**Modern Analysis**

**Problem sheet one.**

(1) Prove that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is uniformly convergent on $\mathbb{R}$.

(2) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on $X \subseteq \mathbb{R}$. How may we use the Weierstrass $M$ test to prove that the sequence converges uninformly?

(3) Prove Riemann's Criterion for the existence of the Riemann integral.

(4) Prove that if $f$ and $g$ are Riemann integrable on $[a, b]$, then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}.$$

(5) Prove Theorem 2.2 in the lecture notes.

(6) From the definition of the Riemann-Stieltjes integral, prove that RS $\int_0^1 x d(x^2) = \mathrm{R} \int_0^1 2x^2 dx$.

(7) Calculate RS $\int_{-1}^1 x^2 d(x|x|)$, RS $\int_{-1}^1 x^2 d(x^2)$ and RS $\int_0^1 \cos x d(\sin x)$.

(8) Prove that if $f$ is continuous and monotone increasing, then

$$\mathrm{RS} \int_a^b f(x) d(f(x)) = \frac{1}{2}(f(b)^2 - f(a))^2.$$

(9) Use the Euler-Maclaurin formula to estimate the following quantities for large $N$.

(i) $\sum_{n=1}^{N} \frac{\sin(\sqrt{n})}{n}$

(ii) $\gamma_N = \sum_{n=1}^{N} \frac{1}{n} - \ln N$.

(10) Use the Euler-Maclaurin formula to prove the integral test for series convergence: If $f$ is a continuous function on $\mathbb{R}$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if R $\int_1^{\infty} f(x)dx < \infty$.

(11) Calculate the sums $\sum_{k=1}^{n} k^2$, and $\sum_{k=1}^{n} k^3$.

(12) Find $\lim_{n\to\infty}$ RS $\int_0^{\frac{\pi}{2}} (1 - \frac{x}{n})^n d(\cos x)$.

**Modern Analysis**

**Problem sheet two.**

(1) Prove Theorem 3.11 in the lecture notes.

(2) Prove that

$$m((0,1]) = \sum_{k=1}^{\infty} m\left(\left(\frac{1}{k+1}, \frac{1}{k}\right]\right).$$

(3) Prove that any countable set is measurable and has measure zero.

(4) What is the measure of the irrational numbers in $[0,1]$?

(5) Prove that if $m^*(A) = 0$ then $A$ is measurable.

(6) Prove Theorem 3.13 in the lecture notes.

(7) Let $A \subset \mathbb{R}$ be a measurable set. For $h \in \mathbb{R}$, define

$$A + h = \{x + h | x \in A\}.$$

Prove that $A + h$ is measurable and $m(A + h) = m(A)$.

(8) Let $E, F$ be measurable and assume $E \subset F$, with $m(F) < \infty$. Prove that $m(F - E) = m(F) - m(E)$.

(9) Show that for any two sets $A, B$ with $A \cup B = [0,1]$,

$$m^*(A) \geq 1 - m^*(B).$$

(10) Let

$$A = \{x \in [0,1] : \text{no 5s occur in the decimal expansion of } x\}.$$

Find $m^*(A)$.

(11) Suppose that $A$ is a bounded set and $m^*(A \cap I) \leq \frac{1}{2} m^*(I)$ for every interval $I$. Prove that $m^*(A) = 0$.

(12) (Hard) Prove that intervals are measurable by verifying that the Caratheodory condition is satisfied.

**Problem sheet three.**

(1) Show that if $A_1, A_2$ are measurable then
$$m(A_1) + m(A_2) = m(A_1 \cup A_2) + m(A_1 \cap A_2).$$

(2) Let $X$ be a nonempty set, and let $f : X \to [0, \infty)$ be a function. Let $P(X)$ be the collection of all subsets of $X$. Define $\mu : P(X) \to [0, \infty)$ by $\mu(A) = \sum_{x \in A} f(x)$ if $A$ is a nonempty, countable set, $\mu(A) = \infty$ is $A$ is uncountable and $\mu(\emptyset) = 0$. Show that $\mu$ is a measure.

(3) Consider the Cantor set. This is formed by taking the interval $C_1 = [0, 1]$ and removing the middle third $(0, 1)$. So $C_2 = [0, 1/3] \cup [2/3, 1]$. Then remove the middle third from each of these intervals. So $C_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 5/9] \cup [8/9, 1]$. Continue this process indefinitely. The Cantor set is defined to be
$$C = \cap_{n=1}^{\infty} C_n.$$
Prove that the Cantor set is nonempty, with infinitely many points and that $m(C) = 0$. In fact the Cantor set is uncountable. So it provides an example of an uncountable set of Lebesgue measure zero.

