to $[a, c]$ except possibly for a set of measure zero. Now given $x \in E$, $f'(x) = 0$, so from the definition of the derivative, on a small interval $[x, x + h]$ we have $|f(x + h) - f(x)| < \epsilon h$.

We can form a finite collection of disjoint closed intervals $[x_1, x_1 + h_1], [x_2, x_2 + h_2], ..., [x_n, x_n + h_n]$, where $a < x_1 < x_1 + h_1 < x_2 < x - 2 + h_2 < \cdots < x_n + h_n < c$. This gives us a decomposition of (a, c) as

$$(a, c) = (a, x_1) \cup [x_1, x_1 + h_1] \cup (x_+ h_1, x_2) \cup \cdots (x_n + h_n, c).$$

and

$$\begin{aligned} m(E - \cup[x_k, x_k + h_k]) &= m(a_1, x_1) + m(x_1 + h_1, x_2) + \cdots \\ &\quad + m(x_n + h_n, c) \\ &< \delta, \end{aligned}$$

since f is absolutely continuous. Thus

$$\begin{aligned} |f(c) - f(a)| &= |f(c) - f(x_n + h_n)| + |f(x_n + h_n) - f(x_n)| + \\ &\quad \cdots + |f(x_1 + h_1) - f(x_1)| + |f(x_1) - f(a)| \\ &\le |f(c) - f(x_n + h_n)| + \cdots + |f(x_1) - f(a)| \\ &\quad + |f(x_1 + h_1) - f(x_1)| + \cdots + |f(x_n + h_n) - f(x_n)| \\ &< \epsilon + \epsilon(h_1 + \cdots + h_n) = \epsilon(1 + c - a). \end{aligned}$$

This holds for every $\epsilon > 0$, so that $f(c) = f(a)$ for every c. Thus f is constant. ☐

If we remove the requirement that f be absolutely continuous, the result is false. There is a function, constructed by Cantor, which has the property that $C'(x) = 0$ a.e. on $[0, 1]$ but $C(0) = 0, C(1) = 1$.

7. Applications of Lebesgue's Integral

We start with yet another version of the Dominated Convergence Theorem.

Theorem 7.1. *Let (X, S, μ) be a measure space, J an interval in $\mathbb{R}$ and $f : X \times J \to \mathbb{R}$ a measurable function such that $f(\cdot, t)$ is a measurable function for each $t \in J$. Assume also that there exists an integrable function g such that $|f(x,t)| \le g(x)$ holds for almost all x and $t \in J$. If for some limit point t_0 (including possibly $\pm\infty$) of J there exists a function h such that $\lim_{t\to t_0} f(x,t) = h(x)$ holds for almost all x, then*

(i) *h is an integrable function, and*
(ii) *$\lim_{t\to t_0} \int f(x,t)dx = \int \lim_{t\to t_0} f(x,t)dx = \int hdx$.*

The proof of this is an exercise. This is a very useful form of the Dominated Convergence Theorem, which has an immediate application.

Theorem 7.2. *Let (X, S, m) be a measure space and let $f : X \times (a, b) \to \mathbb{R}$ be a function such that $f(\cdot, t)$ is Lebesgue integrable for each $t \in (a, b)$. Assume that for some $t_0 \in (a, b)$ the partial derivatives $\frac{\partial f}{\partial t}(x, t_0)$ exists for almost all x. Assume also that there exists an integrable function g and a neighbourhood V of t_0 such that*

$$|D_{t_0}(x,t)| = \left|\frac{f(x,t) - f(x,t_0)}{t - t_0}\right| \le g(x) \tag{7.1}$$

holds for almost all $x, t \in V$. Then

(i) *$\frac{\partial f}{\partial t}(\cdot, t_0)$ defines an integrable function and*
(ii) *The function $F : (a,b) \to \mathbb{R}$ defined by $\int f(x,t)dx$ is differentiable at t_0 and*

$$F'(t_0) = \int \frac{\partial f}{\partial t}(x, t_0)dx.$$

Proof. Clearly $\lim_{t\to t_0} D_{t_0}(x,t_0) = \frac{\partial f}{\partial t}(x,t_0)$ holds for almost all x. Thus by Theorem 7.1, $\frac{\partial f}{\partial t}(\cdot, t_0)$ defines an integrable function and

$$\begin{aligned}\frac{F(t) - F(t_0)}{t - t_0} &= \int \frac{f(x,t) - f(x,t_0)}{t - t_0}dx \\ &= \int D_{t_0}(x,t_0)dx \to \int \frac{\partial f}{\partial t}(x,t_0)dx\end{aligned}$$

as $t \to t_0$. □

Now suppose that $f(x,t)$ is integrable and $g(x,t) = \frac{\partial f}{\partial t}(x,t)$ exists for all $t \in (a,b)$ and is also integrable. By Taylor's Theorem for each x we can write

$$f(x,t) = f(x,t_0) + \frac{\partial f}{\partial t}(x,T)(t - t_0), \ t_0 \in (a,b), \tag{7.2}$$

for some $T \in [a, b]$. We can then say that

$$\begin{aligned}\frac{f(x,t) - f(x,t_0)}{t - t_0} &= g(x, T) \\ &\leq \sup_T |g(x, T)|.\end{aligned}$$

So if $\sup_{t\in[a,b]} |g(x, t)|$ is integrable then the conditions of the Theorem are met and we can differentiate under the integral sign.

The process of differentiating under the integral sign is used extensively (and often without regard to whether the technical conditions we need are actually satisfied). We will have cause to use it a great deal. Let us present a couple of applications to the calculation of integrals.

Example 7.1. Let us show that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$. To proceed, let

$$f(t) = \left(\int_0^t e^{-x^2} dx \right)^2$$

and

$$g(t) = \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2 + 1} dx, t \geq 0.$$

Notice that

$$\frac{\partial}{\partial t} \frac{e^{-t^2(x^2+1)}}{x^2 + 1} = -2te^{-t^2(x^2+1)}, \tag{7.3}$$

which is an integrable function for all t. This allows us to differentiate under the integral sign in the integral defining g. Now $f'(t) = 2e^{-t^2} \int_0^t e^{-x^2} dx$ and

$$g'(t) = \int_0^1 \frac{\partial}{\partial t} \left(\frac{e^{-t(x^2+1)}}{x^2 + 1} \right) dx = -2e^{-t^2} \int_0^1 te^{-x^2t^2} dx.$$

by differentiating under the integral sign. Now let $u = xt$ to give

$$g'(t) = -2e^{-t^2} \int_0^t e^{-u^2} du \tag{7.4}$$

so that $f'(t) + g'(t) = 0$. Integrating gives $f(t) + g(t) = c$ a constant. Observe that

$$f(0) + g(0) = \int_0^1 \frac{dx}{x^2 + 1} = \frac{\pi}{4}.$$

Next, $\lim_{t\to\infty} \frac{e^{-t^2(x^2+1)}}{x^2+1} = 0$ for each x. Also

$$\left| \frac{e^{-t^2(x^2+1)}}{x^2 + 1} \right| \leq \frac{1}{x^2 + 1},$$

for each $t \geq 0$. So by the DCT $\lim_{t\to\infty} g(t) = 0$. Thus

$$\frac{\pi}{4} = \lim_{t\to\infty} (g(t) + f(t)) = \left(\int_0^\infty e^{-x^2} dx \right)^2, \tag{7.5}$$

so $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Example 7.2. Show that $\int_0^\infty e^{-x^2}\cos(2xt)dx = \frac{\sqrt{\pi}}{2}e^{-t^2}$.

To do this, we set $F(t) = \int_0^\infty e^{-x^2}\cos(2xt)dx$. Then observe that $|e^{-x^2}\cos(2xt)| \leq e^{-x^2}$ which is integrable. Further $|2xe^{-x^2}\sin(2xt)| \leq xe^{-x^2}$ and xe^{-x^2} is integrable. So we differentiate under the integral sign to obtain

$$\begin{aligned} F'(t) = \int_0^\infty \frac{\partial}{\partial t}\left(e^{-x^2}\cos(2xt)\right)dx &= -2\int_0^\infty xe^{-x^2}\sin(2xt)dx \\ &= e^{-x^2}\sin(2xt)]_0^\infty \\ &- 2t\int_0^\infty e^{-x^2}\cos(2xt)dx \\ &= -2tF(t). \end{aligned}$$

So that $F'(t) + 2tF(t) = 0$. Solving the ODE gives $F(t) = F(0)e^{-t^2}$. Finally $F(0) = \int_0^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{2}}$.

Now we present a more theoretical application of differentiation under the integral. We prove uniqueness of solutions of the heat equation for certain problems.

Theorem 7.3. *Suppose that u satisfies*

(1) *u is continuous on* $\mathbb{R} \times [0, \infty)$.

(2) $u_t = u_{xx}$ *for* $t > 0$

(3) $u(x, 0) = 0$

(4) $\sup |x|^k \left|\frac{\partial^k}{\partial x^k}u(x,t)\right| < \infty$ *for all k, $0 < t < T$, all $T > 0$.*

Then $u = 0$.

Proof. Introduce $E(t) = \int_{-\infty}^\infty |u(x,t)|^2dx$. Clearly $E(0) = 0$. We will show that E is zero for all t. We differentiate under the integral sign and use condition (4) to guarantee convergence of the integrals. We have

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_{-\infty}^\infty \frac{\partial}{\partial t}u(x,t)\overline{u(x,t)}dx \\ &= \int_{-\infty}^\infty (u_t(x,t)\overline{u(x,t)} + u(x,t)\overline{u_t(x,t)})dx \\ &= \int_{-\infty}^\infty (u_{xx}(x,t)\overline{u(x,t)} + u(x,t)\overline{u_{xx}(x,t)})dx \end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}E(t) &= u_x(x,t)\overline{u(x,t)} + u(x,t)\overline{u_x(x,t)}]_{-\infty}^{\infty} \\ &\quad - \int_{-\infty}^{\infty} (u_x(x,t)\overline{u_x(x,t)} + u_x(x,t)\overline{u_x(x,t)})dx \\ &= -2\int_{-\infty}^{\infty} |u_x(x,t)|^2 dx,\end{aligned}$$

since the boundary terms must be zero by condition (4). Hence $E'(t) \leq 0$ for all t. So E is decreasing or constant. Since it starts at zero and is always non-negative, it must be constant. So $E(t) = 0$ for all $t \geq 0$. Hence $u = 0$.

□

Corollary 7.4. *Suppose that u, w both satisfy*

(1) *u and w are continuous on $\mathbb{R} \times [0, \infty)$.*

(2) *$u_t = u_{xx}$ and $w_t = w_{xx}$*

(3) *u and w both satisfy 4 above.*

(4) *$u(x, 0) = w(x, 0) = f(x)$.*

Then $u = w$.

Proof. We consider the function $v = u - w$. This satisfies the conditions of Theorem 7.3. So $v = 0$. Hence $u = w$. □

The integral used in the proof is called an energy integral. Using similar methods we can prove uniqueness for solutions of other equations, such as the wave equation. This is a large area of research. It is important to know under what conditions the solutions of a partial differential equation are unique. However this is beyond the scope of our subject.

We also consider an application of the monotone convergence theorem.

Example 7.3. Show that $\int_0^1 \frac{dx}{\sqrt{x}} = 2$.

To do this, we set $f_n(x) = x^{-1/2}\chi_{[1/n,1]}$. Plainly, $f_n \to x^{-1/2}$ on $[0, 1]$ pointwise. Now

$$\begin{aligned}\int_0^1 f_n(x)dx = \int_{1/n}^1 x^{-1/2}dx &= 2x^{1/2}]_{1/n}^1 \\ &= 2 - \frac{2}{\sqrt{n}} \to 2,\end{aligned}$$

as $n \to \infty$. The result follows by the monotone convergence theorem.

7.1. **Introduction to the Fourier Transform.** Next we begin our study of the Fourier transform. This is one of the most important tools in mathematics and we will examine it in considerable detail.

Definition 7.5. Let f be an integrable function. The Fourier transform of f is defined to be

$$(\mathcal{F}f)(y) = \widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx. \tag{7.6}$$

There are variants of this definition. Some authors will set

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iyx}dx. \tag{7.7}$$

At a couple of points we will use this because it happens to be more convenient. There are other conventions, but they are all equivalent under a change of variables.

The Fourier transform of an integrable function is not necessarily itself an integrable function. For arbitrary f, the best we can say is

Theorem 7.6 (Riemann-Lebesgue lemma)**.** *Suppose that f is Lebgesgue integrable. Then* $\lim_{|y|\to\infty} |\widehat{f}(y)| = 0$.

The proof is an exercise. The most important result about the Fourier transform is the inversion theorem. This will be proved in the tutorial exercises.

Theorem 7.7 (Fourier inversion)**.** *Suppose that f is an integrable function. Suppose also that $\widehat{f}$ is Lebesgue integrable. Then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(y)e^{iyx}dy. \tag{7.8}$$

A proof of this result will be given in the tutorials. This is the simplest form of the inversion theorem. With additional hypotheses, we can formulate others. For example, if we suppose that f', f'' exist and are integrable, then it follows that $\widehat{f}$ is integrable. This is another tutorial exercise.

We remarked that the Fourier transform of an integrable function is not necessarily itself an integrable function. This is not an example of a situation where one needs to cook up some strange counterexample, as the following demonstrates.

Example 7.4. We compute the Fourier transform of $\chi_{[-1,1]}$. To do this we have

$$
\begin{aligned}
\widehat{\chi}_{[-1,1]} &= \int_{-1}^{1} e^{-iyx} dx \\
&= \frac{1}{-iy} e^{-iyx}]_{-1}^{1} \\
&= \frac{1}{-iy} \left(e^{-iy} - e^{iy} \right) \\
&= \frac{2 \sin y}{y}.
\end{aligned}
$$

As we have seen before, the Fourier transform is not a Lebesgue integrable function.

There is another interesting feature of this example. First, the *support* of a function is defined to be the set of points on which the function value is nonzero. In this example, the support of the original function is bounded. However the support of the Fourier transform is not bounded. This is a general property of Fourier transforms. If f has bounded support, its Fourier transform cannot have bounded support.

There exist large tables of Fourier transforms and the calculation of Fourier transforms is of enormous practicaly importance. A Nobel prize in physics was awarded for work that essentially involves methods for accurate and rapid inversion of certain types of Fourier transform.

The most important example is probably that of the Gaussian.

Example 7.5. Compute the Fourier transform of $f(x) = e^{-x^2}$.

There are various ways of doing this. The direct approach requires Cauchy's Theorem from complex analysis and the integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

We have

$$
\begin{aligned}
\widehat{f}(y) &= \int_{-\infty}^{\infty} e^{-x^2 - ixy} dx \\
&= \int_{-\infty}^{\infty} e^{-(x + \frac{1}{2} iy)^2 - \frac{1}{4} y^2} dx \\
&= e^{-\frac{1}{4} y^2} \int_{-\infty}^{\infty} e^{-(x + \frac{1}{2} iy)^2} dx \\
&= e^{-\frac{1}{4} y^2} \int_{-\infty + \frac{1}{2} iy}^{\infty + \frac{1}{2} iy} e^{-z^2} dz \\
&= e^{-\frac{1}{4} y^2} \int_{-\infty}^{\infty} e^{-z^2} dz \\
&= \sqrt{\pi} e^{-\frac{1}{4} y^2}.
\end{aligned}
$$

To show that $\int_{-\infty+\frac{1}{2}iy}^{\infty+\frac{1}{2}iy} e^{-z^2}dz = \int_{-\infty}^{\infty} e^{-z^2}dz$ we integrate e^{-z^2} around the contour $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$. Here, $\gamma_1(t) = t,\ -R \le t \le R$, $\gamma_2(t) = R + it,\ 0 \le t \le \frac{1}{2}y$, $\gamma_3(t) = t + \frac{1}{2}iy,\ -R \le t \le R$, $\gamma_4(t) = -R + it,\ 0 \le t \le \frac{1}{2}y$. By Cauchy's Theorem $\int_\gamma e^{-z^2}dz = 0$. Thus

$$\int_{\gamma_1} f + \int_{\gamma_2} f = \int_{\gamma_3} f + \int_{\gamma_4} f.$$

Now

$$\left|\int_{\gamma_2} f\right| = \left|\int_0^{\frac{1}{2}y} e^{-(R+it)^2}dt\right| \le \frac{1}{2}ye^{\frac{1}{4}y^2}e^{-R^2} \to 0,$$

as $R \to \infty$ for each fixed y. Similarly for the integral along γ_4. We therefore conclude that for every finite y,

$$\int_{-\infty+\frac{1}{2}iy}^{\infty+\frac{1}{2}iy} e^{-z^2}dz = \int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}.$$

Here we have a function whose Fourier transform decays fast enough for it to be integrable. Is there some kind of rule? Without making further restrictions than integrability, the best we can say is the Riemann-Lebesgue Lemma. This tells us that a Fourier transform decays to zero at infinity. It does not tell us that the Fourier transform is integrable. So what additional conditions do we need in order to get invertibility of the Fourier transform? Reframing this, we can ask is there a space of function V with the property that $\mathcal{F} : V \to V$?

In order to proceed further, we need to introduce some new concepts. In particular we have to make clear what we mean by a 'space of functions.' What properties must V have in order for the Fourier transform to take functions in V and return functions in V? This will require us to study normed spaces. First however we have to define the concept of a metric space.

7.2. **Metric Spaces.**

Definition 7.8. Let X be a non-empty set. A metric on X is a mapping $d : X \times X \to \mathbb{R}$ such that

(i) $d(x, x) = 0$ all $x \in X$.

(ii) $d(x, y) = d(y, x) > 0$ for all $x, y \in X$ and $x \neq y$.

(iii) $d(x, y) \le d(x, z) + d(z, y)$ all $x, y, z \in X$. This is the triangle inequality.

The pair (X, d) is said to be a metric space.

Metric spaces generalise $\mathbb{R}^n$ and the Euclidean distance. We have seen how important the triangle inequality is, so we want it to be retained in any measure of 'distance'.

Example 7.6. Let $X = \mathbb{R}$ and $d(x, y) = |x-y|$. Then $(\mathbb{R}, |\cdot|)$ is a metric space.

Example 7.7. Take $X = \mathbb{R}^d$ and

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

This defines a metric. The proof is an exercise. The triangle inequality can be proved using the Cauchy-Schwartz inequality, which we will introduce later.

Example 7.8. An important, though artificial example is the following. Let X be any set. Define $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y$. This is also a metric. The proof is easy. The only thing we need to check is the triangle inequality. This is trivial.

$$d(x, y) = 1 < d(x, z) + d(z, x) = 1 + 1 = 2.$$

Our final example points to the future.

Example 7.9. Let $C([a, b])$ be the space of continuous functions on $[a, b]$. Then $d(f, g) = \int_a^b |f(t) - g(t)|dt$ is a metric space. Here we use the Riemann integral. As usual, the only thing we need to prove is the triangle inequality. We have

$$\begin{aligned} d(f, g) &= \int_a^b |f(t) - g(t)|dt = \int_a^b |f(t) - h(t) + h(t) - g(t)|dt \\ &\leq \int_a^b |f(t) - h(t)|dt + \int_a^b |h(t) - g(t)|dt \\ &= d(f, h) + d(h, g). \end{aligned}$$

We have the usual definitions.

Definition 7.9. Let $\{x_n\}$ be a sequence in X and d a metric on X. We say that $x_n \to x$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$.

Convergence is an important property of sequences. Of course it possible to have sequences which converge to points not in X. The simplest example is $X = \mathbb{Q}$. Consider the sequence $x_1 = 1.4, x_2 = 1.41, x_3 = 1.414, ...$ These are the first three terms of a sequence of rational numbers which converge to the $\sqrt{2}$. However $\sqrt{2}$ is irrational, so the limit is not in X. This means that, $\mathbb{Q}$ is not *complete*.

Definition 7.10. Let (X, d) be a metric space. We say that X is complete if every convergent sequence has a limit in X.

Example 7.10. $(\mathbb{R}, |\cdot|)$ is complete.

Complete metric spaces are the most useful ones for analysis. The reason for this is that we want to know that the properties that define the metric space are preserved when we take limits. If (X, d) is a metric space and $\{f_n\}_{n=1}^{\infty} \subset X$ is a sequence, we want the limit function f to have the same basic metric properties that the terms of the sequence do. We will see more in the next lecture. In particular we will be interested in so called *Banach spaces*. Recall that we wanted a space of functions V for which $\mathcal{F} : V \to V$. The desired space V will turn out to be a Banach space of a particular type.

Definition 7.11. Let (X, d) be a metric space. We say that $A \subset X$ is dense in X if for every $x \in X$ there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$.

Most of the ideas of analysis on $\mathbb{R}$ can easily be extended to metric spaces. The rule of thumb is to simply replace $|x-y|$ with $d(x, y)$ wherever it occurs. There are of course important caveats. For example, addition of elements is not generally defined on a metric space. We further have to assume that X is also a vector space, so we need to be mindful of whether or not our real analysis concept involves addition or multiplication in some way.

The notion of a Cauchy sequence is important on $\mathbb{R}$ and it is also important in a general metric space setting.

Definition 7.12. Let (X, d) be a metric space. We say that $\{x_n\}$ is a Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n \geq N$.

Of course the following is true.

Theorem 7.13. *Every convergent sequence in a metric space is Cauchy.*

The proof is an exercise. It is similar to the case on $\mathbb{R}$. We can formulate convergence in terms of Cauchy sequences.

Definition 7.14. A metric space (X, d) is complete if and only if every Cauchy sequence is convergent and has limit in X.

Similarly we can formulate the classical ideas about continuous functions.

Definition 7.15. Let $f : X \to Y$ be a function between two metric space (X, d_1) and (Y, d_2). We say that f is continuous at x, if $d_1(x_n, x) \to 0$ as $n \to \infty$ implies $d_2(f(x_n), f(x)) \to 0$.

As an exercise, formulate the definition of continuity in terms of inverse images of open sets.

The most important metric spaces are complete, and their metrics are given by norms. We will discuss these next.

8. Normed Linear Spaces

In these notes we will be concerned with vector spaces (equivalently linear spaces) which also possess a norm. On a general metric space we do not have addition and multiplication by constants defined. Metric spaces which are also vector spaces are therefore of more importance than arbitrary vector spaces. In such a situation the metric is often defined by a norm.

Definition 8.1. Let V be a vector space. A norm on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

(i) $\|v\| \geq 0$ for all $v \in V$.

(ii) $\|v\| = 0$ if and only if $v = 0$.

(iii) $\|v + w\| \leq \|v\| + \|w\|$ all $v, w \in V$

(iv) $\|\lambda v\| = |\lambda|\|v\|$ for all scalars λ and all $v \in V$.

A pair $(V, \|\cdot\|)$ is called a normed linear space or a normed vector space. (Both terms are used). The most basic example is $\mathbb{R}^n$. There are several possible norms.

Example 8.1. Take the linear space to be $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$. The following are norms.

(i) $\|v\|_2 = \sqrt{v_1^2 + \cdots + v_n^2}$

(ii) $\|v\|_1 = |v_1| + \cdots + |v_n|$

(iii) $\|v\|_\infty = \max\{|v_1|, ..., |v_n|\}$

(iv) $\|v\|_a = \left((v_1/a_1)^2 + \cdots + (v_n/a_n)^2\right)^{1/2}$, for $a_i > 0, i = 1, ..., n$.

As in the metric case, the fact that (i) is a norm follows from the Cauchy-Shchwartz inequality.

Any norm automatically gives rise to a metric.

Lemma 8.2. *Let V be a vector space and $\|\cdot\|$ a norm on V. Then $d(u, v) = \|u - v\|$ is a metric on V.*

The proof of this is trivial. All properties of a metric are easily verified. Note that the converse of the result is false. Not every metric gives a norm. Convergence of sequences in normed spaces is defined as we would expect.

Definition 8.3. Let $(V, \|\cdot\|)$ be a normed vector space. We say that a sequence $\{x_n\} \subset V$ converges to x if given any $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that $n \geq N$ implies $\|x_n - x\| < \epsilon$. As usual we write $x_n \to x$.

As with metric spaces, we wish to have a notion of completeness.

Definition 8.4. A normed vector space $X = (V, \|\cdot\|)$ is complete if for every convergent sequence $\{x_n\} \subset V$ with $x_n \to x$, we have $x \in V$. Equivalently, every Cauchy sequence in X converges to a limit in X.

8.1. **Banach Spaces.** The complete normed vector spaces are among the most important structures in mathematics. The real numbers are of course the bedrock of all mathematics. Mimicking important properties of the real number system in a more abstract setting is one of the major pastimes indulged in by mathematicians. The real numbers $\mathbb{R}$ form a complete vector space with the metric $d(x, y) = |x - y|$. We would like to study vector spaces which are also complete with respect to their norm.

Definition 8.5. A complete normed vector space is said to be a Banach space. (Named after Stefan Banach (1892-1945)).

The space $\mathbb{R}^n$ under any of the norms given above is a Banach space. The most important Banach spaces (besides $\mathbb{R}^n$), are the L^p spaces to be discussed in detail in the next lecture. Here are some more examples of Banach spaces.

Example 8.2. Let X be a nonempty set and $B(X)$ the vector space of all bounded, real valued functions on X. Let the infinity norm be given by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Then $\|\cdot\|_\infty$ is a norm and $(B(X), \|\cdot\|_\infty)$ is a Banach space. This follows because a convergent sequence is bounded, so the limit function is bounded.

The space $C(X)$ is also a Banach space with this norm. To prove this, note that under this norm, convergence of sequences is uniform convergence. Since the limit of a sequence of continuous uniformly convergent functions is continuous,it follows that if $\{f_n\}_{n=1}^\infty \subset C(X)$ and $f_n \to f$, then f is continuous so $f \in C(X)$. Therefore $C(X)$ is complete.

Interestingly, if we change the norm, we might lose completeness. Define $\|f\| = R\int_X |f(x)|dx$. So now convergence means $\int_X |f_n - f| \to 0$. But $\int_X |f_n - f| \to 0$ does not imply that $f \in C(X)$. The limit f may not be continuous. So completeness of a vector space generally depends on the norm we choose.

As another example, we introduce the (so called little) l^1 space.

Example 8.3. Let

$$l^1(\mathbb{R}) = \left\{ x = (x_1, x_2, ...), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty \right\}.$$

Then l^1 is a Banach space with the norm $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Proof. That l^1 is a vector space and $\|\cdot\|_1$ is a norm is an easy exercise. We wish to show completeness. We use the completeness of $\mathbb{R}$. Let $\{x^k\}$ be a Cauchy sequence in l^1. Then by definition

$$\|x^k - x^m\|_1 = \sum_{i=1}^{\infty} |x_i^k - x_i^m| \to 0$$

as $m, k \to \infty$. This is only possible if for each i $|x_i^k - x_i^m| \to 0$. Thus $\{x_i^k\}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent by the completeness of $\mathbb{R}$. Now suppose that $x_i^k \to x_i$ for each i. Now every Cauchy sequence is bounded. Let us assume that $\|x^k\| \leq M$. We wish to show that $\sum_{i=1}^{\infty} |x_i| < \infty$, so that $x = (x_1, x_2, x_3, ...) \in l^1$. Observe that

$$\begin{aligned}
\sum_{i=1}^{p} |x_i| &= \sum_{i=1}^{p} |x_i - x_i^k + x_i^k| \\
&\leq \sum_{i=1}^{p} (|x_i - x_i^k| + |x_i^k|) \\
&= \lim_{m\to\infty} \left(\sum_{i=1}^{p} (|x_i^m - x_i^k| + |x_i^k|) \right) \\
&\leq \epsilon + M
\end{aligned}$$

for all p. So $\sum_{i=1}^{\infty} |x_i| < \infty$ and l^1 is complete. □

We can also define the little $l^p(\mathbb{R})$ spaces by

$$l^p(\mathbb{R}) = \left\{ x = (x_1, x_2, ...) : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \right\},$$

with norm $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty$. These are also Banach spaces. Note some authors write l_p instead of l^p.

