2 Lebesgue Measure

2.1) We know about the lengths of intervals in $\mathbb{R}$

$$\lambda((a,b)) = \lambda([a,b]) = b-a$$, If $b=-\infty$ or $a=+\infty$ $\lambda((a,b)) = \infty$

Points have length 0

With disjoint intervals

$$I_1 = (a_1,b_1),\ I_2 = (a_2,b_2) \dots\ I_n = (a_n,b_n)$$

with $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ we want

$$\lambda(I_1 \cup I_2 \cup \dots \cup I_n) = b_1 - a_1 + b_2 - a_2 + \dots + b_n - a_n.$$

We would like to define the length, or "measure" of more general sets

<u>Lemma</u>. Let $U \subseteq \mathbb{R}$ be an open subset. Then $\exists$ a countable collection of open intervals $I_1, I_2, \dots$ with $I_i \cap I_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} I_i = U$.

<u>Proof</u>. Write U as a union of its connected components. Each of these connected components contains a rational number. Therefore the connected components are in 1-1 correspondence with a set of rationals, therefore countable. Each component is an open connected set, so it's an interval. □

Thus, we can define $\lambda(U) = \sum_{i=1}^{\infty} \lambda(U_i)$.

(1.2) Non-measurable sets

In a perfect world, we'd like a function $\lambda : \mathcal{M} \to [0,\infty] = \mathbb{R}^+ \cup \{0\}$, where $\mathcal{M} \subseteq 2^{\mathbb{R}}$ such that

(i) $\lambda(M)$ is well-defined for all subsets of $\mathbb{R}$ ($\mathcal{M} = 2^{\mathbb{R}}$)

(ii) $\lambda(I)$ = length of I if I is an interval

(iii) If $\{E_n\}$ is a sequence of disjoint sets

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n) \qquad \text{(countable additivity)}$$

(iv) If $E \in \mathcal{M}$ and $y \in \mathbb{R}$

$$\lambda(y+E) = \lambda(E) \qquad \text{(translation invariance)}$$

__Lemma__ These properties cannot hold simultaneously. (i.e. there is no such λ.)

__Proof__ (Uses uncountable axiom of choice)

[__AC__. Let $\mathcal{A}$ be a non-empty collection of non-empty sets. Then $\exists$ a set $\mathcal{E}$ so that $\mathcal{E} \cap A = \{x_A\}$ a singleton for all $A \in \mathcal{A}$]

Let $x, y \in [0,1)$ and define
$$x \dot{+} y = x+y-1 \quad \text{if } x+y \geq 1$$
$$= x+y \quad \text{if } x+y < 1$$

[__Exercise__ - check that for $E \subseteq [0,1)$ $m(x \dot{+} E) = m(E)$ using (iv)]

Now define an equivalence relation $\sim$ on $[0,1)$, by

$x \sim y$ if $\exists$ a rational r with $x - y = r$.

Use AC to construct a set P which contains exactly one element of each equivalence class.

Let $\{r_i\}$ be an enumeration of the rationals in $[0,1)$ and let $P_i = P \dot{+} r_i$

<u>Claim</u>: • The P_i are pairwise disjoint

$[P_i \cap P_j \neq \phi \Rightarrow x + r_i = y + r_j$ for some $x, y \in P$

$\Rightarrow x \dot{-} y = r_j \dot{-} r_i \Rightarrow x \sim y \Rightarrow i = j]$

- $\lambda(P_i) = \lambda(P)$ by (iv)
- $\bigcup_{i=1}^{\infty} P_i =$ ~~[illegible]~~ $[0,1]$

Thus, by (iii) $\lambda(\bigcup P_i) = 1 = \sum_{i=1}^{\infty} \lambda(P)$ which is impossible, as the sum is 0 or ∞. $\square$

<u>Discussion</u> We thus have to abandon (i), (ii), (iii) or (iv) or the axiom of choice.

<u>Lebesgue Theory</u>: Abandon (i) – i.e. define the measure of some (relatively large) class of sets with (ii), (iii), (iv) holding.

Can also drop (iii) or (iv).

(1.3) <u>Outer Measure</u> (abandon (iii) – though it's an important step in defining M)

For $A \subseteq \mathbb{R}$, define

$$\lambda^*(A) = \inf \sum_{n=1}^{\infty} \lambda(I_n)$$

where the infimum is taken over all countable coverings of A by open intervals, $A \subseteq \bigcup_{n=1}^{\infty} I_n$.

N.B. $\lambda^*\{x\} = 0$, $\lambda^*(I \cap \mathbb{Q}) = 0$.