(4) Show that a countable union of sets of measure zero has measure zero.

(5) If $m^*$ is Lebesgue outer measure on $\mathbb{R}$ and $A$ is a null set (one with outer measure zero), then
$$m^*(B) = m^*(A \cup B) = m^*(B \setminus A)$$
holds for every subset $B$ of $\mathbb{R}$.

(6) Let $m^*$ be outer measure on $\mathbb{R}$. If a sequence of subsets $\{A_n\}$ of $\mathbb{R}$ satisfies $\sum_{n=1}^{\infty} m^*(A_n) < \infty$, then the set
$$E = \{x \in X : x \in A_n \text{ for infinitely many } n\},$$
is a null set.

(7) Prove that $\chi_A$ is measurable if and only if $A$ is measurable.

(8) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is measurable. Prove that the composition $f \circ g$ is also measurable.

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the derivative $f'$ is Lebesgue measurable.

**Problem sheet four.**

(1) Consider a sequence of functions $(f_n)$, where each $f_n : \mathbb{R} \to \mathbb{R}$ is measurable. Let $f$ be a measurable function. The sequence $(f_n)$ is said to converge in measure to $f$ if, for any $\epsilon > 0$,

$$\lim_{n\to\infty} m[\{x : |f_n(x) - f(x)| \geq \epsilon\}] = 0.$$

Prove that if $f_n \to f$ uniformly, then $(f_n)$ converges in measure to $f$.

(2) Let $(f_n)$ and $(g_n)$ be sequences of almost everywhere real, measurable functions, that converge in measure to $f$ and $g$ respectively. Let $a, b$ be real numbers. Prove the following.

(i) $(af_n + bg_n)$ converges in measure to $af + bg$.

(ii) $(|f_n|)$ converges in measure to $|f|$.

(iii) $(f_n g_n)$ converges in measure to $(fg)$, on $X$ with $m(X) < \infty$.

(iv) $(f_n g)$ converges in measure to $(fg)$, on $X$ with $m(X) < \infty$.

(3) Prove that if $k \leq f \leq K$ a.e. on a measurable set $E$, then

$$km(E) \leq \int_E f \leq Km(E).$$

(4) Let $f$ be an integrable function that is positive everywhere on a measurable set $E$. If $\int_E f = 0$ prove that $m(E) = 0$.

(5) Prove that $\int_0^\infty \sin(x^2)dx$ is not Lebesgue integrable, but exists as an improper Riemann integral.

(6) Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable. Assume that $f$ is differentiable at $x = 0$ and $f(0) = 0$. Show that the function defined by $g(x) = x^{-\frac{3}{2}} f(x)$ for all $x \in (0,1]$ and $g(0) = 0$ is Lebesgue integrable.

(7) Find the Lebesgue integral over $[0,1]$ of the function

$$f(x) = \begin{cases} x^3 + 5x, & x \in [0,1] - \mathbb{Q} \\ 2x^2, & x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

**Problem sheet five.**

(1) Prove that if $\varphi(x)$ is a continuous nondecreasing function in $[a, b]$, then $\varphi'(x)$ is Lebesgue integrable and

$$\int_a^b \varphi'(x)dx \leq \varphi(b) - \varphi(a).$$

(2) Show that

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

(3) Show that for $t \geq 0$

$$\int_0^\infty e^{-xt} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} t.$$

(4) Prove that if $f$ is continuously differentiable, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin(nx)dx = 0$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos(nx)dx = 0.$$

(5) Assume that $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a, \infty)$. Prove that $\int_a^\infty f(x)dx$ exists as an improper Riemann integral, if and only if for every $\epsilon > 0$ there exists an $M$ such that $\left|\int_s^t f(x)dx\right| < \epsilon$ for all $s, t > M$.

The previous result is useful because of the following theorem which you may assume. Let $f : [a, \infty) \to \mathbb{R}$ be Riemann integrable on every closed subinterval of $[a, \infty)$. Then $f$ is Lebesgue integrable if and only if the improper Riemann integral $\int_a^\infty |f(x)|dx$ exists. In this case

$$L\int f dm = R\int_a^\infty f(x)dx.$$

(6) Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

(Integrate by parts and use the convergence theorems).

(7) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with the left and right limits of the derivatives defined at the end points. If the derivative $f'$ is bounded on $[a, b]$, prove that $f'$ is Lebesgue integrable and

$$\int_{[a,b]} f' dm = f(b) - f(a).$$

(8) Show that $f(x) = \frac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and that $\int f dm = 1$.