Familiar concepts from real analysis carry over to the Banach space setting. Let us consider the convergence of series first.

Definition 8.6. A series $\sum_{n=1}^{\infty} v_n$, $v_n \in V$ on a normed vector space is said to be convergent if the sequence of partial sums $S_N = \sum_{n=1}^{N} v_n$ is convergent. If $\{S_N\}$ is convergent with limit S, we say the series is convergent and write

$$\sum_{n=1}^{\infty} v_n = S.$$

If the series is not convergent, we say that it is divergent.

Absolute convergence can also be defined.

Definition 8.7. Let $(V, \|\cdot\|)$ be a normed vector space. The series $\sum_{n=1}^{\infty} v_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} \|v_n\|$ is convergent on $\mathbb{R}$.

As in the real analysis case, absolute convergence on a Banach space implies convergence.

Theorem 8.8. *Let $(V, \|\cdot\|)$ be a Banach space. Then every absolutely convergent series is convergent.*

Proof. Suppose that $\sum_{n=1}^{\infty} v_n$ is absolutely convergent. Define $S_n = \sum_{k=1}^{n} v_k$, $T_n = \sum_{k=1}^{n} \|v_k\|$. Now $T_n \to T \in \mathbb{R}$ by our assumption of absolute convergence. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|T_m - T_n| < \epsilon$. Assume $m > n$. Observe that $T_m > T_n$. Then

$$\begin{aligned} \|S_m - S_n\| &= \|\sum_{i=n+1}^{m} v_i\| \leq \sum_{i=n+1}^{m} \|v_i\| \\ &= T_m - T_n = |T_m - T_n| < \epsilon. \end{aligned}$$

So $\{S_n\}$ is a Cauchy sequence and because we are in a Banach space, it is convergent. □

The converse of this result is also true. The proof requires a few preliminary facts.

Proposition 8.9. *Let (X, d) be a metric space. If $\{x_n\}$ is a Cauchy sequence in (X, d) with a convergent subsequence, then $\{x_n\}$ is itself convergent.*

Proof. Suppose that the convergent subsequence is x_{n_k} and $x_{n_k} \to x \in X$ as $n_k \to \infty$. Now let $\epsilon > 0$ and choose N so that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon/2$. Now choose $K_1 \in \mathbb{N}$ so that $k \geq K_1$ implies $d(x_{n_k}, x) < \epsilon/2$. Pick $K_2 \in \mathbb{N}$ such that $n_{K_2} \geq N$. Then choose $K = \max\{K_1, K_2\}$. Then $n \geq K$ implies

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

So $x_n \to x \in X$. □

The next result will be used in the proof of the Riesz-Fischer Theorem.

Theorem 8.10. *Let $(V, \|\cdot\|)$ be a normed vector space in which every absolutely convergent series is convergent. Then $(V, \|\cdot\|)$ is a Banach space.*

Proof. Let $\{v_n\} \subset V$ be a Cauchy sequence. Choose $N_1 \in \mathbb{N}$ such that $n, m \geq N_1$ implies $\|v_m - v_n\| < 1/2$ and let $n_1 = N_1$. Next choose $N_2 \in \mathbb{N}$ such that $m, n \geq N_2$ implies $\|v_m - v_n\| < 1/2^2$ and let $n_2 = \max\{n_1 + 1, N_2\}$. Then choose $N_3 \in \mathbb{N}$ such that $m, n \geq N_3$

implies $\|v_m - v_n\| < 1/2^3$ and let $n_3 = \max\{n_2+1, N_3\}$. Continuing this process, we have a sequence of integers $n_1 < n_2 < n_3 < \cdots$. Clearly $\{v_{n_k}\}$ is a subsequence of $\{v_n\}$. Now we consider the series

$$v_{n_1} + (v_{n_2} - v_{n_1}) + (v_{n_3} - v_{n_2}) + \cdots .$$

Note that $S_k = v_{n_1} + (v_{n_2} - v_{n_1}) + \cdots + (v_{n_k} - v_{n_{k-1}}) = v_{n_k}$. Now the series of norms

$$\|v_{n_1}\| + \|v_{n_2} - v_{n_1}\| + \cdots < \|v_{n_1}\| + 1/2 + 1/2^2 + \cdots < \infty.$$

Thus the series $v_{n_1} + (v_{n_2} - v_{n_1}) + (v_{n_3} - v_{n_2}) + \cdots$ is absolutely convergent. By assumption the series is therefore convergent. And the sequence of partial sums $\{v_{n_k}\}$ must also be convergent. But this means that $\{v_{n_k}\}$ is a convergent subsequence of the Cauchy sequence $\{v_n\}$. So by Proposition 8.9, $\{v_n\}$ is convergent and $(V, \|\cdot\|)$ is complete. □

Note, when the norm is understood,we will simply write V for the Banach space.

8.2. **Hilbert Spaces.** An important type of Banach space arises when the norm is given by an inner product. Inner products generalise the dot product on Euclidean space.

Definition 8.11. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product on V is a function $(\cdot,\cdot) : V \times V \to \mathbb{R}$ (or $\mathbb{C}$) such that

(i) $(v, v) \geq 0$ all $v \in V$ and $(v, v) = 0$ only if $v = 0$.

(ii) $(v, w) = \overline{(w, v)}$.

(iii) $(av, w) = a(v, w), (v, aw) = \overline{a}(v, w)$ for all $v, w \in V$ and scalars a.

(iv) $(v + w, u) = (v, u) + (w, u)$ all $v, w, u \in V$.

An inner product space is a pair $(V, (\cdot,\cdot))$. The bar denotes the complex conjugate.

Inner products give rise to norms in the following way. It will turn out that

$$\|v\| = \sqrt{(v, v)},$$

is a norm. This requires proof. Some of the properties are obvious. For example

$$\|av\| = \sqrt{(av, av)} = \sqrt{a\overline{a}(v, v)} = |a|\|v\|.$$

For the triangle inequality we require the Cauchy-Schwarz inequality.

Proposition 8.12. *Let V be an inner product and $\|v\| = \sqrt{(v, v)}$. Then $|(v, w)| \leq \|v\|\|w\|$.*

Proof. Suppose $w \neq 0$ and set $u = \frac{w}{\|w\|}$. Then $\|u\| = 1$. So $(u,u) = 1$ and

$$\begin{aligned}
0 \leq \|v - (v,u)u\|^2 &= (v - (v,u)u, v - (v,u)u) \\
&= (v,v) - (v,(v,u)u) - ((v,u)u,u) + ((v,u)u,(v,u)u) \\
&= (v,v) - \overline{(v,u)}(v,u) - (v,u)(u,v) \\
&\quad + (v,u)\overline{(v,u)}(u,u) \\
&= (v,v) - \overline{(v,u)}(v,u) - (v,u)\overline{(v,u)} + (v,u)\overline{(v,u)} \\
&= (v,v) - |(v,u)|^2 \\
&= \|v\|^2 - |(v,u)|^2.
\end{aligned}$$

Thus $|(v,w)|^2 = |(v,\|w\|u)|^2 = \|w\|^2|(v,u)|^2 \leq \|v\|^2\|w\|^2$. Now take square roots. □

We have an immediate consequence of this useful inequality.

Proposition 8.13. *Let $(V,(\cdot,\cdot))$ be an inner product space. Then $\|v\| = \sqrt{(v,v)}$ is a norm.*

Proof. We only check the triangle inequality. We have

$$\begin{aligned}
\|v + w\|^2 = (v+w, v+w) &= (v,v) + (v,w) + (w,v) + (w,w) \\
&= \|v\|^2 + \|w\|^2 + (v,w) + \overline{(v,w)} \\
&= \|v\|^2 + 2\Re(v,w) + \|w\|^2 \\
&\leq \|v\|^2 + 2|(v,w)| + \|w\|^2 \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.
\end{aligned}$$

□

We can immediately deduce from this that for $x \in \mathbb{R}^n$ setting

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2},$$

will produce a norm. This is because the dot product is an inner product and $\|x\|_2 = \sqrt{x \cdot x}$.

An important example on an inner product (in fact the most important apart from the dot product) is the inner product on the space of functions with $\int_a^b |f(x)|^2 dx < \infty$. This is defined by

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx.$$

This is the L^2 inner product and we will have a lot to say about it later.

Definition 8.14. An inner product space which is complete with respect to the norm defined by the inner product is a Hilbert space.

Hilbert spaces are obviously Banach spaces, but because they possess an inner product, they have more structure and are often more useful. They are named for David Hilbert (1862-1943).

Proposition 8.15 (The parallelogram law)**.** *Let* $(V, (\cdot, \cdot))$ *be an inner product space and let* $v, w \in V$*. Then*

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Proof. The proof is an easy exercise.

$$\begin{aligned}\|x - y\|^2 + \|x + y\|^2 &= (x - y, x - y) + (x + y, x + y) \\ &= (x, x) - (x, y) - (y, x) + (y, y) \\ &\quad + (x, x) + (x, y) + (y, x) + (y, y) \\ &= 2(x, y) + 2(y, y) \\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

□

Although elementary, this result is quite useful. It must be satisfied by any inner product space. So if we are given a norm and we wish to know whether or not it is given by an inner product, we check to see if the parallelogram law for the norm holds. If it does not, then the norm does not arise from an inner product.

Every vector space has a basis. This is a fundamental fact of linear algebra. It means that any element of the vector space can be written as a linear combination of basis elements. It turns out that every Hilbert spaces has orthogonal basis.

Theorem 8.16. *Every Hilbert space has a basis* $\{e_n\}$ *with the property that* $(e_n, e_m) = \delta_{nm}$*, where* δ_{nm} *is the Kronecker delta:* $\delta_{nm} = 0$ *if* $n \neq m$ *and* $\delta_{nn} = 1$*. If the basis is countable, the Hilbert space is separable.*

We will prove this result after we have introduced Zorn's Lemma, later in the notes. Most Hilbert spaces encountered in practice are countable. Certainly the Hilbert spaces we will see are separable. Non-separable Hilbert spaces exist, but typically arise in highly specialised circumstances. Actually infinite dimensional separable Hilbert spaces are essentially the same.

Definition 8.17. Let V, W be vector spaces of the same dimension. We say that V, W are isomorphic if there is a one to one and onto linear map $T : V \to W$.

The following result is of importance.

Theorem 8.18. *Let K be an orthonormal set in a Hilbert space H. Then the following conditions are equivalent.*

(a) *K is complete;*

(b) *The closed linear subspace spanned by K is H;*

(c) *K is an orthonormal basis;*

(d) *For any $x \in H$, Parseval's formula holds:*

$$\|x\|^2 = \sum_{y \in K_x} |(x,y)|^2.$$

The proof of this is not difficult, but we will not give it.

Theorem 8.19. *Any two infinite dimensional separable Hilbert spaces are isometrically isomorphic. That is, if H_1 and H_2 are infinite dimensional separable Hilbert spaces, then there is a linear mapping $T : H_1 \to H_2$ such that T is one to one and $\|Tx\|_2^2 = \|x\|_1^2$. Here $\|x\|_2$ is the norm of x in H_2 and $\|\cdot\|_1$ is the norm in H_1.*

Proof. Let $\{x_n\}$ and y_n be orthonormal bases of H_1 and H_2 respectively. Define a linear map $T : H_1 \to H_2$ by $T(x_n) = y_n$. In other words, if $x = \sum_{n=1}^{\infty} c_n x_n$ and $y = \sum_{n=1}^{\infty} d_n y_n$, then $Tx = y$ if and only if $c_n = d_n$ for all $n \geq 1$. By the last part of Theorem 8.18 it follows that

$$\|Tx\|_2^2 = \sum_{n=1}^{\infty} |d_n|^2 = \sum_{n=1}^{\infty} |c_n|^2 = \|x\|_1^2.$$

□

Obtaining minima in Hilbert spaces is not as straightforward as one might hope. This is because of the fact that the Bolzano-Weierstrass Theorem, which says that closed and bounded subsets of $\mathbb{R}$ are compact, has no equivalent in an infinite dimensional space.

Definition 8.20. A metric space is said to be compact if every sequence has a convergent subsequence.

Subsequence arguments are essential in real analysis. As we saw earlier, there are many proofs which require us to use a convergent subsequence, such as when we showed that every Cauchy sequence is convergent. But this method fails in Hilbert and other infinite dimensional spaces.

Example 8.4. The unit ball in a Hilbert space is not compact. To see this, just take an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Then for all n, m, $\|e_n - e_m\| = \sqrt{2}$. So this sequence has no convergent subsequence.

Thus bounded sequences in infinite dimensional spaces do not necessarily have convergent subsequences, so the kinds of arguments that we use in elementary analysis, whereby we pick a convergent subsequence and work with that, do not work in the infinite dimensional case. To get around this obstacle, the projection theorem is used. We need a Lemma that we will not prove. This result is sometimes called the Projection Lemma.

Theorem 8.21. *Let M be a closed convex set in a Hilbert space H. For every point $x_0 \in H$, there exists a unique $y_0 \in M$ such that*

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|.$$

As an application we establish the following result.

Theorem 8.22 (The Projection Theorem)**.** *Let M be a closed linear subspace of a Hilbert space H. Any $x \in H$ can be written in the form $x = y_0 + z_0$, where $x \in M$ and $z_0 \in M^\perp$. The elements y_0, z_0 are uniquely determined.*

Proof. If $x \in M$ then we take $y_0 = x$ and $z_0 = 0$. If $x \notin M$, let y_0 be such that $\|x - y_0\| = \inf_{y \in M} \|x - y_0\|$. This exists and is unique by the Projection Theorem. Now take any point $y \in M$ and any scalar λ. Then $y_0 + \lambda y \in M$, since M is a subspace. Thus

$$\|x - y_0\|^2 \leq \|x - y_0 - \lambda y\|^2 = \|x - y_0\|^2 - 2\Re(\lambda(y, x - x_0)) + |\lambda|^2\|y\|^2.$$

Hence $-2\Re(\lambda(y, x - x_0)) + |\lambda|^2\|y\|^2 \geq 0$. Taking $\lambda = \epsilon$ and letting $\epsilon \to 0$ we have $\Re(y, x - y_0) \leq 0$. If we take $\lambda = -i\epsilon, \epsilon > 0$ we obtain $\Im(y, x - y_0) \leq 0$. Since the two inequalities obtained for y hold also for the point $-y$ (since $-y \in M$) we conclude $(y, x - y_0) = 0$ for any $y \in M$. Thus $z_0 = x - y_0 \in M^\perp$.

To prove uniqueness, suppose that $x = y_1 + z_1, y_1 \in M, z_1 \in M^\perp$. Then $y_0 - y_1 = z_1 - z_0$ lies in both M and $M^\perp$ by the subspace property. Thus $y_0 - y_1 = z_1 - z_0 = 0$. □

If $x = y + z, y \in M, z \in m^\perp$, the point y is called the projection of x in M and the operator $Px = y$ is called a projection operator. These ideas come up in functional analysis and optimisation theory, but we will not pursue them further.

We finish with a major consequence of the previous theorem.

Theorem 8.23. *Let H be a Hilbert space and let V be a closed subspace of H. Then H may be decomposed as the direct sum*

$$H = V \oplus V^\perp. \tag{8.1}$$

This result lies at the heart of Fourier analysis. A Fourier series breaks down a function into a sum of parts which are orthogonal to one another. If H is a Hilbert space of functions, it is often possible to write it (via continued use of the previous theorem), as

$$H = V_1 \oplus V_2 \oplus V_3 \cdots \tag{8.2}$$

where the subspaces V_i and V_j are orthogonal to each other for each i and j. In this way, an element of H can be written as a sum of elements from subspaces. This is what happens when we write a function as a Fourier series. $\sin(nx)$ and $\cos(nx)$ are orthogonal on $[-\pi, \pi]$ with respect to the inner product $(f, g) = \int_{\pi}^{\pi} f(x)g(x)dx$. A function in $L^2[-\pi, \pi]$ (to be defined in the next chapter), can be written as a sum of sines and cosines and this provides us a with an effective decomposition of the Hilbert space into subspaces. Each subspace is spanned by a single element, $\sin(nx)$ or $\cos(mx)$. We will discuss Fourier series later. Now we turn to a very important class of Banach space.

9. L^p Spaces

In the previous lecture we introduced the L^2 inner product. In this lecture we generalise this and show that it actually gives a norm.

Definition 9.1. Let $p \geq 1$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is such that $|f|^p$ is integrable. Then we define the L^p norm of f by

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}. \tag{9.1}$$

It is not immediately clear that this is a norm. Actually, it is not a norm at all, as we have defined it. The first problem is that $\|f\|_p = 0$ does not imply that $f = 0$ only that $f = 0$ a.e. To get around this, we introduce the idea of an equivalence class.

An equivalence relation $\sim$ on a set S must satisfy the following three properties, for all $A, B, C \in S$.

(i) $A \sim A$,

(ii) $A \sim B$ implies $B \sim A$

(iii) $A \sim B$ and $B \sim C$ implies $A \sim C$.

The simplest example is $=$ on the set of real numbers. However we can also think of equivalent matrices. Two matrices A, B are similar if there is an invertible matrix P such that $A = P^{-1}AP$. As an exercise, you can show that this is an equivalence relation.

The idea is to introduce an equivalence relation for integrable functions.

Definition 9.2. Two measurable functions f, g are equivalent if they are equal almost everywhere. We write $f \sim g$.

As an example $f(x) = x$ and $g(x) = x + \chi_{\mathbb{Q}}$ are equivalent because they are equal almost everywhere.

We divide the space of measurable functions up into equivalence classes, which we write $[f]$. Two functions f_1 and f_2 are in the equivalence class if and only if they are equal almost everywhere. We can add equivalence classes together according to the rule $a[f]+b[g] = [af+bg]$ and similarly $[f][g] = [fg]$.

Definition 9.3. We let

$$L^p(\mathbb{R}) = \left\{ [f] : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(x)|^p dx < \infty \text{ for each } f \in [f] \right\} \tag{9.2}$$

In practice we only work with a representative of an equivalence class. So when we say $e^{-x^2} \in L^p(\mathbb{R})$ we mean that every function equivalent to this function is also pth power integrable. We can define $\int [f] = \int f$. Notice that $\|[f]\|_p = 0$ if and only if the functions in the class are equal almost everywhere to zero. That is, $[f] = [0]$, where $[0]$ is the zero equivalance class. One has $[g] + [0] = [g + 0] = [g]$ $[g][0] = [g.0] = [0]$, etc.

Now we prove that the L^p norm is actually a norm for these equivalence classes of functions. Under this norm, we will show that $L^p(\mathbb{R})$ is also a Banach space. We will actually establish two extremely important inequalities satisfied by the L^p norm. First we need a fact about real numbers due to Young.

Theorem 9.4 (Young, 1912)**.** *Let $a, b \geq 0$. For $p > 1$ and q such that $1/p + 1/q = 1$ we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{9.3}$$

Proof. The proof is an elementary application of calculus. Introduce $f : [0, \infty) \to \mathbb{R}$ where

$$f(t) = \frac{1}{p}t^p + \frac{1}{q} - t.$$

Then $f'(t) = t^{p-1} - 1$ and $f''(t) = (p-1)t^{p-2}$. Note that $f'(1) = 0$ and this is the only stationary point and it is a minimum by the second derivative test. Also, $f(1) = 0$. So f increases from zero. Thus

$$\frac{1}{p}t^p + \frac{1}{q} \geq t.$$

Now take $t = \frac{a}{b^{q-1}}$. This gives

$$\frac{1}{p}\frac{a^p}{b^{pq-p}} + \frac{1}{q} \geq \frac{ab}{b^q}.$$

But $pq - p = q$, so cross multiplying gives

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab.$$

□

We use this to prove one of the most important of all inequalities in the subject. The case $p = 2$ is basically the Cauchy-Schwartz inequality. We will use this result to prove the triangle inequality, but it is of real significance in its own right.

Theorem 9.5 (Hölder's inequality)**.** *Let $p > 1$ and q be such that $1/p + 1/q = 1$. Suppose that f, g are measurable functions and that $\|f\|_p < \infty, \|g\|_q < \infty$. Then $\|fg\|_1 < \infty$ and*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

Proof. The proof uses Young's inequality. Observe that fg is measurable. Now if $\|f\|_p = 0$ then $f = 0$ a.e., which gives

$$\int |fg| = 0 = \|f\|_p\|g\|_q.$$

Now suppose that $\|f\|_p > 0$ and $\|g\|_q > 0$. Let

$$h = \frac{|f|}{\|f\|_p}, \; k = \frac{|g|}{\|g\|_q}.$$

Then by Young's inequality we have

$$hk \leq \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q}$$

So

$$\begin{aligned} \int hk &\leq \frac{1}{p}\frac{1}{\|f\|_p^p}\int |f|^p + \frac{1}{q}\frac{1}{\|g\|_q^q}\int |g|^q \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

From the definition of h, k we then conclude $\|fg\| \leq \|f\|_p \|g\|_q$. □

There are many applications of this inequality. The simplest and one of the most useful is when $p = q = 2$. Then $\|f\|_1 \leq \|f\|_2\|g\|_2$. So that the product of two square integrable functions is integrable. Another major application is the proof of the triangle inequality for the L^p norm.

Theorem 9.6 (Minkowski's inequality)**.** *Let f, g be measurable functions. Suppose that $p \geq 1$ and $\|f\|_p < \infty$ and $\|g\|_p < \infty$. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty.$$

Proof. The case $p = 1$ is trivial. So assume $p > 1$. $f + g$ is clearly measurable and so $|f + g|^p$ is measurable. Then

$$\begin{aligned} |f+g|^p &\leq (|f| + |g|)^p \leq (2\max\{|f|, |g|\})^p \\ &\leq 2^p \max\{|f|^p, |g|^p\} \leq 2^p(|f|^p + |g|^p). \end{aligned}$$

This allows us to conclude that

$$\int |f+g|^p \leq 2^p \left(\int |f|^p + \int |g|^p \right) < \infty. \tag{9.4}$$

Hence $\|f + g\|_p < \infty$.

Next we observe that if $\|f + g\|_p = 0$ the result is trivial. Assume not. Choose p, q to satisfy the conditions of the theorem. Then

$$\int ||f+g|^{p-1}|^q = \int |f+g|^{pq-q} = \int |f+g|^p < \infty. \tag{9.5}$$

This tells us that $|f+g|^{p-1} \in L^q(\mathbb{R})$. Thus

$$\begin{aligned}\int |f+g|^p &= \int |f+g||f+g|^{p-1} \\ &\leq \int (|f|+|g|)|f+g|^{p-1} \\ &= \int |f||f+q|^{p-1} + \int |g||f+g|^{p-1} \\ &\leq \|f\|_p \| |f+g|^{p-1}\|_q + \|g\|_p \| |f+g|^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \left(\int (|f+g|^{p-1})^q \right)^{1/q} \\ &= (\|f\|_p + \|g\|_p) \left(\int |f+g|^p \right)^{1/q}.\end{aligned}$$

Thus $\left(\int |f+g|^p\right)^{1-1/q} \leq \|f\|_p + \|g\|_p$. But $1-1/q = 1/p$. So the result is proved. □

It follows from this result that the L^p norm is a genuine norm. Which gives the following:

Corollary 9.7. *The space $L^p(\mathbb{R}), p \geq 1$ is a normed vector space.*

Actually these spaces are very important in analysis because of the following major result. This is one of a number of theorems due to Riesz and Fischer.

Theorem 9.8 (Riesz-Fischer)**.** *For $p \geq 1$ the space $L^p(\mathbb{R})$ is a Banach space.*

Proof. We show that every absolutely convergent series is convergent, which will imply completeness. So suppose that $\sum_{n=1}^{\infty} F_n$ is absolutely convergent in L^p and that $\sum_{n=1}^{\infty} \|F_n\|_p = M$. Here F_n is an equivalence class of functions equal almost everywhere. For each n pick a representative of the class $f_n \in F_n$. Let $g_n = \sum_{k=1}^{n} |f_k|$ and $g = \sum_{k=1}^{\infty} |f_k|$. Note that g, g_n are both measurable. Since $g_n \to g$ monotonically, $g_n^p \to g^p$ monotonically. These functions are positive and so by the monotone convergence theorem

$$\int g^p = \lim_{n\to\infty} \int g_n^p = \lim_{n\to\infty} \|g_n\|_p^p. \tag{9.6}$$

But

$$\|g_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p = \sum_{k=1}^{\infty} \|F_k\|_p = M.$$

So we have $\int g^p \leq M^p < \infty$. So that $g^p < \infty$ a.e. and hence $g < \infty$ a.e.

Now let $A = \{x \in \mathbb{R} : g(x) < \infty\}$. The set A is measurable and $m(A^c) = 0$, since g is finite almost everywhere. On A the series $\sum_{k=1}^{\infty} f_k$ is absolutely convergent and convergent in $\mathbb{R}$. The series $\sum_{k=1}^{\infty} f_k \chi_A$ is also convergent on $\mathbb{R}$. Let its pointwise sum be f. The measurability of A and f_k shows that f is measurable. Next we write

$$|f| \leq \sum_{k=1}^{\infty} |f_k \chi_A| \leq \sum_{k=1}^{\infty} |f_k| \chi_A = g\chi_A. \tag{9.7}$$

So it follows that $\int |f|^p \leq \int g^p \chi_A \leq \int g^p$. Thus $f \in L^p$. In addition $g\chi_A \in L^p$. Now let $F = [f]$ and we see that $F \in L^p$. Now for any $n \in N$,

$$\begin{aligned}
\|F - \sum_{k=1}^{n} F_k\|_p^p &= \|f - \sum_{k=1}^{n} f_k\|_p^p \\
&= \int \left| f - \sum_{k=1}^{n} f_k \right|^p \\
&= \int \left| f - \sum_{k=1}^{n} f_k \chi_A \right|^p .
\end{aligned}$$

Now $\lim_{n\to\infty} |f - \sum_{k=1}^{n} f_k|^p = 0$ pointwise and one can show that

$$\left| f - \sum_{k=1}^{n} f_k \right|^p \leq (2g\chi_A)^p,$$

which is integrable. So we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int \left| f - \sum_{k=1}^{n} f_k \chi_A \right|^p = 0. \tag{9.8}$$

Thus $\lim_{n\to\infty} \|F - \sum_{k=1}^{n} F_k\|_p^p = 0$ and hence $\lim_{n\to\infty} \|F - \sum_{k=1}^{n} F_k\|_p = 0$ is convergent. So $\sum_{n=1}^{\infty} F_n$ is convergent and so completeness follows. □

As a consequence of this L^1, L^2 are Banach spaces. In fact $L^2(\mathbb{R})$ is a Hilbert space and it is the only L^p space which is a Hilbert space. Actually we can say more. $L^p([a, b])$ is a Banach space for a, b possibly infinite. The natural setting for Fourier series is actually $L^2([-\pi, \pi])$ and this is a Hilbert space, just as $L^2(\mathbb{R})$ is.

It is also worth noting that this result is not true if we use the Riemann integral. The point is that we needed the convergence theorems to prove the Riesz-Fischer Theorem and so no theory of L^p spaces is possible with the Riemann integral. The space $\{f : [a, b] \to \mathbb{R} : R\int_a^b |f(x)|dx < \infty\}$ is perfectly well defined, but it is *not* a Banach space. A sequence of Riemann integrable functions converging pointwise or in norm on $[a, b]$ does not have to have a Riemann integrable limit. So this space is not complete. This is one of the major reasons why we use the Lebesgue integral. L^p spaces are immensely important in modern analysis and with the Lebesgue integral they are Banach spaces[4].

A basis for $L^2(\mathbb{R})$ is actually quite easy to obtain. In fact there is more than one, but the best known is given by the Hermite functions.

Consider the differential equation $y''-2xy'+2ny = 0,\ n = 0, 1, 2, 3....$ These have polynomial solutions $H_n(x)$ which are known as Hermite polynomials. They can be found by Rodriguez's formula.

Theorem 9.9. *The nth Hermite polynomial is given by*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{9.9}$$

From these we have the following.

Theorem 9.10. *An orthogonal basis for the Hilbert space $L^2(\mathbb{R})$ is given by the Hermite functions*

$$h_(x) = e^{-x^2/2} H_n(x). \tag{9.10}$$

These satisfy the relation

$$\int_{-\infty}^{\infty} h_n(x)h_m(x)dx = 2^n n!\sqrt{\pi}\delta_{nm}. \tag{9.11}$$

Further, the Fourier transform of h_n satisfies $\widehat{h}_n(y) = (-i)^n\sqrt{2\pi}h_n(y)$.

This gives us a means of defining the Fourier transform of a function in L^2 (the integral transform may not exist in the classical sense for such a function). We set

$$f(x) = \sum_{n=0}^{\infty} c_n h_n(x). \tag{9.12}$$

Then

$$\widehat{f}(y) = \sqrt{2\pi}\sum_{n=0}^{\infty}(-i)^n c_n h_n(y). \tag{9.13}$$

[4]Actually we can use some of the modern extensions of Lebesgue's theory, such as the Henstock integral and we still have a Banach space. But these integrals are not widely used and are even more technical in their construction.

10. Measure Theory and Fourier Analysis

Before proceeding we need a little more measure theory to allow us to handle multiple integrals. The ideas and proofs are essentially as in the one dimensional case, so we will only give the results.

As in the one dimensional case we define measurability using the Caratheodory condition. We introduce outer measure in the obvious way. We let Q_i denote a rectangle in $\mathbb{R}^2$. If $A \subset \mathbb{R}^2$, then the outer measure of A is given by

$$m^*(A) = \inf \left\{ \sum_i |Q_i| : A \subseteq \cup_i Q_i \right\}, \tag{10.1}$$

and $|Q_i|$ is the area of Q_i. We then state Caratheodory's criterion in precisely the same way as in the one dimensional case. A set $A \subset \mathbb{R}^2$ is measurable if and only if for every $E \subset \mathbb{R}^2$,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c). \tag{10.2}$$

Clearly we can extend this definition to $\mathbb{R}^n$ and we leave this to the reader.

Using Caratheodory's condition for measurability, we can prove that the union of two measurable sets is measurable, the compliment of a measurable set is measurable and so on. The proofs are essentially the same as on $\mathbb{R}$.

It turns out that because the area of a rectangle is length times height, we can define Lebesgue measure on $\mathbb{R}^2$ in terms of the product of two one dimensional measures.

Definition 10.1. Let (X, S, μ) and (Y, Σ, ν) be measure spaces and S, Σ are σ algebras. The product σ algebra $S \times \Sigma$ of subset of $X \times Y$ is defined by

$$S \times \Sigma = \{A \times B : A \in S, B \in \Sigma\}. \tag{10.3}$$

The product measure $\mu \times \nu$ is the set function

$$m(A \times B) = \mu \times \nu(A \times B) = \mu(A)\nu(B). \tag{10.4}$$

It is not hard to prove the following. (This is a tutorial exercise).

Theorem 10.2. *The set function $m = \mu \times \nu : S \times \Sigma \to [0, \infty)$ of Definition 10.1 is a measure.*

Some basic facts follow.

Theorem 10.3. *If A is a μ measurable subset of X and B is a ν measurable subset of Y, then $A \times B$ is a $\mu \times \nu$ measurable subset of $X \times Y$.*

Now we define measurable functions.

Definition 10.4. We say that $f : X \times Y \to \mathbb{R}$ is measurable if for each open set $K \subset X \times Y$, $f^{-1}(K)$ is measurable.

The same results on measurability from the one dimensional case also hold in the two dimensional case. For the integral we have a result which does require proof, but we will instead refer the reader to a text on measure theory. In fact we can reformulate the definition of measurability for a function to state that f is measurable if and only if $\{f > a\}$ is a measurable set for every a. So one can see why essentially the same proofs work.

To develop a theory of integration for functions of two variables, we procede by analogy with the one dimensional case. We let

$$\phi = \sum_{i,j} a_{i,j} \chi_{A_i \times B_j}$$

be a simple function. We suppose A_i, B_j are measurable. We can then define the Lebesgue integral of ϕ to be

$$\int_{X\times Y} \phi d(\mu \times \nu) = \sum_{i,j} a_{i,j} m(A_i \times B_j) = \sum_{i,j} a_{i,j} \mu(A_i)\nu(B_j). \quad (10.5)$$

We can prove that this integral is linear and has the properties that we expect.

So next we let $f > 0$ and consider all non-negative integrable simple functions ϕ with the property that $0 \le \phi \le f$ and define

$$\int_{X\times Y} f d(\mu \times \nu) = \sup \left\{ \int_{X\times Y} \phi, 0 \le \phi \le f \right\}. \quad (10.6)$$

Of course we then prove linearity etc. The integral of a general function is defined by decomposing a function into its positive and negative parts. That is we let $f = f^+ - f^-$ as before and define

$$\int f_{X\times Y} d(\mu \times \nu) = \int f^+ d(\mu \times \nu) - \int f^- d(\mu \times \nu). \quad (10.7)$$

A measurable function f is said to be Lebesgue integrable if and only if

$$\int_{X\times Y} |f| d(\mu \times \nu) < \infty.$$

In practice Lebesgue integrals in two dimensions are handled as iterated single integrals. The idea should be familiar from multivariable calculus. The key observation is that we can write

$$\chi_{A_i \times B_j} = \chi_{A_i} \chi_{B_j}.$$

So that

$$\int_{X\times Y} \chi_{A_i \times B_j} d(\mu \times \nu) = \int_Y \int_X \chi_{A_i} \chi_{B_j} d\mu d\nu = \int_Y \left(\int_X \chi_{A_i} \chi_{B_j} d\mu \right) d\nu.$$

One can then prove that for any integrable simple function we have

$$\int_{X\times Y} \phi d(\mu \times \nu) = \int_Y \left(\int_X \phi d\mu \right) d\nu. \quad (10.8)$$

Now since measurable functions can be approximated to arbitary precision by simple functions, this allows us to conclude that for Lebesgue integrable f we have

$$\int_{X\times Y} fd(\mu\times\nu) = \int_Y \left(\int_X fd\mu\right) d\nu. \tag{10.9}$$

We now flesh this out a bit more and address a fundamental question: Does it matter what order we perform the integration in? Intuitively the answer should be no. Indeed this is the case.

Proposition 10.5. *Suppose that f is measurable and for each fixed x, the function $f(x,y)$ is ν integrable. Then*

$$F(x) = \int_Y f(x,y)d\nu(y) \tag{10.10}$$

is μ measurable.

From this we can define the double integral. Let x be fixed and define $f_x = f(x,y)$ for each fixed $y \in Y$ and let $f^y(x) = f(x,y)$ for each fixed $x \in X$.

Definition 10.6. Suppose that f is $\mu\times\nu$ measurable and $f(x,y)$ is ν measurable for each $x \in X$. Then $\int_{X\times Y} fd(\mu\times\nu)$ is said to exist if f^y is an integrable function over X for ν almost all y and $g(y) = \int_X f(x,y)d\mu(x)$ defines an integrable function over Y.

The most important result for double integrals is Fubini's Theorem.

Theorem 10.7 (Fubini)**.** *Let $f : X\times Y \to \mathbb{R}$ be a $\mu\times\nu$ integrable function. Then both iterated integrals exist and*

$$\begin{aligned}\int_{X\times Y} fd(\mu\times\nu) &= \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y)\\ &= \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x).\end{aligned}$$

That is, we may reverse the order of integration.

The condition of integrability is important and cannot be omitted. If $\int |f|d(\mu\times\nu)$ is not finite, then reversing the order of integration may not be valid.

Now let us move on and consider the Fourier transform in some more detail.

10.1. **More on Fourier Transforms.** In this section we consider some of the more important properties of the Fourier transform. The first is a result about continuity.

Theorem 10.8. *Let $f \in L^1(\mathbb{R})$ and $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx$. Then $\widehat{f}$ is uniformly continuous.*

Proof. We can write

$$\widehat{f}(y+\eta)-\widehat{f}(y)=\int_{-\infty}^{\infty} f(x)(e^{-ix(y+\eta)}-e^{-iyx})dx$$
$$=\int_{-\infty}^{\infty} f(x)e^{-iyx}(e^{-ix\eta}-1)dx.$$

We therefore have

$$|\widehat{f}(y+\eta)-\widehat{f}(y)|\leq\int_{-\infty}^{\infty}|f(x)||e^{-ix\eta}-1|dx.$$

Since $|f(x)||e^{-ix\eta}-1|\leq 2|f(x)|$ which is Lebesgue integrable, we may use the dominated convergence theorem to conclude that

$$\lim_{|\eta|\to 0}|\widehat{f}(y+\eta)-\widehat{f}(y)|=\int_{-\infty}^{\infty}\lim_{|\eta|\to 0}|f(x)||e^{-ix\eta}-1|dx=0,$$

and the convergence is independent of y. Thus given $\epsilon>0$ we can find $\delta>0$, independent of y such that for all $|\eta|<\delta$, $|\widehat{f}(y+\eta)-\widehat{f}(y)|<\epsilon$. So $\widehat{f}$ is uniformly continuous. □

Next we present some of the basic operational rules. Proofs of the following are easy exercises.

Theorem 10.9. *Let $f\in L^1(\mathbb{R}), a\in\mathbb{R}$ and $b>0$. Define $f_a(x)=f(x-a)$, $(M_bf)(x)=f(bx)$ and $(S_af)(x)=e^{ixa}f(x)$. Then*

(i) $\widehat{f_a}(y)=e^{-iay}\widehat{f}(y)$

(ii) $\widehat{(S_af)}(y)=\widehat{f}(y-a)$

(iii) $\widehat{(M_bf)}(y)=\frac{1}{b}\widehat{f}\left(\frac{y}{b}\right)$.

The next result is extremely useful.

Proposition 10.10. *Let $f,g\in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty}\widehat{f}(y)g(y)dy=\int_{-\infty}^{\infty}f(y)\widehat{g}(y)dy. \tag{10.11}$$

Proof. Note that $\widehat{f}$ is uniformly continuous and by the Riemann-Lebesgue Lemma $|\widehat{f}(y)|\to 0$ as $|y|\to\infty$. So $\widehat{f}$ is bounded by some constant, say M and thus $|\widehat{f}(y)g(y)|\leq M|g(y)|\in L^1(\mathbb{R})$.

We then have

$$\int_{-\infty}^{\infty}\widehat{f}(y)g(y)dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)e^{-iyx}g(y)dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)g(y)e^{-iyx}dydx$$
$$=\int_{-\infty}^{\infty}f(x)\widehat{g}(x)dx,$$

where we used Fubini's Theorem to reverse the order of integration. Fubini's Theorem applies because $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x)g(y)|dxdy < \infty$. □

10.1.1. *Convolution.*

Definition 10.11. Let $f, g \in L^1(\mathbb{R})$. The convolution of f and g is the function defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy. \tag{10.12}$$

It is not hard to show that the convolution of two L^1 functions is itself L^1.

Proposition 10.12. *Let $f, g \in L^1(\mathbb{R})$. Then $f * g \in L^1(\mathbb{R})$.*

Proof. Let $h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$. Then by Fubini's Theorem

$$\begin{aligned}\int_{-\infty}^{\infty}|h(x)|dx &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(y)g(x-y)|dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(y)g(x-y)|dxdy = \|g\|_1\|f\|_1 < \infty.\end{aligned}$$

So $h \in L^1(\mathbb{R})$. □

The most important fact here is what the Fourier transform does to a convolution.

Theorem 10.13. *If $f, g \in L^1(\mathbb{R})$ then*

$$\widehat{f * g}(y) = \widehat{f}(y)\widehat{g}(y).$$

Proof. The proof is another application of Fubini's Theorem. We have

$$\begin{aligned}\widehat{f * g}(y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-r)g(r)e^{-iyx}drdx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-r)g(r)e^{-iyx}dxdr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(r)e^{-iy(s+r)}dsdr \\ &= \int_{-\infty}^{\infty} f(s)e^{-iys}ds\int_{-\infty}^{\infty} g(r)e^{-iyr}dr,\end{aligned}$$

where we made the substitution $x - r = s$. The application of Fubini's Theorem follows from the fact that $f, g \in L^1(\mathbb{R})$. □

10.2. **Applications of the Fourier Transform.** Let us now consider some of the applications of Fourier analysis. We will begin by solving some of the partial differential equations of classical mathematical physics. We start with the heat equation.

Example 10.1. Suppose that we wish to solve $u_t = u_{xx}, x \in \mathbb{R}, t > 0$, subject to the initial condition $u(x, 0) = f(x)$, with $f \in L^1(\mathbb{R})$ and $|u(x, t)|, |u_x(x, t)| \to 0$ as $|x| \to \infty$. We take the Fourier transform in x. So that

$$\widehat{u}(y, t) = \int_{-\infty}^{\infty} u(x, t)e^{-iyx}dx.$$

Differentiating under the integral sign gives

$$\widehat{u}_t(y, t) = \int_{-\infty}^{\infty} u_t(x, t)e^{-iyx}dx,$$

and we also have using integration by parts

$$\begin{aligned}\int_{-\infty}^{\infty} u_{xx}(x, t)e^{-iyx}dx &= u_x(x, t)e^{-iyx}|_{-\infty}^{\infty} + iy\int_{-\infty}^{\infty} u_x(x, t)e^{-iyx}dx \\ &= u(x, t)|_{-\infty}^{\infty} - y^2\int_{-\infty}^{\infty} u(x, t)e^{-iyx}dx \\ &= -y^2\widehat{u}(y, t).\end{aligned}$$

Our assumptions show that $[u_x(x, t)e^{-iyx}]_{-\infty}^{\infty} = 0$ and similarly for the other boundary terms.

Now we have been a little loose here, so let us do the first line of the previous calculation more precisely.

$$\begin{aligned}\int_{-\infty}^{\infty} u_{xx}e^{-iyx}dx &= \lim_{R\to\infty}\int_0^R u_{xx}e^{-iyx}dx + \lim_{T\to\infty}\int_{-T}^0 u_{xx}e^{-iyx}dx \\ &= \lim_{R\to\infty}[u_xe^{-iyx}]_0^R + \lim_{T\to\infty}[u_xe^{-iyx}]_{-T}^0 + iy\int_{-\infty}^{\infty} u_xe^{-iyx}dx.\end{aligned}$$

Now

$$\begin{aligned}&\lim_{R\to\infty}[u_xe^{-iyx}]_0^R + \lim_{T\to\infty}[u_xe^{-iyx}]_{-T}^0 \\ &= \lim_{R\to\infty} u_x(R, t)e^{-iyR} - u_x(0, t) + u_x(0, t) - \lim_{T\to\infty} u_x(-T, t)e^{iyT} \\ &= 0,\end{aligned}$$

since $|u_x(x, t)| \to 0$ as $|x| \to \infty$. A similar calculation holds for the next integration by parts. However for brevity, we will not spell out every detail when we do these sorts of calculations.

Thus the heat equation becomes the ODE $\widehat{u}_t(y, t) = -y^2\widehat{u}(y, t)$. We solve this to find

$$\widehat{u}(y, t) = \widehat{u}(y, 0)e^{-y^2t}.$$

But $\widehat{u}(y,0) = \widehat{f}(y)$. By the Fourier inversion Theorem we get

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(y)e^{-y^2t+iyx}dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)e^{-y^2t+iy(x-z)}dzdy.$$

To proceed we have to evaluate the integral $\int_{-\infty}^{\infty} e^{-y^2t+iy(x-z)}dy$. We complete the square to obtain

$$\int_{-\infty}^{\infty} e^{-t\left(y^2+iy\left(\frac{x-z}{t}\right)\right)}dy = \int_{-\infty}^{\infty} e^{-t\left[(y+i\frac{(x-z)}{2t})^2+\frac{(x-z)^2}{4t^2}\right]}dy$$
$$= e^{-\frac{(x-z)^2}{4t}}\int_{-\infty}^{\infty} e^{-t(y+i\frac{(x-z)}{2t})^2}dy$$
$$= e^{-\frac{(x-z)^2}{4t}}\int_{-\infty+i\frac{x-z}{2t}}^{\infty+i\frac{x-z}{2t}} e^{-tu^2}du$$
$$= e^{-\frac{(x-z)^2}{4t}}\int_{-\infty}^{\infty} e^{-tu^2}du$$
$$= \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-z)^2}{4t}}.$$

The fact that

$$\int_{-\infty+i\frac{x-z}{2t}}^{\infty+i\frac{x-z}{2t}} e^{-tu^2}du = \int_{-\infty}^{\infty} e^{-tu^2}du.$$

follows from an application of Cauchy's integral theorem to the differentiable function $f(z) = e^{-tz^2}$ and was essentially established previously. This leads to the solution

$$u(x,t) = \int_{-\infty}^{\infty} f(z)K(x-z,t)dz, \tag{10.13}$$

in which $K(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$. This last expression is known as the fundamental solution of the heat equation, or the heat kernel. It plays a major role in many areas of mathematics, not just in the theory of heat conduction.

Example 10.2. We solve a problem for the Laplace equation. Specifically we find u such that

$$\Delta u = u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ y > 0$$
$$u(x,0) = f(x),\ \lim_{y\to\infty} u(x,y), \text{finite},\ u_{xx} \text{ integrable}.$$

Also u, u_x satisfy the same condition as $|x| \to \infty$ as in the heat equation case.

As in the heat equation example we set $\widehat{u}(\xi,y)=\int_{-\infty}^{\infty}u(x,y)e^{-ix\xi}dx$. Then the Laplace equation becomes

$$\frac{d^2\widehat{u}}{dy^2}-\xi^2\widehat{u}=0.$$

The solution of this second order ODE is

$$\widehat{u}(\xi,y)=A(\xi)e^{-|\xi|y}+B(\xi)e^{|\xi|y}. \tag{10.14}$$

We take $B=0$. To see why, observe that $e^{|\xi|y}\geq 1+|\xi|y$. So that if B is non-zero on a set of positive measure we can state

$$\int_{-\infty}^{\infty}B(\xi)e^{|\xi|y}\cos(x\xi)d\xi\geq\int_{-\infty}^{\infty}B(\xi)\cos(x\xi)d\xi+y\int_{-\infty}^{\infty}|\xi|B(\xi)\cos(x\xi)d\xi,$$

and

$$\int_{-\infty}^{\infty}B(\xi)e^{|\xi|y}\sin(x\xi)d\xi\geq\int_{-\infty}^{\infty}B(\xi)\sin(x\xi)d\xi+y\int_{-\infty}^{\infty}|\xi|B(\xi)\sin(x\xi)d\xi.$$

So we have

$$\left|\int_{-\infty}^{\infty}B(\xi)e^{|\xi|y+i\xi x}d\xi\right|^2\geq g_1(x)+y^2g_2(x).$$

Here

$$g_1(x)=\left(\int_{-\infty}^{\infty}B(\xi)\cos(x\xi)d\xi\right)^2+\left(\int_{-\infty}^{\infty}B(\xi)\sin(x\xi)d\xi\right)^2,$$

and

$$g_2(x)=\left(\int_{-\infty}^{\infty}|\xi|B(\xi)\cos(x\xi)d\xi\right)^2+\left(\int_{-\infty}^{\infty}|\xi|B(\xi)\sin(x\xi)d\xi\right)^2.$$

Hence $\int_{-\infty}^{\infty}B(\xi)e^{|\xi|y+i\xi x}d\xi$ is unbounded in y if B is nonzero . Thus we must have $B=0$.

We also know that $\widehat{u}(\xi,0)=\widehat{f}(\xi)$ so $A(\xi)=\widehat{f}(\xi)$. The solution is then

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(z)e^{-iz\xi}e^{-|\xi|y+ix\xi}dzd\xi. \tag{10.15}$$

Using Fubini's Theorem we have

$$u(x,y)=\int_{-\infty}^{\infty}\frac{1}{2\pi}f(\eta)\left(\int_{-\infty}^{\infty}e^{-|\xi|y+i\xi(x-\eta)}d\xi\right)d\eta,$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-|\xi|y+i\xi(x-\eta)}d\xi&=\int_{-\infty}^{0}e^{\xi y+i\xi(x-\eta)}d\xi+\int_{0}^{\infty}e^{-\xi(y-i(x-\eta))}d\xi\\&=\left.\frac{e^{\xi(y+i(x-\eta))}}{y+i(x-\eta)}\right]_{-\infty}^{0}+]+\left.\frac{e^{-\xi(y-i(x-\eta))}}{y-i(x-\eta)}\right]_{0}^{\infty}\\&=\frac{1}{y+i(x-\eta)}+\frac{1}{y-i(x-\eta)}=\frac{2y}{(x-\eta)^2+y^2}\end{aligned}$$

Hence

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(\eta)}{(x-\eta)^2+y^2}d\eta, \tag{10.16}$$

is a solution of this equation. Notice that if $f \in L^1(\mathbb{R})$ then we may use the dominated convergence theorem to take the limit as $y \to \infty$ inside the integral and conclude that $u(x,y) \to 0$ as $y \to \infty$, so the condition on u as $y \to \infty$ is satisfied.

Note also that we say is a solution, not *the* solution. The reason for this is that problems of this type for the Laplace equation do not in general have unique solutions. To obtain uniqueness, we really need to consider additional boundary conditions, but we will not address this question in these notes. We refer the reader to the literature on the Laplace equation for more.

10.3. **The Fourier Transform on L^2.** For $f \in L^1(\mathbb{R})$, the Fourier transform is defined by the integral formula introduced earlier. For a function which is not integrable, but is square integrable, we can also define the Fourier transform using Hermite functions. There is an equivalent way of defining the Fourier transform on $L^2(\mathbb{R})$ which we present here.

Theorem 10.14. *The set $A(\mathbb{R}) = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.*

Proof. Since the Hermite functions are in $A(\mathbb{R})$ and we know these are dense in $L^2(\mathbb{R})$ the result follows. □

If $f \in A(\mathbb{R})$ then its Fourier transform exists. For such functions, we have the following major result.

Theorem 10.15 (Plancherel)**.** *Let $f \in A(\mathbb{R})$. Then $\|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2$.*

Proof. We let $g(x) = \int_{-\infty}^{\infty} \overline{f(y-x)}f(y)dy$ and it is clear that

$$g(0) = \int_{-\infty}^{\infty} \overline{f(y)}f(y)dy = \|f\|_2^2.$$

The idea is to take the Fourier transform of g. So

$$\begin{aligned}
\widehat{g}(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(y-x)}f(y)e^{-ixt}dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\overline{f(y-x)}e^{-ixt}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\overline{f(r)}e^{-i(y-r)t}drdy \\
&= \int_{-\infty}^{\infty} f(y)e^{-iyt}dy \int_{-\infty}^{\infty} \overline{f(r)}e^{irt}dr \\
&= \widehat{f}(t)\overline{\widehat{f}(t)} = |\widehat{f}(t)|^2.
\end{aligned}$$

Now $\widehat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-ixt}dx$, $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(t)e^{ixt}dt$ and it is clear that

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(t)|^2 dt.$$

Hence

$$\|f\|_2^2 = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(t)|^2 dt = \frac{1}{2\pi}\|\widehat{f}\|_2^2.$$

And the result follows from this.

□

We have therefore established that the Fourier transform is a bounded operator on $A(\mathbb{R})$ under the L^2 norm. Since $A(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ we can extend the Fourier transform to $L^2(\mathbb{R})$ by the following construction. We pick $f \in L^2(\mathbb{R})$ and a sequence $f_n \in A(\mathbb{R})$ such that $f_n \to f$ in the sense of L^2. By which we mean $\|f - f_n\|_2 \to 0$ as $n \to \infty$. We then define $\widehat{f}$ to be the L^2 limit of the sequence $\widehat{f}_n$. The full details we will omit, as they require some results on the extension of bounded operators from a dense subspace of a vector space to the whole space. The result is one of the most important in Fourier analysis.

Recall that we wanted a space of functions V with the property that $\mathcal{F} : V \to V$. The following result solves this problem.

Theorem 10.16 (Plancherel)**.** *The Fourier transform can be extended from $A(\mathbb{R})$ to the whole of $L^2(\mathbb{R})$ and for each $f \in L^2(\mathbb{R})$ we have*

$$\|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2. \tag{10.17}$$

Further, the extension is unique. Thus

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

It is worth noting that if we define the Fourier transform by

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx, \tag{10.18}$$

then Plancherel's relation becomes $\|f\|_2 = \|\widehat{f}\|_2$. It is for this reason that many mathematicians prefer this to be the definition of the Fourier transform. On the other hand, with this definition, the Fourier transform of a derivative will be $\widehat{f^{(n)}}(y) = (2\pi iy)^n\widehat{f}(y)$. The moral is that the 2π has to go somewhere. No matter how we define the Fourier transform, a factor of 2π will appear in some formula, somewhere. It is just a matter of taste where we choose the factor to go.

10.4. **The Multi-Dimensional Fourier Transform.** We can extend the Fourier transform to act on functions in $L^1(\mathbb{R}^n)$. We simply take the one dimensional Fourier transform in each variable. This results in the following definition.

Definition 10.17. Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is defined by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot y} dx, \tag{10.19}$$

where $x, y \in \mathbb{R}^n$ and $x \cdot y$ is the usual dot product.

Fourier inversion is carried out as follows.

Theorem 10.18. *Let $\widehat{f}$ be the Fourier transform of $f \in L^1(\mathbb{R}^n)$. Suppose that $\widehat{f}$ is integrable. Then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(y) e^{ix\cdot y} dy. \tag{10.20}$$

The multi-dimensional Fourier transform has properties which are essentially the same as the one dimensional case, though there are obviously some differences. For example we may prove the following by a method which is an obvious extension of the one-dimensional case.

Theorem 10.19. *Let $f \in L^1(\mathbb{R}^n)$. Then $\lim_{|y|\to\infty} |\widehat{f}(y)| = 0$.*

We may establish connections between the Fourier transform and other transforms. As an illustration, consider the Fourier transform on $\mathbb{R}^2$ of a radial function. That is, one of the form $F(x, y) = f(\sqrt{x^2 + y^2})$, where f is a function of a single variable. The Fourier transform of F is

$$\widehat{F}(\xi, \eta) = \int_{\mathbb{R}^2} f(\sqrt{x^2 + y^2}) e^{-ix\xi - iy\eta} dxdy.$$

Now we let $x = r\cos\theta, y = r\sin\theta$ and $\xi = \rho\cos\phi, \eta = \rho\sin\phi$. Then

$$\widehat{F}(\rho, \phi) = \int_0^\infty \int_0^{2\pi} f(r) e^{-ir\rho\cos(\theta-\phi)} r dr d\theta,$$

in which we used the trigonometric expansion of $\cos(\theta - \phi)$. Next observe that

$$\begin{aligned} \frac{\partial}{\partial\phi} \int_0^\infty \int_0^{2\pi} & f(r) e^{-ir\rho\cos(\theta-\phi)} r d\theta dr \\ &= -ir\rho \int_0^\infty \int_0^{2\pi} f(r) \sin(\theta - \phi) e^{-ir\rho\cos(\theta-\phi)} d\theta r dr \\ &= -\int_0^\infty f(r) \left[e^{-ir\rho\cos(\theta-\phi)}\right]_0^{2\pi} r dr = 0. \end{aligned}$$

Hence the Fourier transform is independent of ϕ. Thus we can set $\phi = 0$ in the integral. Actually this holds in higher dimensions as well. The proof is an elementary exercise.