(9) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \delta$. Show that

$$\lim_{n\to\infty} \int_0^a f(nx)dx = a\delta$$

for each $a > 0$.

## 37438 Problem sheet six.

(1) Calculate $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$

(2) Calculate $\displaystyle\int_0^1 \frac{x-1}{\ln x} dx.$ Hint, look at $\int_0^1 \frac{x^p - 1}{\ln x} dx.$

(3) Evaluate $\displaystyle\int_0^\infty e^{-\alpha^2 x^2 - \beta^2/x^2} dx.$

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be $n$ times differentiable and assume that $f^{(n)}$ is integrable for all $n$. Prove that

$$\widehat{f^{(n)}}(y) = (iy)^n \widehat{f}(y).$$

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $xf(x)$ is Lebesgue integrable. Prove that

$$\widehat{xf(x)}(y) = i\frac{d}{dy}\widehat{f}(y).$$

(6) Solve the differential equation

$$u'(x) + xu(x) = 0, \; u(0) = \sqrt{2\pi}.$$

Now take the Fourier transform of this equation and hence show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-iyx} dx = \sqrt{2\pi} e^{-\frac{y^2}{2}}.$$

(7) Compute the Fourier transform of

(i) $f(x) = xe^{-x^2}$.

(ii) $f(x) = e^{-|x|}$.

(iii) $f(x) = \dfrac{1}{x^4 + 1}$.

## 37438 Problem sheet seven.

(1) Show that if $d(x, y)$ is a metric on a space $X$, then

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on $X$.

(2) Assume that two vectors $x, y \in X$ where $X$ is a normed linear space satisfy the relation $\|x + y\| = \|x\| + \|y\|$. Show that for all non negative scalars $\alpha, \beta$ we have

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|.$$

(3) It is true that every norm defines a metric by $d(x, y) = \|x - y\|$. Show by example that not every metric defines a norm.

(4) Let $f : X \to \mathbb{R}$.

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ holds for almost all } x\}.$$

Prove the following.

(i) If $f = g$ a.e., then $\|f\|_\infty = \|g\|_\infty$.

(ii) $\|f\|_\infty \geq 0$ for each function $f$, and $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.

(iii) $\|af\| = |a|\|f\|_\infty$ for all scales $a$.

(iv) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(v) If $|f| \leq |g|$ then $\|f\|_\infty \leq \|g\|_\infty$.

(5) Two norms $\|x\|_1$ and $\|x\|_2$ on a vector space $X$ are said to be equivalent if there exist constants $K > 0$ and $M > 0$ such that

$$K\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Prove that on a finite dimensional vector space all norms are equivalent.

(6) Let $C^1[0, 1]$ be the vector space of all real valued functions on $[0, 1]$ with continuous first derivative. Show that $\|f\| = |f(0)| + \|f'\|_\infty$ is a norm and that it is equivalent to the norm

$$\|f\|_A = \|f\|_\infty + \|f'\|_\infty.$$

Hint: $f(x) = f(0) + \int_0^x f'(t)dt$.

(7) let $X$, $Y$ be normed linear spaces. An operator $T : X \to Y$ is said to be bounded if there exists $K > 0$ such that

$$\|T(x)\|_Y \leq K\|x\|_E.$$

Consider $C[a,b]$ with the norm defined in question 4. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Show that

$$T(f)(x) = \int_a^b K(x,y)f(y)dy$$

is a bounded linear operator. i.e There exists $M > 0$ such that $\|Tf\|_\infty \leq M\|f\|_\infty$ for all $f \in C[a,b]$.

(8) Let $D : C^1[0,1] \to C^1[0,1]$ be given by $Df = f'$. Use the norm of question 4 and show that with respect to this norm, differentiation is an unbounded linear operator. (Hint. Find an example of a function whose norm grows with derivatives).

**37438 Problem sheet eight.**

(1) Show that in a real inner product space $(x, y) = 0$ holds if and only if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Does the same result hold if we allow the inner product to be complex valued?

(2) Assume that the sequence $\{x_n\}$ in an inner product space satisfies $(x_n, x) \to \|x\|^2$ and $\|x_n\| \to \|x\|$. Show that $x_n \to x$.

(3) A sequence $\{x_n\}$ in a Hilbert space $H$ is said to converge weakly to $x$ in $H$ if $(x_n, y) \to (x, y)$ for all $y \in H$.

(i) Show that if a sequence is convergent it is also weakly convergent.

(ii) Show that if a sequence is weakly convergent, then the limit is unique.

(iii) Show by an example that a sequence may be weakly convergent, but not convergent.