Proposition 10.20. *The Fourier transform of a radial function is radial.*

Next observe that

$$\int_0^{2\pi} e^{-ir\rho\cos\theta} d\theta = 2\pi J_0(r\rho) \tag{10.21}$$

where J_0 is the zeroth order Bessel function of the first kind. This means that

$$\widehat{F}(\rho) = 2\pi \int_0^\infty r f(r) J_0(r\rho) dr. \tag{10.22}$$

This is essentially a Hankel transform. We discard the factor of 2π.

Definition 10.21. The zeroth order Hankel transform of a suitable function $f : \mathbb{R}^+ \to \mathbb{R}$ is

$$\widehat{f}(\rho) = \int_0^\infty f(r) r J_0(r\rho) dr. \tag{10.23}$$

Hankel transforms have the nice property that the inversion is the same as the original transform. (Why?)

Theorem 10.22. *Let $\widehat{f}$ be the zeroth order Hankel transform of f. Then*

$$f(r) = \int_0^\infty \widehat{f}(\rho) \rho J_0(r\rho) d\rho, \tag{10.24}$$

provided the integral converges.

More generally we can define a kth order Hankel transform. The condition on f given first is a sufficient but not necessary condition for the Hankel transform to exist.

Definition 10.23. Let f satisfy $\int_0^\infty f(r)\sqrt{r} dr < \infty$. Then if $k > -1/2$, the kth order Hankel transform of f is

$$\widehat{f}(\rho) = \int_0^\infty f(r) r J_k(r\rho) dr \tag{10.25}$$

Inversion is again given by the same integral.

Theorem 10.24. *Let $\widehat{f}$ be the kth order Hankel transform of f. Then*

$$f(r) = \int_0^\infty \widehat{f}(\rho) \rho J_k(r\rho) d\rho, \tag{10.26}$$

provided the integral converges.

Extension to values of k less than $-1/2$ are possible, but we will not discuss them. There are extensive tables of Hankel transforms available as there are for the Laplace and Fourier transforms.

Remark 10.25. Another common definition of the Hankel transform is

$$F_k(\rho) = \int_0^\infty \sqrt{r\rho} f(r) J_k(r\rho) dr. \tag{10.27}$$

It is possible to convert between the two definitions by making a simple change of variables in the integral.

The Hankel transform is used in many areas, notably the solution of PDES on domains involving circular symmetry. The key is the following observation. Now the kth order Bessel functions satisfy the differential equation $x^2y'' + xy' + (x^2 - k^2)y = 0$. This leads to the following result.

Proposition 10.26. *Let F_k denote the kth order Hankel transform of f. Suppose that f is twice differentiable and $f'' + \frac{1}{r}f' - \frac{k^2}{r^2}f$ possesses a Hankel transform. Then*

$$\mathcal{H}_k \left[f'' + \frac{1}{r}f' - \frac{k^2}{r^2}f \right](\rho) = -\rho^2 F_k(\rho). \tag{10.28}$$

Here $\mathcal{H}_k$ is the Hankel transform operator.

The proof of this is a straightforward exercise. We use integration by part, integrating f'' twice and f' once and rewriting the expression in terms of the derivatives of the Bessel function. We then use the fact that Bessel functions satisfy Bessel's equation in order to simplify the resulting expression.

Example 10.3. Let us solve the axially symmetric wave equation

$$u_{tt} = u_{rr} + \frac{1}{r}u_r,$$

subject to the conditions $u(r,0) = f(r)$ and $u_t(r,0) = 0$. We apply the zeroth order Hankel transform to obtain the equation

$$U_{tt} = -\rho^2 U,$$

where U denotes the zeroth order Hankel transform of u. This has the solution

$$U(\rho,t) = A(\rho)\cos(\rho t) + B(\rho)\sin(\rho t).$$

The initial conditions become $U(\rho,0) = F(\rho)$ and $U_t(\rho,0) = 0$. Applying we obtain $U(\rho,t) = F(\rho)\cos(\rho t)$ and hence the solution is

$$u(r,t) = \int_0^\infty F(\rho)\cos(\rho t) J_0(r\rho) r \, d\rho. \tag{10.29}$$

For different choices of f we can evaluate the integral and hence obtain the explicit solution.

11. Fourier Series

Fourier series allow us to expand arbitrary periodic functions on an interval $[-L, L)$ in terms of sines and cosines. The natural setting for Fourier series is $L^2(I)$ where $I = [-\pi, \pi)$ (or equivalently $[0, 2\pi)$.) Throughout we will assume that on $\mathbb{R}$, f is 2π periodic. This is related to a question that bothered Euler. The trigonometric series expansion of a function f will be periodic, therefore f must all be periodic. So we assume throughout that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.

We start with a simple lemma.

Lemma 11.1. *Suppose that $f \in L^2(I)$. Then $f \in L^1(I)$.*

Proof. This is an application of Hölder's inequality. It is obvious that $1 \in L^2(I)$. So we have

$$\int_{-\pi}^{\pi} |1f| \leq \|f\|_2 \|1\|_2 = \sqrt{2\pi}\|f\|_2 < \infty.$$

□

Note that this result only works because $m(I) = 2\pi < \infty$. On a space of infinite measure, this result is false. There are functions in $L^2(\mathbb{R})$ which are not in $L^1(\mathbb{R})$ and vice versa.

Definition 11.2. Let $f \in L^1(I)$. We define the nth Fourier coefficient of f to be

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \tag{11.1}$$

The Fourier coefficients of a function can be used to recover a great deal of information about the function itself. In fact we hope to recover the function from its Fourier coefficients by introducing the concept of a Fourier series.

Definition 11.3. The Fourier series of a function f is the infinite sum

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}. \tag{11.2}$$

We write

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}. \tag{11.3}$$

At this stage we are not claiming that the Fourier series of f is equal to f. In fact for an arbitrary continuous function, this is false. We would like to know under which conditions the result is true. We will turn to that question shortly. Now however, we present some useful properties of Fourier coefficients.

Proposition 11.4. *Suppose that f is an integrable function. Then*

$$|\widehat{f}(n)| \le \frac{1}{2\pi}\|f\|_1, n \in \mathbb{Z}. \tag{11.4}$$

Proof. Suppose that $f \in L^1(I)$. Then

$$|\widehat{f}(n)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| \tag{11.5}$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|dx = \frac{1}{2\pi}\|f\|_1, \tag{11.6}$$

as $|e^{-inx}| = 1$. □

Proposition 11.5. *Suppose that $f \in L^1(I)$ is 2π periodic. Given $a \in \mathbb{R}$ define $f_a(x) = f(x-a)$. Then $\widehat{f_a}(n) = e^{-ina}\widehat{f}(n),\ n \in \mathbb{Z}$.*

Proof. Observe that

$$\begin{aligned}
\widehat{f_a}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_a(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-a)e^{-inx}dx \\
&= \frac{1}{2\pi}e^{-ina}\int_{-\pi-a}^{\pi-a} f(y)e^{-iny}dy \\
&= \frac{1}{2\pi}e^{-ina}\int_{-\pi}^{\pi} f(y)e^{-iny}dy = e^{-ina}\widehat{f}(n).
\end{aligned}$$

Where in the last line we used the periodicity of f and e^{-iny}. □

Next we see what Fourier coefficients do to derivatives.

Proposition 11.6. *Suppose that f is a continuously differentiable 2π periodic function. Then $\widehat{f'}(n) = in\widehat{f}(n)$.*

This is an exercise in integration by parts. Essential to our proof of Dirichlet's Theorem on the convergence of Fourier series is the Riemann-Lebesgue Lemma.

Theorem 11.7 (The Riemann-Lebesgue Lemma)**.** *Suppose that $f \in L^1(I)$. Then $\lim_{|n|\to\infty}|\widehat{f}(n)| = 0$.*

This is a tutorial exercise. Now we come to an important inequality.

Theorem 11.8 (Bessel's Inequality)**.** *If $f \in L^2(I)$ and is periodic, then*

$$\sum_{n=-\infty}^{\infty}|\widehat{f}(n)|^2 \le \|f\|^2,$$

where we define the norm by $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx$.

Proof. Let $(S_N f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}$ and $g = f - S_N f$. Define $e_n(x) = e^{inx}$ and $(h, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\overline{e_n(x)}dx$. Observe that $(e_n, e_m) = \delta_{nm}$, the Kroneckor delta defined by $\delta_{nm} = 1$ if $n = m$ and 0 otherwise.

Now

$$\begin{aligned}\|f\|^2 &= (S_N f + g, S_N f + g)\\ &= (S_N f, S_N f) + (S_N f, g) + (g, S_N f) + (g, g)\\ &= \|S_N f\|^2 + (S_N f, g) + (g, S_N f) + \|g\|^2.\end{aligned}$$

Notice that

$$\begin{aligned}(g, e_n) &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \sum_{m=-N}^{N} \widehat{f}(m)\int_{-\pi}^{\pi} e^{-i(n-m)x}dx\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \widehat{f}(n) = 0\end{aligned}$$

for all $|n| \le N$. Now

$$\begin{aligned}(S_N f, g) &= (S_N f, f - S_N f)\\ &= (S_N f, f) - (S_N f, S_N f) = \left(\sum_{n=-N}^{N} \widehat{f}(n)e^{inx}, f\right) - (S_N f, S_N f)\\ &= \sum_{n=-N}^{N} \widehat{f}(n)(e^{inx}, f) - (S_N f, S_N f)\\ &= \sum_{n=-N}^{N} |\widehat{f}(n)|^2 - (S_N f, S_N f).\end{aligned}$$

Next observe that

$$\begin{aligned}(S_N f, S_N f) = \|S_N f\|^2 &= \left(\sum_{k=-N}^{N} \widehat{f}(k)e^{ikx}, \sum_{m=-N}^{N} \widehat{f}(m)e^{imx}\right)\\ &= \sum_{k=-N}^{N} \widehat{f}(k)\left(e^{ikx}, \sum_{m=-N}^{N} \widehat{f}(m)e^{imx}\right).\end{aligned}$$

But

$$\left(e^{ikx}, \sum_{m=-N}^{N} \widehat{f}(m)e^{imx}\right) = \overline{\widehat{f}(k)},$$

since

$$\begin{aligned}\left(e^{ikx}, \widehat{f}(m)e^{imx}\right) &= \overline{\widehat{f}(m)}\left(e^{ikx}, e^{imx}\right)\\ &= \widehat{f}(k)\delta_{km}.\end{aligned}$$

Thus $(S_N f, S_N f) = \sum_{n=-N}^{N} |\widehat{f}(n)|^2$. So $(S_N f, g) = 0 = \overline{(g, S_N f)}$. Consequently

$$\begin{aligned} \|f\|^2 &= \|S_N f\|^2 + (S_N f, g) + (g, S_N f) + \|g\|^2 \\ &= \|S_N f\|^2 + \|g\|^2 \\ &\geq \|S_N f\|^2 = \sum_{n=-N}^{N} |\widehat{f}(n)|^2. \end{aligned}$$

□

In fact, if $f \in L^2(I)$ then Bessel's inequality is an equality. This is a theorem of Riesz and Fischer, which we will state without proof below.

Now we turn to some of the convergence theorems of Fourier series. The first task will be to prove a version of Dirichlet's Theorem. We first introduce the Dirichlet kernel.

Lemma 11.9. *The Dirichlet kernel is defined by $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ and satisfies*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin([N + 1/2]x)}{\sin(x/2)}.$$

Proof. We use $\sum_{n=-N}^{N} e^{inx} = 1 + \sum_{n=1}^{N} e^{inx} + \sum_{n=-N}^{-1} e^{inx}$. This is the sum of two geometric sums. □

Now observe the following.

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \\ &= \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{in(x-y)} f(y) dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x - y) dy. \end{aligned}$$

We can rewrite this if f is periodic.

Theorem 11.10. *Suppose that f is periodic and integrable. Then the Nth partial sum of its Fourier series $S_N f$ is given by*

$$(S_N f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x - y) dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y) D_N(y) dy.$$

The proof is an exercise. We also have

Lemma 11.11. *The Dirichlet kernel satisfies $\int_{-\pi}^{\pi} D_N(y) = 2\pi$.*

Again this is an easy calculation. Now we can prove Dirichlet's Theorem.

Theorem 11.12 (Dirichlet)**.** *Suppose that $f \in L^1(I)$ and that $f'(x_0)$ exists. Then $\lim_{N\to\infty}(S_N f)(x_0) = f(x_0)$. That is, the Fourier series for f converges to f at a point where f is differentiable.*

Proof. We have

$$\begin{aligned}(S_N f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y) f(x_0 - y)dy - \frac{f(x_0)}{2\pi}\int_{-\pi}^{\pi} D_N(y)dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)[f(x_0 - y) - f(x_0)]dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin([N + 1/2]y) g(y)dy\end{aligned}$$

where $g(y) = \dfrac{f(x_0 - y) - f(x_0)}{y}\dfrac{y}{\sin(y/2)}$. It is not hard to see that $\lim_{y\to 0} g(y) = 2f'(x_0)$. This tells us that g is continuous and hence integrable on I. Now we extend g to $(-2\pi, 2\pi)$ by letting it equal zero outside of I. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin([N + 1/2]y) g(y)dy = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \sin([N + 1/2]y) g(y)dy \tag{11.7}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin([2N + 1]y) g(2y)dy \tag{11.8}$$

$$= \frac{1}{2i}[\widehat{h}(2N + 1) - \widehat{h}(-2N - 1)] \tag{11.9}$$

where $h(x) = 2g(2x)$. By the Riemann-Lebesgue Lemma,

$$\frac{1}{2i}[\widehat{h}(2N + 1) - \widehat{h}(-2N - 1)] \to 0$$

as $N \to \infty$. So we conclude that $(S_N f)(x_0) \to f(x_0)$ as required. □

Dirichlet's Theorem was the first result on the convergence of Fourier series to ever be established. It can be extended to show that if f is piecewise differentiable at x_0 then

$$(S_N f)(x_0) \to \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right].$$

If f is continuous but not differentiable, then it does not follow that the Fourier series for f converges to f. There exist examples of continuous functions with divergent Fourier series. However we can recover f from its Fourier coefficients by a construction due to Fejer.

Definition 11.13. The Fejer kernel is defined by

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1} \frac{\sin([n+1/2]x)}{\sin(x/2)}. \tag{11.10}$$

Lemma 11.14. *The Fejer kernel is given by*

$$F_N(x) = \frac{\sin^2\left(\frac{N}{2}x\right)}{N\sin^2(x/2)}. \tag{11.11}$$

Proof. This is similar to the corresponding result for the Dirichlet kernel. We write

$$\begin{aligned}
F_N(x) &= \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\sum_{n=0}^{N-1}\sin[(n+\frac{1}{2})x)]\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\sum_{n=0}^{N-1}\frac{e^{i(n+\frac{1}{2})x}-e^{-i(n+\frac{1}{2})x}}{2i}\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\left(\sum_{n=0}^{N-1}\frac{e^{ix/2}}{2i}e^{inx}-\sum_{n=0}^{N-1}\frac{e^{-ix/2}}{2i}e^{-inx}\right)\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\left(\frac{e^{ix/2}}{2i}\frac{e^{iNx}-1}{e^{ix}-1}-\frac{e^{-ix/2}}{2i}\frac{e^{-iNx}-1}{e^{-ix}-1}\right)\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\left(\frac{e^{iNx}-1}{2i(e^{ix/2}-e^{-ix/2})}-\frac{e^{-iNx}-1}{2i(e^{-ix/2}-e^{ix/2})}\right)\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\frac{e^{iNX}-2+e^{-iNx}}{(2i)^2\sin(x/2)}\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\frac{(e^{iNx/2}-e^{-iNx/2})^2}{(2i)^2\sin(x/2)}\\
&= \frac{1}{N\sin(\frac{1}{2}x)}\frac{(2i)^2\sin^2(Nx/2)}{(2i)^2\sin(x/2)}
\end{aligned}$$

□

With some work we can establish the following useful properties of the Fejer kernel.

Theorem 11.15. *Let F_N be the Fejer kernel. Then*

(i) $F_N(x) \geq 0$

(ii) $\int_{-\pi}^{\pi} F_N(x)dx = 2\pi$.

(iii) $\lim_{N\to\infty} \int_{\delta<|x|<\pi} F_N(x)dx = 0$ *if* $0 < \delta < \pi$.

(iv) *If* $T_N f = \frac{1}{N}\sum_{n=0}^{N-1} S_n f$, *then*

$$(T_N f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x-y)f(y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)f(x-y)dy.$$

Proof. The first part is obvious. For the second,

$$\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{N}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(x)dx = \frac{1}{N}2\pi N.$$

The proof of (iv) is similar to the corresponding result for the Dirichlet kernel. The proof of (iii) follows from the estimate

$$F_N(x) \le \frac{1}{N\sin^2(\delta/2)},$$

for $\delta < |x| < \pi$.

□

The Fejer kernel is an example of a summability kernel. It allows us to recover f from its Fourier coefficients, even if the Fourier series does not itself converge.

Theorem 11.16 (Fejer)**.** *If f is continuous and periodic, then $T_N f \to f$ uniformly.*

Proof. By the same reasoning as in the proof of Dirichlet's Theorem, we can write

$$\begin{aligned}(T_N f)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)[f(x-y) - f(x)]dy \\ &= \frac{1}{2\pi}\int_{|y|<\delta} F_N(y)[f(x-y) - f(x)]dy \\ &+ \frac{1}{2\pi}\int_{\delta<|y|<\pi} F_N(y)[f(x-y) - f(x)dy.\end{aligned}$$

Now f is continuous and periodic and is therefore uniformly continuous on I. Also

$$|f(x-y) - f(x)| \le 2\sup|f(y)| = C.$$

So

$$\left|\frac{1}{2\pi}\int_{\delta<|y|<\pi} F_N(y)[f(x-y) - f(x)]dy\right| \le \frac{C}{2\pi}\int_{\delta<|y|<\pi} F_N(y)dy \to 0$$

as $N \to \infty$. So we can choose N large enough to make this less than $\epsilon/2$. To estimate the first integral, we use properties of the Fejer Kernel (specifically $\int_{\pi}^{\pi} F_N(y)dy = 2\pi$) to give

$$\left|\frac{1}{2\pi}\int_{|y|<\delta} F_N(y)[f(x-y)-f(x)]dy\right| \leq \sup\{|f(x-y)-f(x)| : |y|<\delta\} \times \frac{1}{2\pi}\int_{|y|<\delta}|F_N(y)|dy \leq \sup\{|f(x-y)-f(x)| : |y|<\delta\}.$$

Since f is uniformly continuous, we choose $\delta > 0$ such that

$$\sup\{|f(x-y)-f(x)| : |y|<\delta\} < \epsilon/2.$$

So we can conclude that for all N large enough

$$|(T_N f)(x) - f(x)| < \epsilon \tag{11.12}$$

for all $x \in I$. Thus $T_N f \to f$ uniformly. □

Fourier series for $f \in L^2(I)$ are very well behaved. The most important result is due to Riesz and Fischer. We will not prove this result.

Theorem 11.17 (Riesz-Fischer)**.** *Suppose that $f \in L^2(I)$. Then*

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0.$$

Further, if we define $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx$, then

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \|f\|^2. \tag{11.13}$$

Conversely, suppose that $\{a_n\}_{n=-\infty}^{\infty}$ is a two sided sequence of complex numbers such that $\sum_{n=-\infty}^{\infty}|a_n|^2 < \infty$. Then there is a unique $f \in L^2(I)$ such that $a_n = \widehat{f}(n)$ for each n.

An interesting question about trigonometric series is this: If the series

$$a_0/2 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

is convergent, is it automatically a Fourier series? That is, does there *always* exist f with Fourier coefficients a_n, b_n which is given by the sum of this series? The answer is no. There are convergent trigonometric series which are not Fourier series. We might think that we can simply define f to be the sum of the series, and then the Fourier coefficients will be given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

However this assumes that we can reverse the order of integration and summation and this is not always true. Indeed much of this course has been devoted to the problem of exactly when we can switch integrals and limits. As an example, the series

$$g(x) = \sum_{n=2}^{\infty} \frac{\sin(nx)}{\ln n}$$

is a convergent trigonometric series, but it is not a Fourier series. This means that the function g is given by a series of sines, but is not a Fourier series expansion, which is a decidedly odd fact.

11.1. **More on the Inversion of Fourier Transforms.** Can we learn any lessons from Fejer's Theorem which can be applied to the recovery of a function from its Fourier transform? Consider the function

$$\widehat{K}_\lambda(y) = \lambda \left(\frac{\sin\left(\frac{\lambda y}{2}\right)}{\frac{1}{2}\lambda y} \right)^2. \tag{11.14}$$

This may be regarded as Fejer's Kernel on $\mathbb{R}$. It is not hard to show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)\widehat{K}_\lambda(y)dy = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \widehat{f}(y)\left(1 - \frac{|y|}{\lambda}\right) e^{iyx}dy. \tag{11.15}$$

As we let $\lambda \to \infty$, we might hope that the integrals in (11.15) will converge to f, at least in some sense, because the resulting integral on the right looks like the Fourier inversion integral.

Indeed there is a version of Fejer summation for the Fourier transform which shows that our hope is justified. We only state the result.

Theorem 11.18. *Let $f \in L^p(\mathbb{R}),\ 1 \le p \le 2$. Then*

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \widehat{f}(y)\left(1 - \frac{|y|}{\lambda}\right) e^{iyx}dy. \tag{11.16}$$

where convergence is in the L^p norm.

This result says that if we set

$$f_\lambda(x) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \widehat{f}(y)\left(1 - \frac{|y|}{\lambda}\right) e^{iyx}dy,$$

then $\lim_{\lambda\to\infty} \|f - f_\lambda\|_p = 0$, which implies that f_λ converges to f almost everywhere, *not* pointwise. Obtaining pointwise convergence is extremely difficult.

What about on $\mathbb{R}^n, n > 1$? We might consider the integrals

$$\frac{1}{(2\pi)^n}\int_{B_\lambda} \widehat{f}(y)\left(1 - \frac{|y|}{\lambda}\right)^{\epsilon} e^{ix\cdot y}dy, \tag{11.17}$$

where $\epsilon > 0$ and B_λ is the ball of radius λ in $\mathbb{R}^n$. Then we ask if this converges to f for each $\epsilon > 0$ for $f \in L^p(\mathbb{R}^n)$ for $\frac{2n}{n+1} < p < \frac{2n}{n-1}$? This is known as the Bochner-Riesz conjecture. For other values of p we need $\epsilon > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. However though it is known to be true for $n = 2$ (a result of L. Carleson and P. Sjolin from 1972), the answer for $n > 2$ is not known.

There are other approaches which turn out rather surprisingly not to work. For example, one might consider the operator T defined by

$$\widehat{(T_R f)}(x) = \chi_{B_R} \widehat{f}(x) \tag{11.18}$$

in which χ_{B_R} is a ball of radius R. We then consider

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_{B_R} \widehat{f}(y) e^{iy \cdot x} dy$$

and hope that this converges to f in $L^p(\mathbb{R}^n)$ for some range of p depending on n. But Charlie Fefferman proved that this is false in 1971 for all $n > 1$ except for $p = 2$ when convergence is not hard to prove. More precisely, Fefferman proved that T_1 is not a bounded operator. Meaning there is no $C > 0$ with the property that $\|T_1 f\|_p \leq C\|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$. This implies our previous statement. This was a startling discovery since it was generally believed that T_1 was a bounded operator on the disc, at least for p in some interval containing 2. Fefferman stated that his result was 'unfortunate'. It is a warning however that just because we think that a result seems eminently plausible, that does not mean that is true.

12. Fourier Analysis and Probability Theory

We continue with some applications of Fourier analysis by introducing the famous Poisson summation formula. Let $f : \mathbb{R} \to \mathbb{R}$ decay rapidly so that the series

$$(Pf)(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n) \tag{12.1}$$

converges. Observe that Pf is 2π periodic, so we can expand it in a Fourier series. In other words we can write

$$(Pf)(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{12.2}$$

where the Fourier coefficients are given by

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (Pf)(x) e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(x + 2\pi n) e^{-ikx} dx. \end{aligned}$$

Now $(Pf)(0) = \sum_{k=-\infty}^{\infty} f(2\pi k) = \sum_{k=-\infty}^{\infty} c_k$. But

$$\begin{aligned} c_k &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-ikx} dx \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi+2\pi n}^{\pi+2\pi n} f(y) e^{-iky+2\pi ink} dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-iky} dy = \frac{1}{2\pi} \widehat{f}(k). \end{aligned}$$

This implies that

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(k). \tag{12.3}$$

This is called the Poisson summation formula. There are different formulations. For example, there is an n dimensional version, but we will not consider it.

Theorem 12.1. *Suppose that f is a bounded, piecewise differentiable L^1 function with Fourier transform $\widehat{f}$. Then*

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(k). \tag{12.4}$$

Example 12.1. Let $f(x) = e^{-a|x|}$ where $a > 0$. Then $\widehat{f}(y) = \dfrac{2a}{a^2 + y^2}$. So we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{2a}{a^2+k^2} &= 2\pi \sum_{k=-\infty}^{\infty} e^{-2\pi a|k|} \\ &= 2\pi\left(1 + \sum_{k=-\infty}^{-1} e^{2\pi ak} + \sum_{k=1}^{\infty} e^{-2\pi ak}\right) \\ &= 2\pi\left(1 + \frac{1}{e^{2\pi a}-1} + \frac{1}{e^{2\pi a}-1}\right) \\ &= 2\pi\mathrm{coth}(\pi a).\end{aligned}$$

It is worth noting that if we define the Fourier transform by

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx,$$

then the Poisson summation formula is the nicely symmetric expression

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

12.1. **The Shannon Sampling Theorem.** A remarkable result due to Shannon in 1948 is essential to much of modern signal processing. This result allows us to recover a continuous, band limited signal by sampling it at discrete points.

Definition 12.2. A function $f \in L^2(\mathbb{R})$ is said to be band limited if there exists a positive number S such that

$$\mathrm{supp}\widehat{f} = \{x \in \mathbb{R} : \widehat{f}(x) \neq 0\} \subseteq [-S, S].$$

For ease of presentation, we will use a slightly different definition of the Fourier transform. We will set

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx. \tag{12.5}$$

In this case the inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y)e^{2\pi ixy}dy. \tag{12.6}$$

Theorem 12.3 (The Shannon Sampling Theorem)**.** *Suppose that $f \in L^2(\mathbb{R})$ is band limited, with support contained in $[-S, S]$. Then we may reconstruct f by sampling at the points, $\left\{\frac{n}{2S}\right\}$, by the formula*

$$f(x) = \sum_{n\in\mathbb{Z}} d_n(x) f\left(\frac{n}{2S}\right), \tag{12.7}$$

where

$$d_n(x) = k_S\left(x - \frac{n}{2S}\right) = \frac{1}{2S}\int_{-S}^{S} e^{2\pi i(x-\frac{n}{2S})y}dy. \tag{12.8}$$

Proof. Suppose that $f : \mathbb{R} \to \mathbb{C}$ with Fourier transform $\widehat{f}(y) = 0$ all $|y| > S$. Then by the Fourier inversion theorem, if

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iyx}dx \tag{12.9}$$

then we have

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y)e^{2\pi iyx}dx = \int_{-S}^{S} \widehat{f}(y)e^{2\pi iyx}dx. \tag{12.10}$$

The idea is to expand the Fourier transform of f in a Fourier series on $[-S, S]$. We have

$$\widehat{f}(y) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi ny/S}, \tag{12.11}$$

where

$$\begin{aligned} c_n &= \frac{1}{2S}\int_{-S}^{S} \widehat{f}(\xi)e^{-in\pi\xi/S}d\xi \\ &= \frac{1}{2S}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2in\pi\xi/(-2S)}d\xi \\ &= \frac{1}{2S}f\left(\frac{-n}{2S}\right). \end{aligned}$$

So

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} e^{2\pi iyx} \sum_{n=-\infty}^{\infty} \frac{1}{2S}f\left(\frac{-n}{2S}\right)e^{in\pi y/S}dy \\ &= \int_{-S}^{S}\sum_{n=-\infty}^{\infty} \frac{1}{2S}f\left(\frac{-n}{2S}\right)e^{2\pi iy(x+\frac{n}{2S})}dy \\ &= \frac{1}{2S}\int_{-S}^{S}\sum_{n=-\infty}^{\infty} f\left(\frac{-n}{2S}\right)e^{2\pi iy(x+\frac{n}{2S})}dy \\ &= \frac{1}{2S}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2S}\right)e^{2\pi iy(x-\frac{n}{2S})}dy \\ &= \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2S}\right)d_n(x). \end{aligned}$$

□

Thus if a signal is band limited, then we can recover it completely if we only know it at equally spaced points. This remarkably fact (and the further developments stemming from it) is the basis for much of the modern theory of telecommunications. These kinds of mathematical techniques are what makes your mobile phone work!

12.2. **Applications in Probability.** Measure theory is fundamental to the modern theory of probability. We define a probability space using the concept of a measure.

Definition 12.4. A probability space $(\Omega, \mathcal{F}, P)$ is a measure space where $P(\Omega) = 1$ and P is a Borel measure. Here P is called a probability measure.

Definition 12.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random variable on Ω is a Borel measurable function $X : \Omega \to \mathbb{R}$.

The elements of the σ algebra are called events and we usually associate the outcome of observations with them. Naturally we are interested in integrable random variables.

Definition 12.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on Ω. We say that

(i) X is an $L^r(\Omega)$ random variable if $\int_\Omega |X|^r dP < \infty$

(ii) If $X \in L^1(\Omega)$, then the expectation of X is $E(X) = \int_\Omega X dP$.

(iii) If X is an $L^2(\Omega)$ random variable, then the variance of X is $\sigma^2(X) = E\left((X - E(X))^2\right)$.

To any random variable we can associate the probability

$$\Pr(X \le x) = \text{Probability}(X \le x). \tag{12.12}$$

Definition 12.7. Let X be a random variable on $(\Omega, \mathcal{F}, P)$. Then

$$\Pr(X \le x) = P(X^{-1}\{(-\infty, x]\}).$$

If $\Pr(X \le x) = F(x)$ is differentiable, then $F'(x) = f(x)$ is called the probability density of X.

Computing expected values is accomplished by integration. We will not prove the next result.

Theorem 12.8. *Suppose that X is a random variable on $(\Omega, \mathcal{F}, P)$ and that $h \in L^1(\Omega)$ and that X has probability density f. Then*

$$E(h(X)) = \int_\Omega h(x) f(x) dx. \tag{12.13}$$

There is a very important example.

Definition 12.9. Let X be a random variable on $\mathbb{R}$ with probability density f. Then the characteristic function of X is

$$E\left(e^{-itX}\right) = \int_{-\infty}^{\infty} e^{-itx} f(x) dx = \varphi_X(t). \tag{12.14}$$

This is of course the Fourier transform of f. One can use it to compute moments. The next result is a simple exercise.

Proposition 12.10. *Let $X \in L^r(\mathbb{R})$ have characteristic function $\varphi_X(t)$. Then*

$$E\left(X^r\right) = (i)^r \frac{d^r}{dt^r}\varphi_X(t)\bigg|_{t=0}.$$

Example 12.2. Let X be a random variable with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{12.15}$$

Then

$$\begin{aligned}\varphi_X(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2} - itx\right)\\ &= e^{-\frac{1}{2}\sigma^2 t^2 - i\mu t}. \end{aligned} \tag{12.16}$$

Thus

$$\varphi_X'(0) = -i\mu, \tag{12.17}$$

so $E(X) = \mu$.

Definition 12.11. Two random variables X, Y are said to be independent if $E\left(h(X)g(Y)\right) = E\left(h(X)\right)E\left(g(Y)\right)$ for all Borel functions X and Y.

A result which we will not prove follows.

Theorem 12.12. *Let X be a continuous random variable on $\mathbb{R}$. Then the characteristic function of X is unique and completely characterises $Pr(X \leq x)$.*

An easy applications of characteristic functions follows.

Theorem 12.13. *Let X, Y be independent random variables on $\mathbb{R}$ with probability densities f_X, f_Y respectively. Then the probability density of $Z = X + Y$ is given by*

$$f_Z(x) = \int_{-\infty}^{\infty} f_X(y) f_Y(x-y)dy.$$

Proof. Let $Z = X + Y$. By independence

$$\begin{aligned}E\left(e^{-itZ}\right) &= E\left(e^{-it(X+Y)}\right)\\ &= E\left(e^{-itX}\right)E\left(e^{-itY}\right).\end{aligned}$$

The result follows from the convolution theorem. □

We can also define

Definition 12.14. Let X be a continuous random variable. The moment generating function $M_X(s)$ is defined by

$$M_X(s) = E\left(e^{-sX}\right). \tag{12.18}$$

Some authors prefer to work with $E\left(e^{sX}\right)$. Notice that if $X \geq 0$, then $M_X(s) = \int_0^\infty f(x)e^{-sx}dx$ is the Laplace transform of the probability density f. The moment generating function gives us quite a bit of information. In fact its very existence tells us something important.

Theorem 12.15. *If the integral defining the moment generating function converges for $s \in (-\gamma, \gamma)$, $\gamma > 0$ then $E(X^r) < \infty$ for all $r \geq 0$.*

Proof. Suppose that X has denisty f. Then

$$M_X(s) = \int_{-\infty}^{\infty} e^{-sx}f(x)dx = \int_0^\infty e^{-sx}f(x)dx + \int_{-\infty}^0 e^{-sx}f(x)dx, \tag{12.19}$$

converges for $-\gamma < s < \gamma$. Thus

$$\int_0^\infty e^{-sx}f(x)dx < \infty, \ -\gamma < s, \tag{12.20}$$

and

$$\int_{-\infty}^0 e^{-sx}f(x)dx < \infty, \tag{12.21}$$

for $\gamma > s$. Now choose $s_0 > 0$ such that $-\gamma < -s_0 < 0$. Then

$$\begin{aligned} M_X(-s_0) &\geq \int_0^\infty e^{s_0 y}f(y)dy \\ &\geq \int_x^\infty e^{s_0 y}f(y)dy, \ x > 0 \\ &\geq e^{s_0 x}\int_x^\infty f(y)dy = e^{s_0 x}\Pr(X \geq x). \end{aligned}$$

Thus

$$\Pr(X \geq x) \leq M_X(-s_0)e^{-s_0 x}$$

and for every $r > 0$

$$\int_0^\infty x^{r-1}\Pr(X > x)dx \leq M_X(-s_0)\int_0^\infty e^{-s_0 x}x^{r-1}dx < \infty. \tag{12.22}$$

Similarly, choose $s_1 \in (0, \gamma)$. Then we arrive at

$$M_X(s_1) \geq \int_{-\infty}^0 e^{-s_1 y}f(y)dy \geq e^{-s_1 x}\Pr(X < x), \tag{12.23}$$

and so for $x < 0$ we have $\Pr(X < x) \leq e^{s_1 x} M_X(s_1)$. Thus

$$\begin{aligned}\int_{-\infty}^{0} |x|^{r-1} \Pr(X < x) dx &\leq M_X(s_1) \int_{-\infty}^{0} e^{s_1 x} |x|^{r-1} dx \\ &= M_X(s_1) \int_{0}^{\infty} e^{-s_1 y} f(y) dy.\end{aligned}$$

Thus for $r > 0$

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^{r-1} \Pr(|X| > x) dx = &\int_{0}^{\infty} x^{r-1} \Pr(X > x) dx \\ &+ \int_{-\infty}^{0} |x|^{r-1} \Pr(X < x) dx < \infty.\end{aligned}$$

And the result follows. □

13. Some Functional Analysis

We begin with one of the major tools of analysis, the Radon-Nikodym theorem, which is fundamental to many areas. It allows us to change from one measure to another, as in for example Girsanov's Theorem. Suppose that $(X, \mathcal{F}, \mu)$ is a measure space. We can define a new measure as follows.

Theorem 13.1. *Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that f is a non-negative, measurable function. Then*

$$\nu(E) = \int_E f d\mu \tag{13.1}$$

is a measure on X.

Proof. It is clear that ν is positive and if $\emptyset$ is the empty set that

$$\nu(\emptyset) = \int_\emptyset f d\mu = 0.$$

Now let $A = \cup_{n=1}^\infty A_n$ where the A_n are disjoint. If $x \in A$, then x is in exactly one A_n. So

$$\chi_A(x) = 1 = \sum_{n=1}^\infty \chi_{A_n}. \tag{13.2}$$

So that

$$\begin{aligned}\nu(A) = \int_A f d\mu &= \int_X f \chi_A d\mu \\ &= \sum_{n=1}^\infty \int_X f \chi_{A_n} d\mu \\ &= \sum_{n=1}^\infty \int_{A_n} f d\mu \\ &= \sum_{n=1}^\infty \nu(A_n).\end{aligned}$$

□

Thus we can obtain infinitely many measures from a given measure.

Example 13.1. Let μ be Lebesgue measure on $\mathbb{R}$. Then

$$P(E) = \int_E \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{13.3}$$

is also a measure on $\mathbb{R}$.

Definition 13.2. Given two measures μ and ν on $(X, \mathcal{F})$, we say that ν is absolutely continuous with respect to μ if $\mu(A) = 0$ implies that $\nu(A) = 0$. We write $\nu \ll \mu$.

Before proceeding we need an important concept. A measure μ is σ-finite if the underlying space X can be written as a countable union of sets X_n and $\mu(X_n) < \infty$ for all n. Thus Lebesgue measure is σ-finite, but it is not finite: $m([0,\infty)) = \infty$, but $[0,\infty) = [0,1]\cup[1,2]\cup[2,3]\cdots$ and $m([n,n+1]) = 1$.

It is clear that if we define two measures as in Theorem 13.1, then they are absolutely continuous with respect to each other. The Radon-Nikodym theorem says that all absolutely continuous measures arise this way. The next result is of great importance, but we will not prove it.

Theorem 13.3 (Radon,Nikodym)**.** *Let $(X,\mathcal{F})$ be a measurable space and let μ and ν be σ-finite measures on $\mathcal{F}$ with $\nu \ll \mu$. Then there is a unique, non-negative, measurable function f such that*

$$\nu(E) = \int_E f\,d\mu, \tag{13.4}$$

all $E \in \mathcal{F}$.

The proof of this result requires some more measure theory. In particular a result known as the *Hahn Decomposition Theorem* which in turn requires the concept of a *signed measure*. These are really beyond the scope of the subject. Nevertheless the Radon-Nikodym theorem is a major tool in modern analysis and it is worthwhile to see it. It is particularly important in stochastic calculus where the Radon-Nikodym derivative of two measures can play a major role.

Definition 13.4. Let $\nu \ll \mu$ and suppose that $\nu(E) = \int_E f\,d\mu$. Then we write

$$\frac{d\nu}{d\mu} = f$$

and call f the Radon-Nikodym derivative of ν with respect to μ.

Radon-Nikodym derivatives have properties similar to the regular derivative. The following is a simple exercise.

Theorem 13.5. *Assume that λ,ν,μ are finite measures, such that $\lambda \ll \mu$ and $\nu \ll \mu$. Then*

(i) *With $\phi = \lambda+\nu$, $\dfrac{d\phi}{d\mu} = \dfrac{d\lambda}{d\mu} + \dfrac{d\nu}{d\mu}$, a.e.*

(ii) *If $\lambda \ll \nu$ then $\dfrac{d\lambda}{d\mu} = \dfrac{d\lambda}{d\nu}\dfrac{d\nu}{d\mu}$, a.e.*

Of course there will be measures μ,ν with the property that $\mu(E) = 0 \neq \nu(E)$. For example, Lebesgue measure and the point measure $\mu_x(E) = 1$ if $x \in E$ and $\mu_x(E) = 0$ if $x \notin E$.

In stochastic calculus, the Radon-Nikodym derivative appears when we change from one Brownian motion to another using Girsanov's Theorem. That is well outside the scope of this subject however.

The Radon-Nikodym theorem has numerous applications in advanced analysis. We will present one below.

13.1. **Representation Theorems.**

Definition 13.6. Let X be a vector space. A linear functional on X is a linear mapping

$$\mu : X \to \mathbb{C}. \tag{13.5}$$

If $\mu(x_n) \to \mu(x)$ whenever $x_n \to x$ we say that μ is a continuous linear functional.

Example 13.2. Let $X = C([a,b])$. Then $\mu(f) = \int_a^b f(x)dx$ is a linear functional on X.

So integration is a linear functional. Actually, it is in a sense the only one. To see this, we first consider the Hilbert space case. We first note that a transformation $T : X \to Y$ where X, Y are normed vector spaces is bounded if there is a constant C such that $\|Tx\|_Y \le C\|x\|_X$.

Theorem 13.7 (The Riesz Representation Theorem I)**.** *Let H be a Hilbert space and $F : H \to \mathbb{C}$ a bounded linear functional on H. Then there is a unique element $z \in H$ such that for all $x \in H$, $F(x) = (x, z)$.*

Proof. Denote by $N = \{x \in H : F(x) = 0\}$. Then N is clearly a closed linear subspace of H. To see that it is a subspace notice that if $x, y \in N$, and λ, μ are scalars, then

$$F(\lambda x + \mu y) = \lambda F(x) + \mu F(y) = 0,$$

so that $\lambda x + \mu y \in N$. Plainly $F(0) = F(0.0) = 0F(0) = 0$. Thus we also have $0 \in N$, so N is a subspace. To show that it is closed let $\{x_n\}_{n=1}^{\infty} \subset N$. Then $F(x_n) = 0$ for all n. Suppose $x_n \to x$. Then $|F(x_n) - F(x)| = |F(x_n - x)| \le C\|x_n - x\|_H \to 0$ as $x_n \to x$ in H. Thus $F(x_n) \to F(x)$ (which implies F is continuous). Since $F(x_n) = 0$ for all x we have $F(x) = 0$, so $x \in N$ and N is closed.

If $H = N$, then $F(x) = 0$ for all x. So we can take $z = 0$. If $N \ne H$, then let

$$z_0 \in N^{\perp} = \{y \in H : (x, y) = 0, x \in N, y \ne 0\}.$$

Suppose $\alpha = F(z_0) \ne 0$. Now

$$x - F(x)z_0/\alpha \in N \tag{13.6}$$

since

$$F\left(x - F(x)\frac{z_0}{\alpha}\right) = F(x) - F(x)\frac{F(z_0)}{F(z_0)} = 0. \tag{13.7}$$

Thus by definition of $N^{\perp}$,

$$\left(x - \frac{F(x)z_0}{\alpha}, z_0\right) = 0.$$

Hence

$$\frac{F(x)}{\alpha}(z_0, z_0) = (x, z_0).$$

Thus $F(x) = (x, z)$ where $z = \dfrac{\overline{\alpha} z_0}{(z_0, z_0)}$.

Now suppose that there is a $z' \in H$ such that $(x, z) = (x, z')$ for all $x \in H$. Then $(x, z - z') = 0$ for all x. Take $x = z - z'$. Then we have $(z - z', z - z') = \|z - z'\|^2 = 0$, but this implies that $z = z'$ by a basic property of norms. □

If we take $H = L^2(\mathbb{R})$, then every linear functional is given by

$$F(f) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \tag{13.8}$$

for some $g \in L^2(\mathbb{R})$. So this linear functional is given by an integral. In fact the following is true.

Theorem 13.8 (The Riesz Representation Theorem II)**.** *Let (X, s, μ) be a measure space and F a continuous linear functional on $L^p(X, \mu), p > 1$. Let q be such that $1/p + 1/q = 1$. Then there is a $g \in L^q(X, \mu)$ such that for all $f \in L^p(X, \mu)$*

$$F(f) = \int_X fgd\mu$$

and $\|F(f)\| \leq \|f\|_p \|g\|_q$.

The proof is involved, but the basic idea is to define a measure $\nu(A) = F(\chi_A)$ for every finite set A. It is not hard to establish that this is a measure, and $\nu \ll \mu$. So by the Radon-Nikodym theorem we have the existence of a g such that

$$\nu(A) = \int_A gd\mu. \tag{13.9}$$

By linearity, if $\phi = \sum_i a_i \chi_{A_i}$, then $F(\phi) = \int \phi g d\mu$. Now given $\phi \in L^p$, we may find an increasing sequence of step functions $\phi_n \to f$. Using the Dominated Convergence Theorem we have

$$\begin{aligned} F(f) &= \lim_{n\to\infty} \int_X \phi_n g d\mu \\ &= \int_X \lim_{n\to\infty} \phi_n g d\mu \\ &= \int_X fgd\mu. \end{aligned}$$

One then establishes that $g \in L^q$ and the final result is an application of Hölder's inequality.

Actually we can say even more.

Theorem 13.9 (The Riesz Representation Theorem III). *Let $X = \mathbb{R}^n$ or $\mathbb{C}^n$. Let $C_c(X)$ be the space of all compactly supported functions on X. Then for every continuous linear functional F on $C_c(X)$, there is a measure μ such that*

$$F(f) = \int_X f d\mu.$$

Thus linear functionals are essentially given by measures.

The role that linear functionals play is absolutely fundamental in many branches of analysis. Let us give some examples, to see how they arise.

Example 13.3 (The Bergman kernel). In the study of functions of a single complex variable, a crucial result is the Cauchy integral formula. This says that if f is an analytic function in a doman $D \subseteq \mathbb{C}$ and γ is a simple, smooth closed curve bounding D, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi. \tag{13.10}$$

The *Cauchy kernel*

$$K(z, \xi) = \frac{1}{2\pi i}\frac{1}{\xi - z}$$

is an example of a reproducing kernel. That is, if one integrate a function against the kernel, the function is returned.

What happens if we move from $\mathbb{C}$ to $\mathbb{C}^n$? Is there an analogue of the Cauchy integral formula? Actually there are several, but explicit descriptions are hard except in some special cases. One such special case is due to Stefan Bergman. For $\Omega \subseteq \mathbb{C}^n$, define for some positive function V

$$A^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \left(\int_\Omega |f(z)|^2 dV(z) \right)^{1/2} \right\}, \tag{13.11}$$

and each f is analytic in Ω. One can prove that if

$$(f, g) = \int_\Omega f(z)\overline{g(z)} dV(z),$$

then $A^2(\Omega)$ is a Hilbert space.

Now we define a continuus linear function on $A^2(\Omega)$ by setting

$$l_z f = f(z). \tag{13.12}$$

That is, l_z evaluates f at the point $z \in \Omega$. The Riesz representation theorem says that there is an element k_z of $A^2(\Omega)$, such that for all $f \in A^2(\Omega)$

$$\begin{aligned} l_z f = f(z) &= (f, k_z) \\ &= \int_\Omega f(\zeta)\overline{K(z, \zeta)} dV(\zeta), \end{aligned}$$

where $K(z,\zeta) = \overline{k_z(\zeta)}$. K is called the Bergman kernel. Thus the Riesz representation tells us that a reproducing kernel exists. This is a higher dimensional analogue of the Cauchy integral formula. However actually computing the form of the kernel is currently impossible, except for some special cases. For example, if $\Omega = B_n = \{z \in \mathbb{C}^n : |z| \leq 1\}$ then the kernel can be shown to be

$$K_B(z,\zeta) = \frac{n!}{\pi^n(1 - z \cdot \bar{\zeta})^{n+1}}. \tag{13.13}$$

Here $z \cdot \bar{\zeta} = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$.

Example 13.4 (The Dirac delta function). While studying problems in Quantum mechanics, Dirac discovered that he needed a function δ with the property that for every f,

$$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a).$$

Unlike the situation in complex variable theory, on the real line one can prove that there is no such function.

However we can regard the mapping $f \to f(z)$ as a linear functional, in which case there is a measure μ_a with the property that for each f

$$\int_{\mathbb{R}} f(x)d\mu_a = f(a).$$

This is the basis for the theory of distributions, developed by Laurent Schwartz. In the next section we will consider distributions in detail.

14. Distributions

The theory of distributions was invented to extend the range of functions for which the Fourier transform is defined. Distributions are also called generalised functions. Originally, distributions arose in physics. In the 1930's Paul Adrian Maurice (PAM) Dirac was studying problems in quantum mechanics. In order to make one of his calculations work, he introduced a "function" which he called the delta function. The delta function had the following property. Let $\delta(x)$ be the delta function. For every function f,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

or more generally,

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Such a function would be a tremendously useful thing to have. Unfortunately, no such functions exists. What $\delta(x)$ would look like if it did exist, is not hard to imagine. If we were to plot the graph of such a beast, we would see an infinite mass concentrated at a single point a and zero everywhere else. This would be a very strange function indeed.

The fact that no such function exists should not hinder us. We have seen behaviour similar to the desired behavior of the delta function already. Approximate identities do a very similar job to the one that Dirac was interested in.

Remember our solution of the heat equation derived in a previous chapter. Notice that for any $f \in L^1(\mathbb{R})$,

$$\lim_{t\longrightarrow 0} \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{4\pi t}}e^{\frac{-(x-y)^2}{4t}}dy = f(x). \tag{14.1}$$

So in some sense we must have

$$\lim_{t\longrightarrow 0} \frac{1}{\sqrt{4\pi t}}e^{\frac{-(x-y)^2}{4t}} = \delta(x-y) \tag{14.2}$$

This is tantalisingly close to what Dirac wanted. Can we make sense of the connection? In other words, is there some way to make the statement in equation (14.2) precise? The answer is yes. There are several, equivalent ways of doing it. In fact one can build up an entire Calculus for the delta function and related "functions" ie distributions. The theory is due mainly to Laurent Schwartz. Our aim is to develop a theory of distributions which will include things like the Dirac delta function in a rigorous way.

For our study of distributions, we will be concerned with the space $\mathcal{D}(\Omega)$. Let $\Omega \subset \mathbb{R}^n$. Let $K \subset \Omega$ be a compact set and define,

$$\mathcal{D}_K = \{f \subset C^\infty(\Omega) | \text{supp} f \subset K\} \tag{14.3}$$

Then

$$\mathcal{D}(\Omega) = \bigcup_{K \subset \Omega} \mathcal{D}_K.$$

Now we introduce the *dual* $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$.

Definition 14.1. The dual of $\mathcal{D}(\Omega)$, denoted $\mathcal{D}'(\Omega)$, is the space of linear functionals on $\mathcal{D}(\Omega)$. That is, $\mathcal{D}'(\Omega)$ consists of linear functionals I defined on $\mathcal{D}(\Omega)$. So if $I \in \mathcal{D}'(\Omega)$, then $I(\phi) \in \mathcal{C}$ for all $\phi \in \mathcal{D}(\Omega)$.

We will need multi-index notation. A multi-index α is an n-tuple of non-negative integers. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Example 14.1. Take $\alpha = (1,1)$ and let the variables be x and y. Then $D^\alpha = \frac{\partial^2}{\partial x \partial y}$. If instead we take $\alpha = (2,0)$ then $D^\alpha = \frac{\partial^2}{\partial x^2}$.

To define a distribution we need a *semi-norm* on $\mathcal{D}(\Omega)$.

Definition 14.2. For $N = 0, 1, 2, \ldots$ and $\phi \in \mathcal{D}(\Omega)$. We set

$$||\phi||_N = \sup_\Omega \{|D^\alpha \phi| : |\alpha| \leq N\}.$$

A semi-norm has all the properties of a norm, except that some non-zero elements may have zero semi-norm.

It is not hard to prove that $||\phi||_N \leq ||\phi||_{N+1}$. A distribution is essentially a linear functional. However, to make the technical details work, we need to be a little more precise. The definition of a distribution is as follows.

Definition 14.3. A distribution is a linear map $\Lambda : \mathcal{D} \longrightarrow \mathbb{C}$ such that for all compact sets $K \subseteq \Omega$ there is an integer N and a constant $c < \infty$ such that,

$$|\Lambda(\phi)| \leq c||\phi||_N \quad \forall \phi \in \mathcal{D}_K$$

Functions ϕ for which $\Lambda\phi$ is defined are called *test functions*.

Remark 14.4. Λ is a bounded linear functional. The constant c and N may depend on K

If the same N will do for all K and is the smallest integer with this property, we say Λ has order N. However c might still vary with K. If there is no smallest N, we say that Λ has infinite order.

Example 14.2. Let $x \in \Omega$ and set $\delta_x(\phi) = \phi(x)$ and $\phi \in \mathcal{D}$. Then δ_x is a distribution of order 0 since

$$|\delta_x(\phi)| = |\phi(x)| \leq \sup_{y\in\Omega} |\phi(y)|$$

This is the Dirac delta.

Example 14.3. Let f be a locally integrable function on Ω. (i.e. for all K compact, $\int_K |f(x)|dx < \infty$). We define a distribution in the following way.

$$\Lambda_f(\phi) = \int_\Omega f(x)\phi(x)dx, \quad \text{for all test functions } \phi \in \mathcal{D} \tag{14.4}$$

Then this is a linear mapping from $\mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ and for $\phi \in \mathcal{D}_K$ we have the estimate

$$\begin{aligned} |\Lambda_f(\phi)| = |\int f(x)\phi(x)dx| &\leq \int_\Omega |f(x)||\phi(x)|dx \\ &\leq \sup_{y\in\Omega} |\phi(y)| \int_K |f(x)|dx \\ &= c||\phi||_0. \end{aligned}$$

Where $c = \int_K |f(x)|dx$. Therefore Λf is a distribution of order 0.

Example 14.4. Let μ be a measure on Ω such that $|\mu|(k) < \infty$ for all compact K. Then $\Lambda_\mu\phi = \int \phi d\mu$ is a distribution of order 0.

We conclude this section with a theorem due to Schwartz, the creator of the theory of distributions. This theorem tells us precisely what a distribution is. Essentially it says that the space of distributions $\mathcal{D}$ is exactly equal to the set of linear functionals on $\mathcal{D}$ and that $\mathcal{D}'$ is a vector space, but it cannot be made into a metric space. We will not prove this result.

Theorem 14.5 (L Schwartz)**.** *There is a topology on $\mathcal{D}(\Omega)$ which makes it into a locally convex topological vector space, and such that $\mathcal{D}'(\Omega)$ is precisely the set of continuous linear functionals on $\mathcal{D}(\Omega)$. $\mathcal{D}$ is complete in this topology but not metrizable.*

Proof. See Rudin. □

14.1. **Calculus with distributions.** We have so far presented a couple of examples of distributions. One of the advantages of distributions is that one can do calculus with them, as we shall see below. Let us look at the problem of how we would define a derivative for a special class of distributions.

Let Λ_f be as in (14.4), where f is locally integrable. Let α be a multi-index. We define the α-th derivative of Λ_f by

$$D^\alpha \Lambda_f(\phi) = (-1)^{|\alpha|}\Lambda_f(D^\alpha \phi).$$

This is a distribution of order $|\alpha|$, since

$$|D^\alpha \Lambda_f \phi| = |\Lambda_f D^\alpha \phi| \leq c||D^\alpha \phi||_0 \leq c||\phi||_{|\alpha|}$$

Why is this the natural way to define $D^\alpha \Lambda_f$? It is simply the integration by parts formula in disguise. This requires a bit of explanation.

Remember what the integration by parts formula says. If h and g are two differentiable functions, then

$$\int_K h'g = -\int_K hg' \tag{14.5}$$

assuming g has compact support contained in K. If you do not quite understand this formula, think of the one dimensional case.

We know that

$$\int_a^b h'(x)g(x)dx = [h(x)g(x)]_a^b - \int_a^b h(x)g'(x)dx. \tag{14.6}$$

However, if h and g are zero at $x = a$ and $x = b$, then this reduces to

$$\int_a^b h'(x)g(x)dx = -\int_a^b h(x)g'(x)dx.$$

This is the same as (14.5). The point is that if h and g are zero on the boundary of the region of integration, which is exactly what happens in the case where they have compact support, then the first term on the right hand side of (14.6) will equal zero. This justifies (14.5).