(4) Let $H$ be a Hilbert space with inner product $(\cdot, \cdot) : H \times H \to \mathbb{C}$. Prove that for all $x, y \in H$

$$(x, y) = \frac{1}{4}\left([\|x + y\|^2 - \|x - y\|^2] + i[\|x + iy\|^2 - \|x - iy\|^2]\right).$$

This is known as the polarisation identity and it used to recover the inner product from the norm.

(5) Given an example of a function $f$ such that $f \in L^2(\mathbb{R})$ but $f \notin L^1(\mathbb{R})$. Then find an example of a function such that $f \in L^1(\mathbb{R})$, but $f \notin L^2(\mathbb{R})$.

(6) Let $[a, b]$ be a closed, bounded interval. Define the spaces $L^p([a, b])$ in the obvious way. If $f \in L^1([a, b])$ does it follow that $f \in L^2([a, b])$? What about the converse?

(7) Let $p > 1$, $p \neq 2$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

(8) Let $f \in L^2([0, 1])$ satisfy $\|f\|_2 = 1$ and $\int_0^1 f(x)dx \geq \alpha > 0$. For each $\beta \in \mathbb{R}$, define $E_\beta = \{x \in [0, 1] : f(x) \geq \beta\}$. If $0 < \beta < \alpha$ show that $m(E_\beta) \geq (\beta - \alpha)^2$. (Hint: This uses Hölder's inequality. Note that $f - \beta \leq (f - \beta)\chi_{E_\beta} \leq f\chi_{E_\beta}$.)

(9) Suppose that $\{\phi_n\}$ is an orthonormal set in the Hilbert space $L^2([-1, 1])$. Show that the sequence $\{\psi_n\}$ defined by

$$\psi_n(x) = \left(\frac{2}{b - a}\right)^{1/2} \phi_n\left(\frac{2}{b - a}\left(x - \frac{b + a}{2}\right)\right)$$

is an orthonormal set in $L^2([a,b])$.

(10) Let $1 \leq p < \infty$ and suppose $f \in L^p([a,b])$, where $a < b$, $a, b \in \mathbb{R}^*$ and let $\epsilon > 0$. Show that

$$m^*(\{x \in [a,b] : |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int_a^b |f(x)|^p dx.$$

where $m^*$ is Lebesgue outer measure.

(11) Let $1 \leq p < \infty$ and suppose $\{f_n\}$ is a sequence in $L^p([a,b])$, where $a < b$, $a, b \in \mathbb{R}^*$. Prove that if $\|f_n - f\|_p \to 0$ as $n \to \infty$, then $f_n \to f$ in measure.

**37438 Modern Analysis**

**37438 Problem sheet nine.**

(1) Solve the PDE

$$u_t = u_{xx} + h(x), \quad h \in L^1(\mathbb{R}), \quad x \in \mathbb{R}$$

subject to the initial condition $u(x,0) = f(x)$ and the assumption that $u(x,t), u_x(x,t) \to 0$ as $|x| \to \infty$.

(2) Solve the Poisson equation

$$u_{xx} + u_{yy} = h(x), \quad h \in L^1(\mathbb{R}) \quad x \in \mathbb{R},\ y \geq 0$$

subject to the condition $u(x,0) = f(x)$ and the assumption that $u(x,y), u_x(x,y) \to 0$ as $|x| \to \infty$.

(3) Solve the integral equation

$$u(x) = h(x) + \int_{-\infty}^{\infty} k(x-y)u(y)dy,$$

where $h$ and $k$ and their Fourier transforms are integrable.

(4) Use Parseval's identity to evaluate the integrals

(a) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

(b) $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$

(5) Define the function $h_\lambda$ by

$$h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|y|} e^{iyx} dy.$$

Prove that

(a) $h_\lambda(x) = \dfrac{2\lambda}{\lambda^2 + x^2},$

(b) $\displaystyle\int_{-\infty}^{\infty} h_\lambda(x)dx = 2\pi.$

(c) $h_\lambda(x) = \dfrac{1}{\lambda} h_1\left(\dfrac{x}{\lambda}\right).$

The convolution of two functions $f$ and $g$ is defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

(d) Prove that if $f$ is integrable, then for every $\lambda > 0$

$$f * h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|y|} \hat{f}(y) e^{ixy} dy.$$

(e) Prove that for all $f \in L^1(\mathbb{R})$, $\lim_{\lambda \to 0} f * h_\lambda = 2\pi f$. Deduce from this the Fourier inversion Theorem. (Hint: Consider $f * h_\lambda - 2\pi f$ ).

The function $h_\lambda$ is known as an approximation of the identity.