In general if we integrate by parts $|\alpha|$ times, we will have

$$\int_K (D^\alpha h)g = (-1)^{|\alpha|} \int_K hD^\alpha g.$$

What does this do for us? It is actually a very powerful idea. Notice that the function f which defines our distribution Λ_f is simply any locally integrable function. We never claimed that it was differentiable, let along $|\alpha|$ times differentiable.

What we want to do is define the derivative of the distribution Λ_f. Since $\Lambda_f(\phi) = \int f\phi$ we ought to define the derivatives of Λ_f, $D^\alpha \Lambda_f$ according to the rule

$$D^\alpha \Lambda_f \phi = \int (D^\alpha f)\phi.$$

The problem with this is that $D^\alpha f$ may not exist, since we have not assumed that f is differentiable. However $D^\alpha \phi$ exists, since the test function $\phi \in C^\infty(\Omega)$. The integration by parts formula says that if $D^\alpha f$ exists and ϕ has compact support, then

$$\int_k (D^\alpha f)\phi(x)dx = (-1)^{|\alpha|} \int_k fD^\alpha \phi dx \tag{14.7}$$

But the integral on the right hand side of (14.7) exists even if $D^\alpha f$ does not. Hence it is logical to define the derivative of a distribution Λ_f according to the rule

$$(D^\alpha \Lambda_f)(\phi) = (-1)^{|\alpha|}\Lambda_f(D^\alpha \phi). \tag{14.8}$$

Define $L^1_{loc}(\Omega)$ to be the space of locally integrable functions in Ω. That is, there is a space $K \subseteq \Omega$ such that $\int_K f < \infty$.

We know that if $f \in L^1_{loc}(\Omega)$, then Λ_f is a distribution, and its derivative is defined by (14.8). It is not hard to extend this definition to a more general distribution.

For a general distribution $\Lambda \in \mathcal{D}'$ we define

$$(D^\alpha \Lambda)(\phi) = (-1)^{|\alpha|}\Lambda(D^\alpha \phi). \tag{14.9}$$

Example 14.5. By the Riesz representation theorem we think of distributions as coming from integration against a measure. The standard notation for the Dirac measure δ_x is to write it as an integral. Let us calculate $\delta'(x)$ on Ω. Let $f \in \mathcal{D}(\Omega)$, and let $a \in \Omega$. Let δ_a be the Dirac function with mass concentrated at $x = a$. In other words

$$\int_\Omega \delta_a(x) f(x) dx = f(a).$$

Note that it is often the case that we prefer to write this as

$$\int_\Omega \delta_a(x) f(x) dx = \int_\Omega \delta(x-a) f(x) dx = f(a).$$

In order to calculate the derivative of δ_a, we apply the formula (14.9). This gives us

$$\int_\Omega \delta_a'(x) f(x) dx = -\int_\Omega f'(x)\delta_a(x) dx = -f'(a)$$

In general it is not hard to show that

$$\int_\Omega \delta_a^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(a).$$

Example 14.6. The Heaviside step function at a is defined by.

$$H_a(x) = \begin{cases} 1 & : \quad x \geq a \\ 0 & : \quad x < a \end{cases}$$

It turns out that the derivative of the distribution defined by H_a is the Dirac delta function.

To see this, let $\phi \in \mathcal{D}(\mathbb{R})$. Define

$$\Lambda_H(\phi) = \int_{-\infty}^{\infty} H_a(x)\phi(x) dx = \int_a^\infty \phi(x) dx.$$

Now,

$$(D\Lambda_{H_a})(\phi) = -\Lambda_{H_a}(D\phi) = -\int_a^\infty \phi'(x)dx = -\left[\phi(x)\right]_a^\infty = \phi(a),$$

since $\lim_{x\to\infty}\phi(x) = 0$. So $D\Lambda_{H_a} = \delta_a$.

Let us now present a few rules for dealing with distributions.

14.1.1. *Multiplication by functions.* Let $\Lambda \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$. Then $f\Lambda$ is defined by

$$(f\Lambda)(\phi) = \Lambda(f\phi). \tag{14.10}$$

It is not hard to show that $f\Lambda$ is a distribution in $\mathcal{D}(\Omega)$.

14.1.2. *Limits.* The weak * topology on $\mathcal{D}'$, is defined as follows. We say the sequence $\{\Lambda_i\}$, $\Lambda_i \in \mathcal{D}$ converges (in the weak * topology) to a distribution $\Lambda \in \mathcal{D}'(\Omega)$ if,

$$\Lambda_i\phi \longrightarrow \Lambda\phi$$

for all $\phi \in \mathcal{D}(\Omega)$. We write $\Lambda_i \longrightarrow \Lambda$.

Example 14.7. If $\{f_i\}$ is a sequence of locally integrable functions, we say $f_i \longrightarrow \Lambda$ in the sense of distributions, if for all $\phi \in \mathcal{D}(\Omega)$

$$\int f_i\phi \longrightarrow \Lambda\phi$$

If $f_n(x) = \sqrt{n\pi}e^{\frac{-x^2}{n}}$. Then $f_n \longrightarrow \delta_0(x)$, the Dirac delta function, in the sense of distributions.

Remark 14.6. The family $\{f_i\}$ is an approximation of the identity. So what we are saying is that for any $\phi \in \mathcal{D}(\Omega)$, with $0 \in \Omega$,

$$\lim_{n\to\infty}\int_\Omega \phi(x)\sqrt{n\pi}e^{\frac{-x^2}{n}}dx = \int_\Omega \delta_0(x)\phi(x)dx = \phi(0). \tag{14.11}$$

Taking limits of distributions works better than taking limits of ordinary sequences of functions. Basically all the properties that we would like to hold, do in fact hold. That is the content of the next result.

Theorem 14.7. *Suppose* $\Lambda_i \in \mathcal{D}'(\Omega)$ *for* $i = 1, 2, 3, \ldots$ *and suppose that for all* $\phi \in \mathcal{D}(\Omega)$, $\lim_{i\longrightarrow\infty}\Lambda_i\phi$ *exists as a complex number. Write* $\Lambda\phi$ *to denote this limit. Then* $\Lambda \in \mathcal{D}'(\Omega)$ *and furthermore* $D^\alpha\Lambda_i \longrightarrow D^\alpha\Lambda$.

Note this is not in general true for ordinary functions. For example, if we take a sequence of differentiable functions f_n, with $f_n \to f$ the limit function f need not even be differentiable. It is certainly not true without added conditions on the f_n that the derivatives of the f_n converge to the derivative of f. ie, $f_n' \to f'$. However this is automatically true for distributions.

14.2. **Supports and Structure Theorems.** What do distributions actually look like? We have seen some examples of useful distributions, such as the Dirac Delta function. Is the Dirac Delta in some sense typical of what distributions look like? Addressing that question is the purpose of this next section.

Let $\Lambda \in \mathcal{D}'(\Omega)$ and let $\omega \subseteq \Omega$ be an open set. We say that Λ vanishes on ω if $\Lambda\phi = 0$ for all ϕ with support in ω. The support of Λ is the complement of the largest open set on which Λ vanishes. If $\Lambda = \Lambda_f$ where f is locally integrable function, then this definition coincides with the usual definition of the support of f.

To see this, let $\Lambda_f\phi = \int_\Omega f\phi$ and supp $\phi = \omega$. Thus

$$\int_\Omega f\phi = \int_\omega f\phi = 0,$$

for all ϕ. Which implies that $f = 0$ on ω. Hence $\text{supp} f \subseteq \omega^c$.

Theorem 14.8. *The following holds for distribution in $\mathcal{D}'$.*

(a) *If $\phi \in \mathcal{D}(\Omega)$ and $\Lambda \in \mathcal{D}'(\Omega)$, and*

$$\text{supp}\phi \bigcap \mathit{supp}\Lambda = \emptyset,$$

then $\Lambda\phi = 0$.

(b) *If* supp Λ *is empty, then $\Lambda = 0$.*

c) *If $\psi \in C^\infty(\Omega)$ and $\psi = 1$ in an open set V containing* supp Λ, *then $\psi\Lambda = \Lambda$*

(d) *If* supp Λ *is compact then Λ has finite order. In fact there exists $c < \infty$ and N such that $|\Lambda\phi| \le c\|\phi\|_N$ $\forall\phi \in \mathcal{D}(\Omega)$. Further more Λ extends in a unique way to a continuous linear functional on $C^\infty(\Omega)$. ie $\Lambda\phi$ exists for $\phi \in C^\infty \backslash \mathcal{D}^\infty$.*

The aim of this section is to describe a generic distribution. That is, say what a distribution looks like. For distributions which are supported at a single point, the picture is particularly simple as the following result shows.

Proposition 14.9. *Suppose that $\Lambda \in \mathcal{D}'(\Omega)$ and that the support of $\Lambda = \{p\}$ for $p \in \Omega$. Suppose the order of $\Lambda = N$. There exists a constants $c_\alpha \forall |\alpha| \le N$ such that,*

$$\Lambda = \sum_{|\alpha| \le N} c_\alpha D^\alpha \delta_p$$

where δ_p is the Dirac measure at p. Conversely every distribution of this form is supported at p.

This result tells us that the distributions supported at a single point are linear combinations of derivatives of Dirac delta functions. For distributions which are not supported at a single point, the situation is somewhat more complicated. Nevertheless, the following result gives us the characterisation that we seek.

Theorem 14.10 (Structure Theorem)**.** *Let $\Lambda \in \mathcal{D}'(\Omega)$. For each multi-index α there is a continuous function g_α on Ω such that,*

a) Each compact subset of K of Ω intersects the support of only finitely many g_α's.

b) $\Lambda = \sum_\alpha (D^\alpha \Lambda_{g_\alpha})$. Furthermore if Λ has compact support, then we can choose the g_α's so that only finitely many are non-zero.

Remark 14.11. This theorem says that if $\phi \in \mathcal{D}_K$ then $\Lambda\phi = \sum_\alpha (D^\alpha \Lambda_{g_\alpha})\phi$ and only finitely many of the g_α's are non zero.

Remark 14.12. This shows that the class of distributions we have introduced is minimal in the sense that it contains locally integrable functions, their derivatives, linear combinations and nothing else. In other words, distributions in $\mathcal{D}'(\Omega)$ are made up from locally integrable functions.

14.3. **Convolutions of distributions.** The problem is to extend convolutions to distributions. Suppose $\Lambda \in \mathcal{D}', \phi \in \mathcal{D}$.

Definition 14.13. We define the convolution of a distribution Λ and a function $\phi \in \mathcal{D}(\Omega)$ by

$$\Lambda * \phi = \Lambda(T_x \phi^\nu), \tag{14.12}$$

where $(T_x f)(y) = f(y+x)$ and $\phi^\nu(y) = \phi(-y)$.

Let us unravel this and see that it makes sense. Observe that

$$(T_x\phi^\nu)(y) = T_x(\phi)(-y) = \phi(-y+x) = \phi(x-y).$$

So if we put all the pieces together, and take $\Lambda = \Lambda_f$. Then,

$$\begin{aligned}
\Lambda_f * \phi = \Lambda_f(T_x\phi^\nu) &= \int f(y)(T_x\phi^\nu)(y)dy \\
&= \int f(y)T_x(\phi(-y))dy \\
&= \int f(y)\phi(x-y)dy \\
&= f * \phi(x)
\end{aligned}$$

So this gives us the convolution of f and ϕ, which is what we should get in this case.

The following Lemma is easy to establish.

Lemma 14.14. *If $\Lambda \in \mathcal{D}$ then $\Lambda * \phi \in C^\infty$ and for all α,*

$$D^\alpha(\Lambda * \phi) = (D^\alpha \Lambda) * \phi = \Lambda * (D^\alpha \phi)$$

Now that we have defined the convolution of a distribution and a function in $\mathcal{D}(\Omega)$ we consider the convolution of two distributions.

Definition 14.15. Let Λ_1 and Λ_2 be distributions, and at least one of them has compact support. Then $\Lambda_1 * \Lambda_2$ is defined by

$$(\Lambda_1 * \Lambda_2)(\phi) = \Lambda_1 * (\Lambda_2 * \phi).$$

Remark 14.16. We have defined the convolution of two distributions where at least one has compact support. However, there is no notion of the convolution of arbitrary distributions. Indeed there are theorems which show that you cannot define such a thing.

Remark 14.17. If Λ_1 or Λ_2 has compact support, then
a) $\Lambda_1 * \Lambda_2 = \Lambda_2 * \Lambda_1$
b) $D^\alpha(\Lambda_1 * \Lambda_2) = (D^\alpha \Lambda_1) * \Lambda_2 = \Lambda_1 * (D^\alpha \Lambda_2)$
c) For all $\Lambda, D^\alpha \Lambda = (D^\alpha \delta_0) * \Lambda$

Part (c) gives us the useful fact that differentiation in $\mathcal{D}'$ is given by convolution with the Derivatives of the Dirac delta

14.4. **Some useful theorems.**

Theorem 14.18. *Let k be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that for some $c > 0$,*

$$\int_{\mathbb{R}^n} |k(x,y)|dy \leq c \text{ for a.e } x$$
$$\int_{\mathbb{R}^n} |k(x,y)|dx \leq c \text{ for a.e } y$$

if $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Then the function defined by,

$$Tf(x) \int_{\mathbb{R}^n} |k(x,y)|f(y)dy \leq c \text{ for a.e } x$$

belongs to $L^p(\mathbb{R}^n)$ and $\|Tf\|_p \leq \|f\|_p$.

Proof. This is an application of Holders inequality. □

Theorem 14.19 (Young's inequality)**.** *Let $f \in L^1(\mathbb{R}^n), g \in L^p(\mathbb{R}^n)$. Then $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Proof. Take $k(x,y) = f(x,y)$ in the previous theorem. □

14.5. **Tempered Distributions.**

We would like to define the Fourier transform of a distribution. We saw earlier that if $f, g \in L^1(\mathbf{R}^n)$ then,

$$\int_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy$$

This suggests that to define a Fourier transform of a distribution Λ we should set

$$\widehat{\Lambda}(\phi) = \Lambda(\widehat{\phi}) \quad , \Lambda \in \mathcal{D}'$$

The difficulty with this is that if we work with the space $\mathcal{D}'$, this does not work. The reason is that if $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\widehat{\phi} \notin \mathcal{D}(\mathbb{R}^n)$ unless $\phi = 0$. This means that $\Lambda(\widehat{\phi})$ may not exist because $\widehat{\phi}$ is no longer compactly supported. So it is not a test function for Λ. To get around this problem, we need $\mathcal{S}$, the space of rapidly decreasing functions, which we introduced earlier. For convenience we give the definition of $\mathcal{S}$ once more.

Definition 14.20. We say that $f \in C^\infty(\Omega)$ is rapidly decreasing if for all $N \in \mathbf{N}$, and all multindices α, there is a constant c_N^α such that

$$|D^\alpha f(x)| \leq c_N^\alpha (1 + |x|^2)^{-N}$$

where $|x|^2 = \sum_{i=1}^n x_i^2$. We denote the rapidly decreasing functions by $\mathcal{S}(\Omega)$. The space $\mathcal{S}(\Omega)$ is known as Schwartz space.

We have already seen that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ then $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$. So to extend the Fourier transform to distributions we introduce a new class of distributions. These are the *tempered distributions*.

Let $\Omega \subseteq \mathbb{R}^n$. The space of tempered distributions on Ω, written $\mathcal{S}'(\Omega)$ is constructed in the same way as the space $\mathcal{D}'(\Omega)$. That is, $\mathcal{S}'$ is the dual of $\mathcal{S}(\Omega)$. So if $\phi \in \mathcal{S}$, and $\Lambda \in \mathcal{S}'(\Omega)$, then $\Lambda(\phi) \in \mathbb{C}$. As before, if we take f to be a locally integrable function, then the linear functional defined by

$$\Lambda_f \phi = \int_\Omega f\phi$$

is a tempered distribution.

The formal definition of a tempered distribution may be stated as follows.

Definition 14.21. $\Lambda \in \mathcal{D}'$ is a tempered distribution, if Λ extends to continuous linear functional on $\mathcal{S}$.

So the tempered distributions are distributions $\Lambda \in \mathcal{D}'$ with an additional property. Namely that we can take Schwartz functions as the test functions and the linear functional is still defined. That is

$$|\Lambda(\phi)| < \infty$$

for all $\phi \in \mathcal{S}(\Omega)$.

We saw earlier that for a compactly supported distribution Λ, can be extended to $(C^\infty)'$. Since $\mathcal{S}(\Omega) \in C^\infty(\Omega)$ every compactly supported distribution is tempered.

The Fourier transform of a tempered distribution is easy to define.

Definition 14.22. Let Λ be a tempered distribution on $\Omega \subseteq \mathbb{R}^n$. The Fourier transform of Λ, denoted by either $\mathcal{F}\Lambda$ or $\widehat{\Lambda}$, is defined by the expression

$$\mathcal{F}\Lambda(\phi) = \widehat{\Lambda}(\phi) = \Lambda(\widehat{\phi}), \tag{14.13}$$

for all test functions $\phi \in \Omega$.

Example 14.8. The Dirac delta function at a has compact support, hence $\delta_a \in \mathcal{S}'$. To calculate $\widehat{\delta_a}$ we write

$$\widehat{\delta_a}(\phi) = \delta_a(\widehat{\phi}) = \widehat{\phi}(a),$$

for every $\phi \in \mathcal{S}(\Omega)$.

Because of the way the Dirac delta function is defined, it is easier to see the Fourier transform, just by applying the usual Fourier transform definition. More precisely, if $\Omega = \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \delta_a(x)e^{-ixy}dx = \int_{-\infty}^{\infty} \delta(x-a)e^{-ixy}dx = e^{-iay}. \tag{14.14}$$

Taking $a = 0$ gives $\mathcal{F}^{-1}(1) = \delta(x)$. So the Fourier transform of the Dirac delta centered at zero is 1, and the inverse Fourier transform of 1 is the Dirac delta.

We can compute the Fourier transform of the derivative of the Delta function in the same way. Taking $a = 0$ gives

$$\int_{-\infty}^{\infty} \delta'(x)e^{-ixy}dx = -\int_{-\infty}^{\infty} (-iy)e^{-ixy}\delta(x)dx = iy. \tag{14.15}$$

Exercise. Compute by the above method the Fourier transform of $\delta''(x-a)$.

The reader should see the significance of this extension of the Fourier transform. As a regular function, the Fourier transform of 1 does not exist, since it is not integrable. However, if we think of 1 as a distribution, its Fourier transform is defined. We can also compute the Fourier transform of a polynomial. We will simply calculate the Fourier transform of x.

We want to make sense of the expression

$$\int_{-\infty}^{\infty} xe^{-ixy}dx.$$

This makes no sense as an ordinary integral. If we treat it as a distribution however, we note that $x \in \mathcal{S}'(\mathbb{R})$, since for any $\phi \in \mathcal{S}, x\phi \in \mathcal{S}$. Hence $\int_{-\infty}^{\infty} xe^{-ixy}dx$ exists as a distribution. Now, we can perform the following manipulation.

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-ixy}dx &= -\int_{-\infty}^{\infty} \frac{1}{i}\frac{d}{dy}e^{-2\pi ixy}dx \\ &= i\frac{d}{dy}\int e^{-ixy}dx \\ &= i\delta'(y),\end{aligned}$$

since $\int e^{-ixy}dx = \delta(x)$. Thus the Fourier transform of x is defined as a distribution and we have

$$\mathcal{F}(x)(y) = i\delta'(y). \tag{14.16}$$

Using these methods it is possible to compute the Fourier transform in the sense of distributions for a whole range of functions which lie well outside the usual domain of the Fourier transform.

We conclude our section on distributions with a brief discussion of the Payley-Wiener theorems. These characterise the behaviour of the Fourier transform of particular functions.

Theorem 14.23 (Paley-Wiener 1). *(1) If $\phi \in \mathcal{D}(\mathbb{R}^n)$ is supported in $B_r = \{x : |x| < r\}$ and*

$$f(z) = \int_{-\infty}^{\infty} \phi(t)e^{-izt}dt$$

then f is an entire function and for $N = 1, 2, 3, \dots$ there exists γ_N such that

$$|f(z)| \leq \gamma_N(1 + |z|^2)^{-N}e^{r|\Im(z)|}$$

(2) The converse holds. So if

$$|f(z)| \leq \gamma_N(1 + |z|^2)^{-N}e^{r|\Im(z)|}$$

is true then there exists some ϕ such that,

$$f(z) = \int_{-\infty}^{\infty} \phi(t)e^{-izt}dt$$

This theorem says that the Fourier transform of a compactly supported function C^∞ function is analytic everywhere. The Fourier transform has to satisfy a particular growth condition. Conversely any entire function satisfying the given growth condition, has to be the Fourier transform of a compactly supported, C^∞ function. There is a corresponding theorem for distributions.

Theorem 14.24 (Paley-Wiener 2). *(a) Suppose $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$ is supported in B_r and has order N. Set $f(z) = \Lambda(e^{-iz(\cdot)})$. Then $f(z)$ is an entire function and $f|_{\mathbb{R}^n}$ is the Fourier transform of Λ. Further there is a constant c such that*

$$|f(z)| \leq c(1 + |z|^2)^{-N}e^{r|\Im(z)|}$$

(b) The converse is true. ie if f satisfies

$$|f(z)| \leq c(1 + |z|^2)^{-N}e^{r|\Im(z)|}$$

then there exists some $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$ with support in B_r such that,

$$f(z) = \Lambda e^{-iz(\cdot)}dt$$

15. Partial Differential Equations

In this chapter we will take a more theoretical look at the theory of Partial Differential Equations, (PDEs). In particular we are interested in solving PDE's with (in general) complex coefficients.

15.1. **Local Solvability.** To simplify notation let $D_\alpha = (2\pi i)^{-|\alpha|}D^\alpha$, where α is a multi-index. So in this notation

$$\widehat{(D_\alpha f)}(\xi) = \xi^{|\alpha|}\widehat{f}(\xi).$$

A partial differential operator with constant coefficients is an expression of the form,

$$L = \sum_{|\alpha|\le k} a_\alpha D_\alpha, \quad a_\alpha \in \mathbb{C}$$

If $\sum_{|\alpha|=k} |a_\alpha| \neq 0$ then L is said to have order k.

This just means that the order of the highest derivative in L is k. If $p(\xi)$ denotes the polynomial $p(\xi) = \sum_{|\alpha|\le k} a_\alpha \xi^\alpha$, then we have $L = p(D)$ and $(\widehat{Lf})(\xi) = p(\xi)\widehat{f}(\xi)$. We can also consider nonconstant coefficient, linear partial differential operators (PDO's) by allowing the a_α to be functions of $x \in \mathbb{R}^n$. ie let

$$L = \sum_{|\alpha|\le k} a_\alpha(x) D_\alpha$$

Studying operators with variable coefficients is substantially harder. We will consider constant coefficient equations unless otherwise specified. Our primary concern is the following.

Problem Given $f \in C^\infty(\Omega), \Omega \subseteq \mathbb{R}^n$, find a distribution Λ such that

$$L\Lambda = f.$$

We say that L is locally solvable at $x_0 \in \Omega$ if there is a neighborhood V of x_0 and a distribution Λ such that $L\Lambda = f$ holds for all points $x_0 \in V$.

Remark We may as well assume for local solvability, that f has compact support. If not, just multiply f by a function $\phi \in \mathcal{D}(\Omega)$ which takes the value 1 near x_0.

Aside Given a compact set $K \in \Omega$, it is possible to construct a function such that,

$$\phi = \begin{cases} 1 & : \quad x \in K \\ 0 & : \quad x \text{ sufficiently outside } K \end{cases}$$

and $\phi \in C^\infty(\Omega)$. Functions like ϕ are sometimes called mollifiers. Mollifiers are very useful. Notice that if we take a function f which does not have compact support, and multiply by a mollifier ϕ with support K, then $f(x)\phi(x) = f(x)$ for all $x \in K$ and $f(x)\phi(x) = 0$ for all $x \notin K$. Thus we can construct a compactly supported function which is equal to f on some specified compact set K.

One of the two most important results in the theory of linear constant coefficient operators on $\mathbb{R}^n$ is the following. It tells us when it is possible to solve a given PDE.

Theorem 15.1. *Let L be a PDO with constant coefficients. If $f \in \mathcal{D}(\mathbb{R}^n)$, then there is a $u \in C^\infty(\mathbb{R}^n)$ satisfying,*

$$Lu = f$$

Proof. (The idea). We take the Fourier transform. Since $f \in \mathcal{D}(\mathbb{R}^n)$, $\widehat{f}$ exists and $\widehat{f}$ is an entire function by the Paley-Wiener theorem. Also,

$$(\widehat{Lu})(\xi) = p(\xi)\widehat{u}(\xi).$$

So taking the Fourier transform of the PDE $Lu = f$, we have,

$$(\widehat{Lu})(\xi) = \widehat{f}(\xi),$$

which implies that

$$p(\xi)\widehat{u}(\xi) = \widehat{f}(\xi). \tag{15.1}$$

Hence

$$\widehat{u}(\xi) = \frac{\widehat{f}(\xi)}{p(\xi)}. \tag{15.2}$$

So by the Fourier inversion theorem we should have

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\widehat{f}(\xi)}{p(\xi)} e^{i\xi\cdot x} d\xi \tag{15.3}$$

There is an obvious difficulty with the expression (15.3). Since $p(\xi)$ is a polynomial, with potentially many real zeros, there is no reason to suppose that the integral will actually converge.

But $\widehat{f}(\xi)$ is entire and so is $p(\xi)$, because it is a polynomial, and all polynomials are entire. So by Cauchy's theorem, we can deform the contour of integration to miss the poles of $1/p(\xi)$. The technical details of this are quite involved.

The idea is to show that if we change the contour of integration in (1) we get a distribution that is in fact a C^∞ function solving out PDE. This actually works. The details are in Rudin's book on Functional Analysis. □

15.2. **Fundamental Solutions.** Given a linear partial differential operator $P(D_\alpha)$ can you (locally) solve the PDE

$$P(D_\alpha)\Lambda = f,$$

where f is now a distribution? As before, we may assume that f has compact support. A distribution Λ is called a fundamental solution for $L = P(D)$ if it satisfies

$$P(D_\alpha)\Lambda = \delta_0.$$

Lemma 15.2. *If $f \in \mathcal{D}'$ has compact support and Λ is a fundamental solution, then $u = \Lambda * f$ is a distribution which is a solution of $Lu = f$. We also say u is a distributional solution, or a solution in the sense of distributions.*

Proof. This is an easy calculation.

$$Lu = L(\Lambda * f) = (L\Lambda) * f = (\delta * f) = f$$

□

Remark If you can find a fundamental solution with compact support then $P(D_\alpha)u = f$ is globally solved by $u = \Lambda * f$.

The most important fact about linear constant coefficient Partial Differential Operators is that they all have fundamental solutions. This is a very famous result in the theory of PDEs. It is a deep result and we will not prove it.

Theorem 15.3 (Malgrange-Ehrenpreis)**.** *Every linear PDO with constant coefficients has a fundamental solution.*

The existence of fundamental solutions was established independently by L.Ehrenpreis (Amer.J.Math, col.76, pp.883-903, 1954) and by B.Malgrange in his thesis (Ann. Inst. Fourier, vol 6, pp. 271-355,1955-1956).

It is imperative that the PDO has constant coefficients, for if it does not the theorem fails. The first example of this fact can be found in Annals of Mathematics 66, 1957 p155-156 where H.Lewy showed that if

$$L\phi = -\phi_x - i\phi_y + 2i(x + iy)\phi_z$$

then,

Theorem 15.4. *There exists a C^∞ function $f(x, y, z)$ such that the equation $Lu = f$ has no local solution anywhere.*

The moral of this is that constant coefficient operators are 'easy' and variable coefficient operators are hard. In fact constant coefficient operators have their own unique features, which is why the word 'easy' is in quotation marks. However, we will not go into this any further.

16. The Axiom of Choice and Zorn's Lemma

An important though rarely remarked upon feature of mathematics is that we like to be able to pick elements from different sets and combine them into new sets. For example, we might like to take two whole sets A and B and combine them by taking their Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$. Now here is a question. If A and B are non empty sets, does it follow that $A \times B$ is nonempty? The answer would seem to be fairly obviously yes. What about if we have an infinite family A_i where i ranges over some index set I. (For example we might take $I = \mathbb{N}$.) If each A_i is nonempty, does it follow that $\prod_{i \in I} A_i$ is also nonempty? The answer again seems to obviously be yes, but the fact is, there is no way of proving this using the axioms of set theory. Because this fairly important property of Cartesian products cannot be deduced from the axioms of set theory, it must itself be taken as an axiom. It is known as the *axiom of choice.*
Axiom of Choice. If $\{A_i, i \in I\}$ is a nonempty family of sets such that $A_i \neq \emptyset$, then $\prod_{i \in I} A_i \neq \emptyset$. (Here $\emptyset$ is the empty set).

There are a number of equivalent formulations for the axiom of choice. Probably the most useful is the following.

If $\{A_i, i \in I\}$ is a nonempty family of pairwise disjoint sets such that $A_i \neq \emptyset$ for each $i \in I$, then there exists a set $E \subseteq \cup_{i \in I} A_i$, such that $E \cap A_i$ consists of precisely one element for each $i \in I$.

This formulation says that given a family of sets, it is possible to select an element from each one and form a new set. This statement seems obvious, but cannot be proved and it is has some profoundly unsettling consequences. The proof of the existence of a non Lebesgue measurable set depends upon the axiom of choice and the existence of a nonmeasurable set in turn leads to the *Banach-Tarski paradox.* Recall that this says that a sphere can be decomposed into non measurable sets and the pieces reassembled into two spheres of exactly the same size as the original sphere, without stretching or deforming the pieces.

Many mathematicians would like to remove the axiom of choice from mathematics altogether, but this is not an easy thing to do. Attempts to do mathematics without the axiom of choice have been made, but are not generally considered to be terribly useful.

Equivalent to the axiom of choice is something called Zorn's lemma. To formulate this, we introduce the idea of a *partial order.*

Definition 16.1. A relation on a set denoted $\leq$ is a partial order if it satisfies

(a) $x \leq x$ holds for all x. (reflexivity)

(b) If $x \leq y$ and $y \leq x$ then $x = y$. (antisymmetry)

(c) If $x \leq y$ and $y \leq z$ then $x \leq z$. (transitivity).

A set with a partial order is called partially ordered.

We could equally well use $\geq$ but this doesn't make any logical difference

. The simplest example of a partially ordered set is the real numbers. We can say that one real number is less than or equal to another real number. The real numbers are actually totally ordered. (However the Complex numbers are not ordered. We cannot say that one complex number is bigger than another complex number. Suppose we could. Now suppose that $i > 0$. Then if we multiply both sides by i we get $i^2 = -1 > 0$ which is false. So i is not greater than zero. Okay, so now take $i < 0$. If we now multiply by i, the inequality must reverse because i is negative and so we again have $-1 > 0$. A contradiction. Thus we cannot order the complex numbers).

We can also introduce partial orders on other structures, such as groups, but we will not go into that here.

Now suppose that X is a partially ordered set. A subset $Y \subset X$ is said to be a *chain*, if for each pair $x, y \in Y$ either $x \leq y$ or $y \leq x$. A chain is also referred to as a totally ordered set. Now suppose that there is an element $u \in Y$ with the property that for all $x \in Y$ we have $x \leq u$. Then we call u an upper bound for Y. An element $m \in X$ is called a *maximal* element of X if $m \leq x$ always implies $x = m$. In other words, nothing is 'above' m in the partial order of the set. (Note: In a partially ordered set, a maximal element *does not* have to be unique).

An obvious question to ask is when a partially ordered set has a maximal element?

Zorn's Lemma. If every chain of a maximally ordered set X has an upper bound in X then X has a maximal element.

Zorn's Lemma is logically equivalent to the axiom of choice. If we assume Zorn's lemma, then the axiom of choice can be proved from it. If we conversely assume the axiom, then we can prove Zorn's Lemma from it.

Zorn's Lemma is used in the proof of many fundamental results of mathematics. For example, the proof of the deeply important result that every Hilbert space has an orthonormal basis uses Zorn's Lemma. Zorn's Lemma is extensively used in a great deal of modern analysis.

16.1. **More on Hilbert Spaces.** We will now illustrate one of the many uses of Zorn's Lemma by proving an important fact about Hilbert spaces. Previously we stated the result that every Hilbert space has an orthonormal basis. In fact if the Hilbert space is *separable*, the basis is countable.

First, we define the term *separable*

Definition 16.2. Let X be a metric space. (For example a normed vector space). If there exists a countable subspace $K \subset X$ such that every element of X is a limit of a sequence in K, then X is said to be separable.

Most Hilbert and Banach spaces that we encounter are separable. There do exist nonseparable Hilbert spaces however. The best known example of a nonseparable Hilbert space was constructed by Harald Bohr, the brother of the physcist Niels Bohr. This Hilbert space consists of so called continuous, *almost periodic functions* on $\mathbb{R}$. We need not worry too much about the precise definition of almost periodic function, but the idea is that although there does not exist a $T > 0$ such that $f(x+T) = f(x)$ for all x, we can find a $T > 0$ such that for all x, $|f(x+T) - f(x)| < \epsilon$, where ϵ is small.

The commonly encountered Hilbert spaces, such as $L^2(\mathbb{R})$ and l^2 are separable. Every element of these spaces is the limit of a linear combination of basis vectors. For example, in l^2 every element is a linear combination of elements of the form

$$(1, 0, 0, 0, ...),\ (0, 1, 0, 0, 0, ...),),\ (0, 0, 1, 0, 0, 0, ...)$$

etc. We also know that a basis for $L^2(\mathbb{R})$ consists of the Hermite functions $H_n(x)e^{-\frac{1}{2}x^2}$, where $H_n(x)$ is the nth Hermite polynomial,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

forms a basis for $L^2(\mathbb{R})$. The solution of a Sturm-Liouville problem on some interval $[a, b]$ will give an orthonormal basis for $L^2[a, b]$.

In fact every Hilbert space has an orthonormal basis. How do we know this? To prove this, we define an orhonormal set in a Hilbert space H to be one in which $\|e\| = 1$ for each $e \in K$ and $(e, f) = 0$ for each $e, f \in K$ with $e \neq f$. An orthonormal set is complete if every element of H can be written as a linear combination of elements of K.

Now suppose that K is an orthonormal set in a Hilbert space H. We say that K is an orthonormal basis if for every $x \in H$ we can write

$$x = \sum_{y \in K_x} (x, y) y \tag{16.1}$$

where $K_x = \{y; y \in K \text{ and } (x, y) \neq 0\}$. i.e. K is complete. A result stated earlier is

Theorem 16.3. *Let K be an orthonormal set in a Hilbert space H. Then the following conditions are equivalent.*

(a) *K is complete;*

(b) *The closed linear subspace spanned by K is H;*

(c) *K is an orthonormal basis;*

(d) *For any $x \in H$, Parseval's formula holds:*

$$\|x\|^2 = \sum_{y \in K_x} |(x, y)|^2.$$

The proof of this is not difficult.

Theorem 16.4. *In every Hilbert space there is an orthonormal basis.*

Proof. Consider the class of all linearly independent orthonormal sets in a Hilbert space H. We define a partial order on this set by inclusion. i.e $K_1 \leq K_2$ if $K_1 \subset K_2$. Since the largest possible such set is H itself, each orthonormal set is a chain with an upper bound. So by Zorn's Lemma, there exists a maximal orthonormal set K. Since K is maximal, it must be complete and hence it is an orthonormal basis. □

We will not prove the following result, though the proof is not very difficult.

Lemma 16.5. *If H is a separable Hilbert space, then it has a countable orthonormal basis.*

Countable simply means that the basis elements can be listed as e_1, e_2, e_3 etc.

17. The Baire Category Theorem

How strange can a function be? This was a question that was asked by analysts in the 19th century after the discovery of a series of increasingly pathological functions. In the time of Euler, little attention was paid to the notion of concepts like continuity and smoothness. The very concept of a function was not properly formulated and many ideas about the behavious of functions were wrong. For example, it was believed that if two analytic expressions agreed on an interval, then they agreed everywhere, a belief that strikes modern mathematicians as astonishingly naive. It was also assumed that if a function was continuous, it more or less had to also be differentiable. This turned out to be completely wrong, when Weierstrass constructed an example of a function that was continuous everywhere, and nowhere differentiable. Weierstrass proved the following remarkable result.

Theorem 17.1. *If $a \geq 3$ is an odd integer and if $0 < b < 1$ such that $ab > 1 + \frac{3\pi}{2}$, then the function*

$$f(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x)$$

is continuous everywhere and differentiable nowhere.

The continuity of the function follows from the fact that it is a sum of continuous functions. It is not hard to prove that the series converges. This is an easy consequence of the Weierstrass M test.

Since $|b^k \cos(\pi a^k x)| < b^k$ and

$$\sum_{n=0}^{\infty} b^k = \frac{b}{1-b},$$

the series defining f in the theorem is uniformly convergent. A sequence of continuous functions which converges uniformly, must converge to a continuous limit. So f is continuous. However the series of derivatives $-\sum_{k=0}^{\infty} b^k \pi a^k \sin(\pi a^k x)$ is a divergent series, suggesting that the function is not differentiable. This is true, but the proof is quite difficult.

Following Weierstrass's discovery, pathological functions were constructed with disturbing frequency. Quite bland looking examples often had hidden complexities. This was very important in the development of Fourier analysis, where we often need to consider sequences of continuous functions, and perform operations where we reverse the order of integration and summation. In Euler's time, this operation was carried out without even considering whether such a thing is possible. This is because they were considering only sequences of polynomials and these behave as well as a mathematician could desire. Of course, reversing the order of summation and integration cannot be done with impunity, as we have seen.

Even sequences of well behaved functions can converge to very badly behaved functions. Recall the example

$$f_{k,j}(x) = (\cos(\pi k! x))^{2j}.$$

This is a sequence of continuous, indeed analytic functions. Yet the limit function is not even continuous. (It is called Dirichlet's function). This is quite bad behaviour contained from functions which are themselves all analytic.

The kind of behaviour that we can expect from a function turns out to be quite bizarre, but are there limits? This was one of the most important questions in analysis in the 19th century. In this section we will discuss some aspects of it, and present a major result that emerged from the consideration of the problem. We begin by focusing on a question about continuity. How discontinuous can a function be?

It is not difficult to find a function which is continuous at every irrational number, but discontinuous at every rational number. Here is an example due to Johannes Karl Thomae (1840-1921). Let $x \in (0,1)$ and define

$$r(x) = \begin{cases} \frac{1}{q} & x = p/q \textit{ in lowest terms} \\ 0 & x \notin \mathbb{Q}. \end{cases} \tag{17.1}$$

This function is continuous at every irrational number and discontinuous at every rational! To see this, we need a lemma.

Lemma 17.2. *If* $a \in (0,1)$*, then* $\lim_{x\to a} r(x) = 0$.

Proof. The proof is rather subtle and can be skipped on a first reading.

Let $\epsilon > 0$ and choose an integer $N > 0)$ such that $1/N < \epsilon$. Now in $(0,1)$, there are only finitely many rationals in lowest terms, whose denominator is less than or equal to N. This is obvious with a little thought. The only fractions with denominator 5 or smaller are $1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5$. Because the collection of such fractions is finite, given a, we can find an interval around a, $(a-\delta, a+\delta) \subset (0,1)$ such that none of the given fractions with denominator less than N lies in this interval. (Except possibly a, if it is rational). Now choose $x \in (a-\delta, a+\delta)$. If $x = p/q$, then $|r(x) - 0| = |r(p/q)| = 1/q < \epsilon$. This is because $q > N$ if $p/q \neq a \in (a-\delta, a+\delta)$. If x is irrational, then $|r(x) - 0| = 0 < \epsilon$. Either way, if $\epsilon > 0$, then we have a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|r(x) - 0| < \epsilon$ and so $\lim_{x\to a} r(x) = 0$. □

This lemma immediately implies that r has the above stated property. If a is rational, then $r(a) \neq 0$. So let x_n be a sequence in $(0, 1$ such that $x_n \to a$. Then $\lim_{n\to\infty} r(x_n) = 0 \neq r(a)$. So r is not continuous at any rational a. If a is irrational however, then $\lim_{n\to\infty} r(x_n) = 0 = r(a)$. So r is continuous at a, if a is irrational.

The discovery of this function led to the question of whether or not it is possible to find a function with the opposite property. Can we find a function which is continuous at the rationals, but discontinuous at the irrationals? This is not an easy question at all, and getting any intuition about the problem is tough. One can try and construct such a function, but attempts to do so all failed. Eventually Volterra proved that no such function actually exists. In fact he did something deeper. He proved the following truly remarkable result. The proof requires the Baire Category Theorem.

Theorem 17.3 (Volterra)**.** *Let* $C_f = \{x \in \mathbb{R} : f(x) \text{ is continuous at } x\}$ *and* $D_f = \{x \in \mathbb{R} : f(x) \text{ is discontinuous at } x\}$. *Then there cannot exist a function* g *such that* $C_f = D_g$ *and* $D_f = C_g$.

In other words, there do not exist two functions f and g such that the set of points where f is continuous is the set of points where g is discontinuous and vice versa. This is a remarkable result and is one of the theorems which puts some kind of limit on the question of how strange a function can be. The theorem immediately implies that there is no function continuous on the rationals and discontinuous on the irrationals. To see this, suppose that such a function does exist and call it g. Now recall our function $r(x)$. We would then have the following situation: $D_g = C_r$ and $C_g = D_r$. This is impossible by Volterra's theorem.

It turns out that there is an easier way to prove that there is no function continuous on the rationals and discontinuous on the irrationals. We will present this later. The idea is due to Renè Baire (1874-1932). Baire had the brilliant insight that many questions about functions reduce to questions about sets and proved a deep theorem in his PhD thesis, which is now called the Baire Category Theorem. Baire worked with real numbers, but his ideas actually immediately extend to complete metric spaces and have many applications in Functional Analysis. Some of the major results in Functional Analysis use the Baire Category Theorem for their proof.

Recall that a subset X of a metric space M is said to be dense in M if every point $x \in M$ is the limit of a sequence $\{x_n\} \in X$. The most obvious example is the rational numbers: $\mathbb{Q}$ is dense in $\mathbb{R}$ as every real number is the limit of a sequence of rational numbers. Baire moved from the idea of a dense set to the idea of a *nowhere dense* set.

Definition 17.4. Let X be a subset X of a metric space M We say that X is nowhere dense if $M \setminus \bar{X}$ is dense in M. Here $\bar{X}$ is the closure of X.

What this means is that if we take the closure of X away from M, we are still left with a subset of M which is dense in M. In other words, the set X is very small in some sense. If you remove the closure of a nowhere dense set from a metric space, you don't take out very much of the mass of the space. Hankel preferred the term a sparse set to a nowhere dense set. They are also called meager sets. Baire's terminology was incredibly bland and has often been criticised, because it makes it difficult to get any insight into what is actually going on

An example of a nowhere dense set is provided by considering the interval $[0, 1]$ and the set $X = \{\frac{1}{k}, k = 1, 2, 3, , , \}$. If we take the closure of X we get X together with the point 0. Now remove this from $[0, 1]$. We end up with a set that has a countable number of points removed and the closure of this set is again $[0, 1]$. So X is nowhere dense.

The union of two nowhere dense sets is again nowhere dense. However a countable union does not have to be nowhere dense. A single point is nowhere dense. But the rationals are dense and countable, and hence a union of nowhere dense sets. Baire gave the following definition.

Definition 17.5. A set is of the first category if it is a countable union of nowhere dense sets. A subset that is not of the first category is said to be of the second category.

This terminology was criticised for being so bland as to be almost useless. The term "first category" tells you nothing at all about what is happening, but it is still used.

Theorem 17.6 (Baire Category Theorem)**.** *Any nonempty, complete metric space is of the second category.*

There are equivalent formulations which give more insight than the version stated above. Here is one of the more useful formulations. The proof is actually not that difficult.

Theorem 17.7. *If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of a complete metric space M, then $\bigcap_{n=1}^{\infty} U_n$ is also dense in M.*

This tells us that countable intersections of dense subsets of metric spaces are still dense. This means that if X and Y are dense in M, then $X \bigcap Y$ is still dense in M. This is of great importance in functional analysis, where we often deal with dense subsets. (A basis for example). The Baire category theorem has many applications in analysis. We first give another formulation of the Baire Category Theorem, which is actually how Baire formulated the result on $\mathbb{R}$.

Theorem 17.8 (Baire)**.** *Let $F = \bigcup_{n=1}^{\infty} P_n$ be a union of nowhere dense subsets of (α, β). Then there is a point in (α, β) which is not in F.*

Here is a simple application of this result.

Theorem 17.9 (Cantor)**.** *A sequence of points cannot exhaust an interval. That is, the set of points in any interval (α, β) is uncountable.*

Proof. Let $\{x_k\}$ be a sequence of points in (α, β). The set of points in the sequence is then $F = \{x_1, x_2, x_3, ...\}$, which is a countable union of single points, and hence nowhere dense sets. Baire's theorem says that there is a point in (α, β) not contained in F. Hence no countable sequence of numbers can give every point in an interval. □

We now give a proof of the result first established by Volterra. We define the oscillation of a function f on an interval I by

$$\omega_f(I) = \sup\{f(x) : x \in I\} - \inf\{f(x) : x \in I\}. \tag{17.2}$$

So the oscillation of a function is (essentially) the difference between its maximum and minimum values on the given interval. We can define the oscillation at a point as well. At a single point $a \in I$ we define the oscillation of f at a by

$$\omega_f(a) = \inf\{\omega_f(J) : J \subseteq I, J \text{ is an open interval containing } a\}. \tag{17.3}$$

A technical lemma is needed.

Lemma 17.10. *A bounded real valued function f defined on an open interval I is continuous at $a \in I$ if and only if $\omega_f(a) = 0$ and the set $\{x \in I : \omega_f(x) < \epsilon\}$ is an open set.*

Proof. First suppose that f is continuous. Recall that f is continuous at a if given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. This means that if $x \in (x - \delta, x + \delta)$, then $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Now by definition

$$\begin{aligned}\omega_f(a) &\leq \omega_f((a - \delta, a + \delta)) \\ &= \sup\{f(x) : x \in (a - \delta, a + \delta)\} - \inf\{f(x) : x \in (a - \delta, a + \delta)\}. \\ &\leq f(a) + \epsilon - (f(a) - \epsilon) = 2\epsilon.\end{aligned}$$

This holds for all $\epsilon > 0$, so that if f is continuous at a, then $\omega_f(a) = 0$.

To prove the converse, suppose that $\omega_f(a) = 0$. Then for any $\epsilon > 0$, there is an open interval $J \subseteq I$, such that $\omega_f(J) < \epsilon$. As J is open, then there exists a $\delta > 0$ such that $J \subseteq (a - \delta, a + \delta)$. Hence $|x - a| < \delta$ implies that

$$\begin{aligned}|f(x) - f(a)| &\leq \sup\{f(x) : x \in (a - \delta, a + \delta)\} \\ &\quad - \inf\{f(x) : x \in (a - \delta, a + \delta)\} \\ &= \omega_f((a - \delta, a + \delta)) \leq \omega_f(J) < \epsilon.\end{aligned}$$

Hence f is continuous at a.

Finally, we need to prove that $\{x \in I : \omega_f(x) < \epsilon\}$ is open. Recall that a set A is open if for any $x \in A$ we can find an open interval J, such that $x \in J$ and $J \subseteq A$.

Now let $\epsilon > 0$ and take $x_0 \in \{x \in I : \omega_f(x) < \epsilon\}$. Suppose that J is an open interval containing x_0 with $\omega_f(J) < \epsilon$. For any other $y \in J$ we have $\omega_f(y) \leq \omega_f(J)\epsilon$. Thus $J \subseteq \{x \in I : \omega_f(x) < \epsilon\}$. This proves that $\{x \in I : \omega_f(x) < \epsilon\}$ is open. $\square$

With this result and the Baire Category Theorem, we can now prove the following result.

Theorem 17.11 (Volterra)**.** *There is no function defined on* $(0, 1)$ *that is continuous at each rational point of* $(0, 1)$ *and discontinuous at each irrational point* $(0, 1)$.

Proof. We want to proceed by contradiction. The question is what sort of contradiction can we find? The idea is to write the interval $[0, 1]$ as a union of nowhere dense subsets. This would then prove that $[0, 1]$ is of the first category. Since $[0, 1]$ is complete, it must be of the second category, by Baire's Theorem. Hence we have a contradiction.

The sets we work with are defined as follows. Assume that such a function exists. Let

$$U_n = \left\{ x \in (0, 1) : \omega_f(x) < \frac{1}{n} \right\}. \tag{17.4}$$

By the previous lemma, this set is open. Now the function is supposed to be continuous at each rational number in $(0, 1)$ and hence at each rational x, we have by the lemma $\omega_f(x) = 0 < \frac{1}{n}$. Thus if $x \in \mathbb{Q}$, then certainly $x \in U_n$. The function is not continuous at any irrational point, so given x_0 irrational, there is a number N, such that $\omega_f(x_0) > 1/N$, since $\omega_f(x_0)$ cannot be zero. So certainly x_0 is not in every U_n. It thus follows that the only points in every U_n are the rational numbers. Hence

$$\bigcap_{n=1}^{\infty} U_n = \mathbb{Q} \cap (0, 1).$$

The rational numbers are dense in $(0, 1)$, so each U_n is also dense, since each U_n contains the rationals in $(0, 1)$. Now define $V_n = (0, 1) \setminus U_n, n = 1, 2, 3, ...$ Each V_n is nowhere dense in $(0, 1)$, (and hence also in $[0, 1]$.) To see why, observe that

$$(0, 1) \setminus V_n = (0, 1) \setminus ((0, 1) \setminus U_n,) = U_n \tag{17.5}$$

and U_n is dense in $(0, 1)$. Thus by definition, V_n is nowhere dense.

Further

$$(0, 1) \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} V_n.$$

Now $\mathbb{Q}$ is a countable set, so it can be written as $\mathbb{Q} = \cup_{n=1}^{\infty} r_n$ for some sequence $\{r_n\}$. Each point $\{r_n\}$ is nowhere dense. We thus have

$$[0, 1] = (\cup_{n=1}^{\infty} r_n) \cup (\cup_{n=1}^{\infty} V_n) \cup \{0\} \cup \{1\}. \tag{17.6}$$

In other words, we have written $[0, 1]$ as a countable union of nowhere dense sets, proving that it is of the first category. This is a contradition, as $[0, 1]$ is of the second category by Baire's Theorem. This proves the result. There does not exist a function continuous at the rational numbers and discontinuous at the irrational numbers. □

The Baire Category Theorem is used in the proofs of some of the major results of functional analysis, including the Open Mapping Theorem and the Banach-Steinhaus Theorem.

Theorem 17.12 (The Open Mapping Theorem). *Let X and Y be Banach spaces and suppose that $T : X \to Y$ is bounded and linear. If T is also onto, (i.e. $T(X) = Y$), then $T(U)$ is open in Y whenever U is open in X. That is, onto and bounded operators map open sets to open sets.*

Note that the set of bounded linear operators from X to Y is denoted $\mathcal{B}(X, Y)$. It is also a normed linear space and has its own norm. So we may define a norm $\|T\|_{\mathcal{B}(X,Y)}$ for elements. This norm appears in the Banach-Steinhaus Theorem. This theorem gives conditions on a family of opeators which guarantee that each operator in the family is bounded in norm by the same constant.

The Banach-Steinhaus Theorem relates to families of operators. It is also called the principle of uniform boundedness.

Theorem 17.13 (Banach-Steinhaus). *Consider a Banach space X and a normed linear space Y. If T_α is a family of bounded linear operators from X to Y is such that for each α,*

$$\sup\{\|T_\alpha x\|_Y\} < \infty$$

for every $x \in X$, then for each α,

$$\sup\{\|T_\alpha\|_{\mathcal{B}(X,Y)} < \infty.$$

These are two of the big four theorems of functional analysis, the other two being the Hahn-Banach Theorem and the Closed Graph Theorem. We will state and prove all four of these results.

18. Some basic facts about linear operators

Now we will prove some of the major results of functional analysis and give some applications. We need some properties of linear operators first.

On normed linear spaces, we are often interested in bounded linear operators. Consider the mapping $T : X \to Y$, where X and Y are linear spaces. We say T is a bounded linear operator if for all $x, y \in X$ and all scalars a, b, we have

$$T(ax + by) = aT(x) + bT(y),$$

and there exists a constant $M > 0$ sucht that for all $x \in x$ we have

$$\|Tx\|_Y \leq M\|x\|_X.$$

In this notation $\|\cdot\|_X$ is the norm on X, and $\|\cdot\|_Y$ is the norm on Y. These are not necessarily the same norm. Let

$$K = \inf\{M : \|Tx\|_Y \leq M\|x\|_X\}.$$

We say that K is the norm of T and write $\|T\| = K$. It is not hard to show that we also have

$$\|T\| = \inf\{\|Tx\|_Y : \|x\|_X = 1\} = \inf\left\{\frac{\|Tx\|_Y}{\|x\|_X} : \|x\|_X \neq 0\right\}. \quad (18.1)$$

In what follows we will abuse notation and drop the X and Y subscripts from the norm symbols, understanding that $\|Tx\| = \|Tx\|_Y$ and $\|x\| = \|x\|_x$ etc. So we take the norm of z according to the space where z lives.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for every sequence $x_n \to x$, we have $f(x_n) \to f(x)$. By analogy, an operator on a normed space X is said to be continuous if for every $x_n \to x$ we have $Tx_n \to Tx$, where convergence is defined in terms of the norm. It is immediately apparent that a bounded linear operator is continuous. This follows because if $x_n \to x$, then for every $\epsilon > 0$, we can find an N such that for $n \geq N$, we have

$$\|x_x - x\| < \frac{\epsilon}{K}.$$

Now by linearity and boundedness

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq K\|x_n - x\| < K\frac{\epsilon}{K} = \epsilon.$$

Thus $Tx_n \to Tx$ and so T is continuous at x.

Lemma 18.1. *A linear operator continuous at one point in its domain is continuous at all points.*

Proof. Suppose that T is continuous at z. Let $x_n \to x$. Then

$$x_n - x + z \to z.$$

Since T is continuous at z, $T(x_n - x + z) \to Tz$. Now

$$T(x_n - x + z) = Tx_n - Tx + Tz \to Tz.$$

So we must have $Tx_n \to Tx$ and T is also continuous at x. □

One might ask if it is possible to have an operator which is continuous and unbounded? The answer is no.

Theorem 18.2. *A linear transformation $T : X \to Y$ is continuous if and only if it is bounded.*

Proof. We have already seen that boundedness implies continuity. Suppose now that T is continuous and unbounded. Observe that $T0 = 0$. This is because by linearity

$$T(0) = T(0x) = 0T(x) = 0.$$

Now, since T is unbounded, for every n, there is a point x_n' such that

$$\|Tx_n'\| \geq n\|x_n'\|.$$

If there were no such point, n would be a bound.