(6) Suppose that $f \in L^1(\mathbb{R})$ and that both $f', f''$ exist and are continuous and integrable. Prove that the Fourier transform $\widehat{f} \in L^1(\mathbb{R})$.

(7) Suppose that $f, f_n \in L^1([-\pi, \pi])$ and that $f_n \to f$. Prove that the Fourier coefficients satisfy $\widehat{f_n} \to \widehat{f}$.

(8) Prove the Weierstrass approximation theorem: If $f \in C([-\pi, \pi])$, then, given any $\epsilon > 0$, there is a polynomial $p(x)$ such that

$$\sup_{x \in [-\pi, \pi]} |f(x) - p(x)| < \epsilon.$$

(Hint: Use a Fourier series).

(9) Let $f$ be $2\pi$ periodic and integrable on $[-\pi, \pi]$. Let $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixn} dx$.

(a) Show that $\widehat{f}(n) = -\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) e^{-inx} dx$.

(b) Use the result of (a) and the DCT to prove that if $f$ is continuous, then $\widehat{f}(n) \to 0$ as $n \to \infty$. (Hint: Find an expression for $2\widehat{f}(n)$).

(c) Prove that if for all $x$ there is a $C > 0$ and $0 < \alpha \leq 1$, such that $|f(x+h) - f(x)| \leq C|h|^\alpha$, then $\widehat{f}(n) = O(1/|n|^\alpha)$.

(10) Let the Theta function be defined by

$$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-t\pi k^2}.$$

Prove that $\Theta(t) = \dfrac{1}{\sqrt{t}} \Theta(\dfrac{1}{t})$.

**37438 Modern Analysis**

**Problem sheet ten.**

(1) Prove that the product measure of two measures $\mu$ and $\nu$ is a measure. That is, prove that $(\mu \times \nu)(A \times B) \geq 0$ and

$$(\mu \times \nu)(\cup_{i=1}^{\infty}(A_i \times B_i) = \sum_{i=1}^{\infty}(\mu \times \nu)(A_i \times B_i).$$

(2) Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ with $f(0, 0) = 0$. Evaluate

$$\int_0^1 \int_0^1 f(x, y)dxdy \text{ and } \int_0^1 \int_0^1 f(x, y)dydx.$$

Explain your answer.

(3) If $\lambda, \mu$ and $\nu$ are finite measures such that $\lambda << \nu$ and $\nu << \mu$, show that

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}.$$

(4) Suppose that $\mu$ and $\nu$ are finite measures and that $\mu << \nu$ and $\nu << \mu$. Prove that

$$\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1.$$

(5) Calculate the moment generating function of a normal random variable.

(6) Let $\lambda$ be Lebesgue measure. Suppose that $\mu(E) = \int_E f d\lambda$ and that the measure $\mu$ satisfies $\mu(aE) = \mu(E)$ for all $a > 0$ and each measurable subset $E$ of $(0, \infty)$. The aim of this question is to compute the Radon-Nikodym derivative $f$.

(i) Let $E = [1, x]$. What is $aE$?

(ii) Show that $f$ satisfies $\int_1^x f(t)dt = \int_a^{ax} f(t)dt$

(iii) Show that $f(x) = c/x$ for some constant $c$.

(7) Let $X$ be a Banach space. Show that if $L$ is a linear functional on $X$ and $L$ is continuous at $a \in X$, then $L$ is uniformly continuous on the whole of $X$.

(8) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of subsets of $\Omega$. Define

$$\overline{\lim_{n\to\infty}} A_n = \bigcap_{m=1}^{\infty} \bigcup_{n\geq m} A_n$$
$$= \{\omega : \omega \in A_n \text{for infinitely many } n\}.$$

Now suppose that each $A_n$ is measurable. Prove the Borel-Cantelli Lemma:

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(\overline{\lim}_{n\to\infty} A_n) = 0.$$

Conversely, if the $A_n$ are $P$ independent sets: $P(A_n \cap A_m) = P(A_n)P(A_m)$; then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P(\overline{\lim}_{n\to\infty} A_n) = 1.$$

(9) If $X$ and $Y$ are independent prove that $Var(X+Y) = Var(X) + Var(Y)$.

(10) If $X_n$ are independent random variables on $(\Omega, \mathcal{F}, P)$, with $E(X_n) = \mu$, $Var(X_n) \leq K < \infty$ prove the weak law of large numbers: $\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu$ in $L^2$.

(11) Show that if $X$ and $Y$ are random variables on a probability space, then $d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)$ is a metric and that convergence in $d$ is equivalent to convergence in the probability measure $P$.