Now we let $x_n = \frac{x_n'}{n\|x_n'\|}$. Plainly $\|x_n\| = \frac{1}{n} \to 0$ and so $x_n \to 0$. But $\|Tx_n\| > n\|x_n\| > n\frac{1}{n}\|x_n'\|/\|x_n'\| = 1$. This holds for all n. So Tx_n does not converge to 0, because $\|Tx_n\| > 1$ and hence $\|Tx_n\| \nrightarrow 0$. . Hence T is not continuous at 0. This contradicts the previous lemma, so T cannot be continuous.

□

Another important and completely equivalent definition of continuity is the following. T is continuous if and only if the set $\{x \in X : Tx \in U\}$ is open whenever U is open. That is, the inverse image under T of an open set is open.

If we define the operator norm as in (18.1), then the space

$$\mathcal{B}(X, Y) = \{T : X \to Y, \text{bounded and linear}\} \tag{18.2}$$

is a normed vector space. In fact, if Y is a Banach space, then $\mathcal{B}(X, Y)$ is also a Banach space.

Theorem 18.3. *If X is a normed linear space and Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{B}(X, Y)$. Every Cauchy sequence is bounded and hence there is a constant K such that for every x,

$$\|T_n x\| \leq K\|x\|.$$

Now by boundedness and linearity

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| \to 0.$$

Thus $\{T_n x\}$ is a Cauchy sequence in Y. Now Y is Banach and hence complete. So there exists a point $y \in Y$ such that $T_n x \to y$. We define an operator T by $Tx = y$.

Since $\|T_n x\| \leq K\|x\|$ it follows that

$$\|Tx\| \leq \lim_{n\to\infty} \|T_n x\| \leq K\|x\|.$$

It is also clear that T is linear, since $T_n(x + x') = T_n x + T_n x'$ and so $T_n(x + x') \to Tx + Tx'$. Thus T is a bounded linear operator. So $T \in \mathcal{B}(X, Y)$. We now have to prove that $\|T - T_n\| \to 0$. That is, T_n converges to T. We know that $\{T_n\}$ is a Cauchy sequence, so given $\epsilon > 0$ we may find an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $\|T_m - T_n\| < \epsilon$. Thus for all $m, n > N$,

$$\|T_m x - T_n x\| \leq \epsilon\|x\|.$$

Let $m \to \infty$. We then have $\|Tx - T_n x\| \leq \epsilon\|x\|$ if $n > N$. Thus $\|T_n - T\| \to 0$. This completes the proof. $\square$

19. The Hahn-Banach Theorem

Probably the most important result in the subject, this theorem tells us when we can extend operators defined on subspaces to the whole space. We begin with a lemma, that some authors consider to be a version of the Hahn-Banach Theorem. It is important to note that there are different formulations of this result, in different settings.

Theorem 19.1 (Hahn-Banach Lemma)**.** *Let X be a real vector space and let p be a real functional on X satisfying*

$$p(x+y) \leq p(x) + p(y), \;\; p(\lambda x) = \lambda p(x)$$

for all $\lambda \geq 0$, $x, y \in X$. Let f be a linear functional F on a subspace $Y \subset X$ such that

$$f(x) \leq p(x), \tag{19.1}$$

for all $x \in X$. Then there exists a real linear functional F on X such that

$$F(x) = f(x) \tag{19.2}$$

for all $x \in Y$ and $F(x) \leq p(x)$ for all $x \in X$.

Proof. We let $\mathcal{K}$ be the set of all pairs (Y_α, g_α), where Y_α is a subspace of X containing Y and g_α is a real linear functional on Y_α satisfying $g_\alpha(x) = f(x)$, for all $x \in X$, $g_\alpha(x) \leq p(x)$ for all $x \in Y_\alpha$.

We will use Zorn's Lemma, so we define a partial order by inclusion. That is $(Y_\alpha, g_\alpha) \leq (Y_\beta, g_\beta)$ if $Y_\alpha \subset Y_\beta$ and $g_\alpha = g_\beta$ on Y_α. Now every totally ordered subset $\{(Y_\beta, g_\beta)\}$ has an upper bound (Y', g') with $Y' = \bigcup Y_\beta, g' = g_\beta$ on Y_β. There is by Zorn's Lemma a maximal element (Y_0, g_0). To prove the result we must show that $Y_0 = X$ and we take $F = g_0$.

Now we proceed by contradiction, so we assume $Y_0 \neq X$. Suppose that there is an element $y_1 \in X$, but $y_1 \notin Y_0$. Consider the set given by the span of points of the form $x = y + \lambda y$ where λ is real and $y \in Y_0$. Now define a linear functional g_1 on Y_1 by $g_1(y + \lambda y) = g_0(y) + \lambda c$. To derive a contradiction we want to choose the constant c so that

$$g_0(y) + \lambda c \leq p(y + \lambda y_1)$$

for all $\lambda \in \mathbb{R}, y \in Y_0$, because this will give $(Y_1, g_1) \in \mathcal{K}$ and $(Y_0, g_0) \leq (Y_1, g_1)$, $Y_0 \neq Y_1$, which contradicts the maximality of $Y_0, g_0)$.

Now note that for any two points $x, y \in Y_0$,

$$g_0(y) - g_0(x) = g_0(x-y) \leq p(y-x) \leq p(y+y_1) + p(-y_1 - x).$$

So

$$-p(-y_1 - x) - g_0(x) \leq p(y + y_1) - g_0(y).$$

We then have

$$A = \sup_{x \in Y_0}\{-p(-y_1 - x) - g_0(x)\} \leq \inf_{y \in Y_0}\{p(y+y_1) - g_0(y)\} = B.$$

To get our contradiction, take c to be any number $A \leq c \leq B$. Then

$$c \leq p(y + y_1) - g_0(y), \text{all } y \in Y_0 \quad (19.3)$$

$$-p(-y_1 - y) - g_0(y) \leq c, \text{all } y \in Y_0. \quad (19.4)$$

Multiplying the first of these by $\lambda > 0$ and replacing y by y/λ gives

$$\lambda c \leq p(y + \lambda y_1) - g_0(y). \quad (19.5)$$

Now multiply the second equation by $\lambda < 0$ and make the same replacement to get the same result. This tells us that c has the desired property and the result is proved. □

To state the Hahn-Banach Theorem we need the notion of a dual.

Definition 19.2. Let X be a vector space. Then the dual of X, denoted X^* consists of the set of all linear functionals on X. Elements of X^* will be denoted x^*.

Using the Riesz representation theorems, we can identify the duals of various spaces, such as $L^p(X), p > 1$. We can also consider the dual of a dual, which we will briefly mention below. First note that

Theorem 19.3. *If X is a Banach space, then so is X^*.*

Proof. This follows immediately from Theorem 18.3. □

The Hahn-Banach Theorem may be stated as follows.

Theorem 19.4. *Let X be a normed vector space and let Y be a subspace. Then to every $y^* \in Y^*$ there corresponds an $x^* \in X$ such that*

$$\|x^*\| = \|y^*\|, \ x^*(y) = y^*(y), \textit{all } y \in Y.$$

Proof. We only consider the real case, which happens immediately from the Hahn-Banach Lemma. Take $p(x) = \|y^*\|\|x\|$, $f(x) = y^*(x)$ and $x^* = F$. It is obvious that $\|x^*\| \geq \|y^*\|$. Now for any $x \in X$, write $x^*(x) = \theta|x^*(x)|$ where $\theta = \pm 1$. Then

$$|x^*(x)| = \theta x^*(x) = x^*(\theta x) \leq p(\theta x) = \|y^*\|\|\theta x\| = \|y^*\|\|x\|.$$

So $\|x^*\| = \|y^*\|$. □

The dual of a dual is defined as we would expect. Certainly if $x^* : X \to \mathbb{R}$, then we may define an element x^{**} by $x^{**} = x^*(x)$. This leads to an obvious question. First a definition.

Definition 19.5. A vector space is said to be reflexive if $(X^*)^* = X$.

What vector spaces are reflexive? It is not hard to show that $L^p(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$, is reflexive for finite $p > 1$. This is an immediate consequence of the Riesz Representation Theorem. But this fails for $p = 1$.

Theorem 19.6. *The space $L^1(\Omega)$ is not reflexive.*

The fact that L^1 is not reflexive has important consequences in analysis, but a discussion is beyond the scope of the current notes.

The Hahn-Banach Theorem is used extensively in functional analysis, because it allows us to extend an operator from a subspace to the whole space. Let us consider an example. Closely related to the concept of a fundamental solution is that of a Green's function.

Definition 19.7. A function $G(x,y)$ defined for all $y \in \Omega \subset \mathbb{R}^n$ and $x \in \Omega - \{y\}$, is called a Green's function for the Laplace equation $\Delta u = 0$ in Ω if

(i) $G(x,y) = k(x,y) + h(x,y)$, is harmonic in x, for $x \in \Omega$,

(ii) $G(x,y)$ is continuous in x when $x \in \bar{\Omega} - \{y\}$,

(iii) $G(x,y) = 0$ for $x \in \partial\Omega$, the boundary of Ω.

Here $\bar{\Omega}$ is the closure of Ω.

We recall the following result.

Theorem 19.8 (Maximum Principle)**.** *Let u be harmonic in Ω, continuous in $\bar{\Omega}$. Then the maximum and minimum values of u occur on $\partial\Omega$.*

As an application of the Hahn-Banach Theorem, we prove the existence of a Green's function in the case $n = 2$. The extension of a linear functional in the following proof is accomplished by the Hahn-Banach Theorem. The proof requires the extension, and it is only done once. This is typical of the applications of the four major theorems of functional analysis. They are technical results that allow us to make important conclusions in our proofs.

Theorem 19.9. *If $n = 2$ and $\partial\Omega$ is C^1, then Green's function exists.*

Proof. We know that the space of all continuous functions on $\partial\Omega$ is a Banach space under the supremum norm. Denote this space by X and by X' the subspace of X consisting of functions f for which the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on Ω has a solution u. We introduce the linear functional L_y on X' by $L_y(f) = u(y)$. By the maximum principle, L_y is bounded and has norm 1. So we can apply the Hahn-Banach Theorem to extend L_y to a linear functional on X, which we also denote by L_y.

For $z \notin \partial\Omega$, consider $f_z \in X$ defined by

$$f_z(x) = \ln|x - z|, \ x \in \partial\Omega. \tag{19.6}$$

Let $k_y(z) = L_y(f_z)$. We show that $k_y(z)$ is harmonic. The fact that f_z satisfies Laplace's equation in the z variable is clear. Now let $z' =$

$(z_1 + \delta, z_2)$, then by the fact that L_y is continuous we have

$$\lim_{\delta \to 0} \frac{k_y(z') - k_y(z)}{\delta} = \lim_{\delta \to 0} L_y \left(\frac{f_{z'} - f_z}{\delta} \right) = L_y \left(\frac{\partial f_z}{\partial z_1} \right) \tag{19.7}$$

Continuing in the same vein we have $\delta k_y(z) = L_y(\Delta f_z) = 0$, since f_z is harmonic and L_y is continuous, so $L_y(0) = 0$. Since $\ln|x - z|$ is harmonic in $x \in \Omega$, when $z \notin \bar{\Omega}$ we have

$$k_y(z) = L_y(f_z) = \ln|x - z|, \; z \notin \bar{\Omega}.$$

Now take a point $z \in \Omega$ and near $\partial\Omega$. Denote by z' its reflection with respect to the tangent plane to $\partial\Omega$ at the point of $\partial\Omega$ nearest to z. Then one can show that

$$\max_{x \in \partial\Omega} \frac{|z - x|}{|z' - x|} \to 1,$$

if $z \to z_0 \in \partial\Omega$. Thus

$$\|f_z - f_{z'}\| = \max_{x \in \partial\Omega} \ln \left(\frac{|z - x|}{|z' - x|} \right) \to 0$$

if $z \to z_0 \in \partial\Omega$. From which we have $L_y(f_z - f_{z'}) \to 0$, that is

$$k_y(z) - k_y(z') \to 0$$

if $z \to z_0 \in \partial\Omega$. Now $k_y(z')$ exists and equals $\ln|y - z_0|$. Hence $k_y(z)$, $z \in \Omega$ can be extended into a continuous function $\tilde{k}_y(z)$ in $\bar{\Omega}$ and $\tilde{k}_y(z) = \ln|y - z|$ if $z \in \partial\Omega$. One can then see that $\tilde{k}_y(x) = \ln|x - y|$ is a Green's function. □

20. The Open Mapping Theorem

This result was stated earlier. Before we proceed to the proof, which relies on the Baire Category Theorem, we introduce some notation. If $A \subseteq X$ and a is a scalar, we denote $aA = \{ax : x \in A\}$. We define the open ball

$$B_r(x_0) = \{x \in X : \|x - x_0\| < r\}.$$

We will define $aB_r(x) = B_{ar}(x)$. We observe that if $x \in B_r(x_0)+B_s(x_0)$, then $x \in B_{r+s}(x_0)$. Here $A + B = \{a + b, a \in A, b \in B\}$.

Convergence of a sequence can be reformulated in terms of open balls. We say that $x_n \to x$ if for every $\epsilon > 0$ there exists an N such that for all $n \geq N$ we have $x_n \in B_\epsilon(x)$.

In preparation we recall two previous results.

Theorem 20.1. *Suppose that X is a Banach space and $\sum_{n=1}^\infty \|x_n\| < \infty$. i.e $\sum_{n=1}^\infty x_n$ is absolutely convergent. Then $\sum_{n=1}^\infty x_n$ is convergent. In other words, there exists $x \in X$, such that $x = \sum_{n=1}^\infty x_n$.*

It is important to note that this result holds if and only if X is a Banach space.

Theorem 20.2. *Let $\sum_{n=1}^\infty x_n$ be a series in a normed linear space X. Then absolute convergence of the series implies convergence if and only if X is a Banach space.*

Finally, recall that a set U in a normed linear space is open if and only given any $x \in U$, we can surround it with an open ball which is entirely contained within U. That is, there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. The proof of the open mapping theorem relies on this.

Theorem 20.3 (The Open Mapping Theorem)**.** *Let X and Y be Banach spaces and suppose that $T : X \to Y$ is bounded and linear. If T is also onto, (i.e. $T(X) = Y$), then $T(U)$ is open in Y whenever U is open in X. That is, T maps open sets in X onto open sets in Y.*

The idea of the proof is to show that if U is open, then for any $x \in T(U)$ we can find an open ball $B_r(x) \subseteq T(U)$ where r is suitably small. The first two parts of the proof establish some technical properties about open balls that we need.

Proof. The proof proceeds in three stages. We consider an operator T with the required properties. We first prove that if $B_{\frac{1}{2}}(0)$ is a ball of radius $1/2$ centered on 0 in X, then we can find a ball in Y of radius ϵ which is contained in $\overline{T(B_{\frac{1}{2}}(0))}$. This is a technical detail which we need for the rest of the proof.

We assumed that Y is complete and hence by Baire's Theorem, it cannot be written as the union of nowhere dense sets. T is also onto, and hence

$$Y = T(X) = T\left(\bigcup_{n=1}^{\infty} B_{\frac{n}{2}}(0)\right) = \bigcup_{n=1}^{\infty} T\left(B_{\frac{n}{2}}(0)\right). \tag{20.1}$$

Baire's Theorem implies that at least one of the sets $B_{n/2}(0)$ isn't nowhere dense. So there is some integer N such that one of the sets $T\left(B_{\frac{N}{2}}(0)\right)$ isn't nowhere dense. Since $T\left(B_{\frac{N}{2}}(0)\right)$ isn't nowhere dense, the interior of its closure cannot be empty. We may thus find a $y_0 \in Y$ and an $r > 0$ such that

$$B_r(y_0) \subseteq \overline{T\left(B_{\frac{N}{2}}(0)\right)}. \tag{20.2}$$

Now let $\epsilon = \frac{r}{2N}$. If $y \in B_{2\epsilon}(0)$ then $\frac{y_0}{N} + y \in \overline{T(B_{\frac{1}{2}}(0))}$. Next let $y \in B_\epsilon(0)$. By symmetry, both y_0/N and $-y_0/N$ are contained in $\overline{T(B_{\frac{1}{2}}(0))}$. Thus

$$\begin{aligned} y &= -\frac{y_0}{2N} + (\frac{y_0}{2N} + y) \\ &= -\frac{y_0}{2N} + \frac{1}{2}(\frac{y_0}{N} + 2y) \\ &\in \overline{T(B_{\frac{1}{4}}(0))} + \overline{T(B_{\frac{1}{4}}(0))} \\ &\subseteq \overline{T(B_{\frac{1}{2}}(0))}. \end{aligned}$$

So if $y \in B_\epsilon(0)$, then $y \in \overline{T(B_{\frac{1}{2}}(0))}$. Hence $B_\epsilon(0) \subseteq \overline{T(B_{\frac{1}{2}}(0))}$.

What does this do for us? It shows us that for any $y \in B_\epsilon(0)$, we can find a point $x_1 \in B_{\frac{1}{2}}(0)$ such that Tx_1 and y are as close together as we desire. So choose an x_1 such that

$$\|y - Tx_1\| < \frac{\epsilon}{2}.$$

This means that

$$y - Tx_1 \in B_{\frac{\epsilon}{2}}(0) = \frac{1}{2}B_\epsilon(0) \subseteq \frac{1}{2}\overline{T(B_{\frac{1}{2}}(0)} = \overline{T(B_{\frac{1}{4}}(0))}.$$

So now we choose $x_2 \in B_{\frac{1}{4}}(0)$ such that Tx_2 and $y - Tx_1$ are as close as we like. In particular

$$\|(y - Tx_1) - Tx_2\| < \frac{\epsilon}{4}.$$

We repeat this to generate a sequence x_n with $x_n \in B_{\frac{1}{2^n}(0)}$ and

$$\left\| y - \sum_{k=1}^{n} Tx_k \right\| < \frac{\epsilon}{2^n}.$$

Now the sequence $\sum_{k=1}^n x_k$ is convergent. To see this, observe that by the triangle inequality

$$\left\| \sum_{k=1}^n x_k \right\| \leq \sum_{k=1}^n \|x_k\| \leq \sum_{k=1}^n \frac{1}{2^k} \to 1.$$

Thus $\sum_{k=1}^n x_k$ is absolutely convergent and hence convergent. Let $x = \sum_{k=1}^\infty x_k$. By the above we have $\|x\| \leq \sum_{k=1}^\infty \| = 1$. This shows that $x \in B_1(0)$.

We are now in a position to prove the open mapping theorem. Let U be an open set. We want to show that $T(U)$ is open. Let $Tx \in T(U)$. Because U is open, we may find an open ball centered on x of radius $\delta > 0$ such that $B_\delta(x) \subseteq U$. To show that $T(U)$ is open, we need to show that we can surround any point $Tx \in T(U)$ with an open ball contained entirely within $T(U)$. We show that $B_{\epsilon\delta}(Tx) \subseteq T(U)$. To this end, let $y \in B_{\epsilon\delta}(Tx)$. Since y is near the point Tx we can write $y = Tx + y_1$ for some $y_1 \in B_{\epsilon\delta}(0)$. It is clear that

$$y = \frac{y_1}{\delta} \in B_\epsilon(0) \subseteq T(B_1(0)).$$

Now write $y_2 = Tx_2$ for some $x_2 \in B_1(0)$. Then by linearity

$$y = Tx + \delta T x_2 = T(x + \delta x_2),$$

with $x+\delta x_2 \in U$. Thus $y \in B_{\epsilon\delta}(Tx)$ implies $y \in T(U)$ and so $B_{\epsilon\delta}(Tx) \subseteq T(U)$. Thus $T(U)$ is open and this proves the theorem.

□

Notice that the Baire Category Theorem is used only at one point in the proof, but the proof simply doesn't work without it. This is typical of the applications of Baire's theorem.

As a corollary of the Open Mapping Theorem, we have an easy proof of the following result due to Banach.

Theorem 20.4. *Let X and Y be Banach spaces and let T be a one to one bounded linear map from X onto Y. Then the inverse of T, T^{-1} is also a bounded linear mapping.*

Proof. Because the map is one to one and onto, the inverse exists. This is just algebra. That T^{-1} is linear is trivial. We know that $T^{-1}T(z) = z$ for all $z \in X$. Suppose that $Tx_1 = y_1$ and $Tx_2 = y_2$. Then $x_1 = T^{-1}y_1$ and $x_2 = T^{-1}y_2$. Linearity follows from

$$\begin{aligned} T^{-1}(ay_1 + by_2) &= T^{-1}(aTx_1 + bTx_2) = T^{-1}(T(ax_1 + bx_2)) = ax_1 + bx_2 \\ &= aT^{-1}y_1 + bT^{-1}y_2. \end{aligned}$$

So T^{-1} is linear, one to one and onto. Hence it maps open sets to open sets. Thus for any open set $U \in X$, $\{y \in Y : T^{-1}(y) \in U\}$ is open. Hence T^{-1} is continuous and therefore it is bounded. □

21. The Principle of Uniform Boundedness

As with many of the theorems in functional analysis, there are different versions of the Banach-Steinhaus Theorem, which is often called the Principle of Uniform Boundedness. A version of the theorem was stated earlier. Here we will prove the result in a slightly different, but completely equivalent form. There are also versions of the theorem which hold in topological vector spaces. All the different versions of the theorem are related to the problem of determining when a family of operators is uniformly bounded.

That is, given a family of operators $\mathcal{A} \subseteq \mathcal{B}(X,Y)$, when does there exist a constant $K > 0$ such that $\|T\| < K$ for all $T \in \mathcal{A}$? This might seem to be an odd question to ask, because every T under consideration is bounded. The space $\mathcal{B}(X,Y)$ is after all defined to be the space of bounded linear operators from X to Y. The difficulty lies in the fact that $\mathcal{A}$ may be an infinite set. Consider the operators $T_n : C[a,b] \to C[a,b]$, (where we equip $C[a,b]$ with the supremum norm), defined by $T_n f = nf$, $n = 1, 2, 3, 4, ...$ Each operator in this family is bounded.

We have

$$\|T_n f\|_\infty = n\|f\|_\infty$$

for all n. So $\|T_n\| = n$. The problem is that there does not exist a constant K such that $\|T_n\| \leq K$ for all n. If there were such a constant then we would have $K > n$ for all $n = 1, 2, 3, ...$ which is plainly impossible. The Banach-Steinhaus Theorem tells us when we can bound an entire family of operators by the same norm. Here we will pay close attention to which norm is which, since this is a theorem about norms.

Theorem 21.1 (Banach-Steinhaus)**.** *Consider a Banach Space X and a normed linear space Y. If $\mathcal{A} \subseteq \mathcal{B}(X,Y)$ is such that*

$$\sup\{\|Tx\|_Y : T \in \mathcal{A}\} < \infty$$

for each $x \in X$, then

$$\sup\{\|T\|_{\mathcal{B}(X,Y)} : T \in \mathcal{A}\} < \infty.$$

So if the set $\|Tx\|$ is bounded by a constant for every x, T, then there is a uniform bound for the norms of the operators in $\mathcal{A}$. Note that $\|T\|_{\mathcal{B}(X,Y)}$ is the operator norm defined previously.

Proof. We begin by defining the sets

$$E_n = \{x \in X : \|Tx\|_Y \leq n \text{ for all } T \in \mathcal{A}\} = \bigcap_{T \in \mathcal{A}} \{x \in X : \|Tx\|_Y \leq n\}.$$

Each set E_n is closed, since if $T_j \in E_n$ and T_j is Cauchy, then T_j converges to a $T \in E_n$. Also, since T is a bounded linear operator, each $\|Tx\|_Y$ is bounded by some constant, which may be arbitrarily large, so given an $x \in X$, we will eventually find an n such that $\|Tx\|_Y \leq n$. From this it follows that $X = \bigcup_{n=1}^{\infty} E_n$.

Now we assumed that X is Banach and hence complete. So by the Baire Category Theorem it cannot be written as the union of nowhere dense sets, which implies that at least one E_n is nowhere dense. Thus there is an N such that the interior of E_N is not empty. If x_0 is a point in the interior (i.e. not on the boundary) of E_N, then we can find an open ball $B_r(x_0) \subseteq E_N$, for some $r > 0$.

From the definition of E_n, this tells us that $\|Tx\|_Y \leq N$ for every $x \in B_r(x_0)$ and every $T \in \mathcal{A}$. Now recall the definition of the operator norm from the previous section. We had

$$\|T\|_{\mathcal{B}(X,Y)} = \inf \{\|Tx\|_Y : \|x\|_X = 1\}. \tag{21.1}$$

We are trying to show that the family of norms is bounded. In order to do this, we need only consider elements of norm 1, by (21.1). More precisely, we want to find a $K > 0$ such that $\|Tx\|_Y \leq K$ for every element $x \in X$ of norm 1 and each $T \in \mathcal{A}$.

To do this, first notice that if we take an x with $\|x\|_X = 1$, then $y = \frac{r}{2}x + x_0 \in B_r(x_0)$. To see this, note that $\|x_0 - y\| = \frac{r}{2}\|x\| = \frac{r}{2} < r$, so

$$y \in \{x \in X : \|x - x_0\|_X < r\} = B_r(x_0).$$

Since $y \in B_r(x_0)$, it follows that $\|Ty\|_Y \leq N$. We therefore have

$$\begin{aligned} N \geq \|Ty\|_Y = \|T(\frac{r}{2}x + x_0)\|_Y = \|\frac{r}{2}Tx + Tx_0\| \\ \geq \frac{r}{2}\|Tx\|_Y - \|Tx_0\|_Y \end{aligned}$$

where we used the reverse triangle inequality in the last line. Rearranging this gives

$$\|Tx\|_Y \leq \frac{2}{r}(N + \|Tx_0\|_Y).$$

This holds for every x of norm 1 and every T and K is independent of x. We have thus computed an upper bound for the set $\{\|T\|_{\mathcal{B}(X,Y)} : T \in \mathcal{A}\}$ and this completes the proof. □

The Banach-Steinhaus Theorem is used extensively in operator theory, Now however we turn to the final of the big four theorems, the Closed-Graph Theorem.

The Closed Graph Theorem gives a condition guaranteeing that an operator T is continuous in terms of its graph. The graph of a function is a familiar object. We define it as the set $(x, f(x))$, where x ranges over the domain of f. For an operator $T : X \to Y$, with domain D_T, we similarly define the graph as the set of points $G_T = (x, Tx)$ as x ranges over D_T. The graph is a subset of $X \times Y$. If this subset is closed, then we say that T is a closed operator. So T is closed if and only if G_T is a closed linear subspace of $X \times Y$. Another way of looking at this, is that if $x_n \in D_T$ and $x_n \to x, Tx_n \to y$, then T is closed if $x \in D_T$ and

$y = Tx$. Note the similarity to a continuous operator. In fact closed operators are continuous.

Theorem 21.2 (The Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $T : X \to Y$ be linear, (with $D_T = X$). If T is closed, then it is continuous.*

Proof. To prove that T is continuous, it is enough to prove that it is bounded, since bounded linear operators are continuous. The key to the proof is to turn the graph G_T into a Banach space and use a corollary of the Open Mapping Theorem.

The graph G_T is a closed linear subspace of $X \times Y$. Now given two Banach spaces, we may equip $Z = X \times Y$ with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. Under this norm, Z is itself a Banach space. Since G_T is a closed subspace of Z, then G_T must also be a Banach space.

We now define the operator $J : G_T \to X$ given by $J(x, Tx) = x$. The operator J is one to one and onto, hence J^{-1} exists and is bounded, by the Banach inverse Theorem. Thus there is a $K > 0$ such that $\|J^{-1}x\| \le K\|x\|_X$ for all $x \in X$. Now

$$(x, Tx) = J^{-1}x$$

by definition of J and so

$$\|x\|_X + \|Tx\|_Y = \|(x, Tx)\| = \|J^{-1}x\| \le K\|x\|_X, \tag{21.2}$$

for all $x \in X$. This tells us that $\|Tx\|_Y$ is bounded for all $x \in X$ and thus T is continuous. □