Lemma. If $\{A_n\}$ is a sequence of sets

$$\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$$

Proof Let $\varepsilon > 0$

For each A_n, chose a covering of A_n by intervals $\{U_{n,i} : i = 1, 2, \dots\}$ s.t.

$$\lambda^*(A_n) \geq \left[\sum_{i=1}^{\infty} \lambda^*(U_{n,i})\right] - \frac{\varepsilon}{2^n}$$

Let $A = \bigcup_{n=1}^{\infty} A_n$, so that $A \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} U_{n,i}$, so

$$\begin{aligned}
\lambda^*(A) &\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \lambda^*(U_{n,i})\right) \\
&\leq \sum_{n=1}^{\infty}\left(\lambda^*(A_n) + \frac{\varepsilon}{2^n}\right) \\
&= \sum_{n=1}^{\infty} \lambda^*(A_n) + \varepsilon
\end{aligned}$$

$\square$

However, we have no reason to assume that $\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda^*(A_n)$.

In fact, λ^* satisfies (i), (ii), ~~(iii)~~ (iv) so it can't satisfy (iii).

Aim. Restrict λ^* to some class of sets so that countable additivity does hold on that class.

(1.4) Algebras and σ-algebras of sets. Msble sets

Definition A class of sets $\mathcal{M} \subseteq 2^{\mathbb{R}}$ is called a σ-algebra of sets if

(i) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$

(ii) If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of sets in $\mathcal{M}$ then $\bigcup_{i=1}^{\infty} A_i$ also belongs to $\mathcal{M}$.

It's called an algebra of sets if (ii) holds only for finite collections of sets. Clearly the set of finite unions of intervals is an algebra of sets.

Definition Let $E \subseteq \mathbb{R}$. We say that E is (Lebesgue) measurable if $\forall A \subseteq \mathbb{R}$

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$$

[Clearly ϕ and $\mathbb{R}$ itself are both msble !]

Theorem The class $\mathcal{M}$ of Lebesgue measurable sets is a σ-algebra.

Proof (i) If $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$ (as $E^{cc} = E$)

(ii) Lemma If $\lambda^*(E) = 0$ then E is msble

Pf. Let $A \subseteq \mathbb{R}$ then $A \cap E \subseteq E$ so $\lambda^*(A \cap E) \leq \lambda^*(E) = 0$
Thus $\lambda^*(A \cap E) = 0$

Also $A \cap E^c \subseteq A$ so $\lambda^*(A) \geq \lambda^*(A \cap E^c) = \lambda^*(A \cap E^c) + \lambda^*(A \cap E)$

But also $\lambda^*(A) \leq \lambda^*(A \cap E^c) + \lambda^*(A \cap E)$

Thus $\lambda^*(A) = \lambda^*(A \cap E^c) + \lambda^*(A \cap E)$ and E is msble $\square$

<u>Lemma</u>. If E_1 and E_2 are msble, then $E_1 \cup E_2$ is msble (i.e. $\mathcal{M}$ is an algebra of sets)

<u>Proof</u> Let $A \subseteq \mathbb{R}$. Since E_2 is msble

$$\lambda^*(A \cap E_1^c) = \lambda^*((A \cap E_1^c) \cap E_2) + \lambda^*((A \cap E_1^c) \cap E_2^c)$$

Now $A \cap (E_1 \cup E_2) = A \cap E_1 \cup (A \cap E_2 \cap E_1^c)$

Thus $\lambda^*(A \cap (E_1 \cup E_2)) \leq \lambda^*(A \cap E_1) + \lambda^*(A \cap E_2 \cap E_1^c)$

Thus

$$\boxed{\lambda^*(A \cap (E_1 \cup E_2))} + \lambda^*(A \cap E_1^c \cap E_2^c)$$

$$\leq \lambda^*(A \cap E_2 \cap E_1^c) + \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c \cap E_2^c)$$

$$\leq \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c) = \lambda^*(A)$$ as E_1 msble

But clearly

$$\lambda^*(A \cap (E_1 \cup E_2)) + \lambda^*(A \cap (E_1 \cup E_2)^c) \geq \lambda^*(A)$$

So we have equality and $E_1 \cup E_2$ is msble. $\square$

Lemma Let A be a set and $\{E_i\}_{i=1}^n$ a finite sequence of disjoint msble sets.

Then $\lambda^*(A \cap (\bigcup_{i=1}^n E_i)) = \sum_{i=1}^n \lambda^*(A \cap E_i)$

Proof By induction on n.
S'pose true for any collection of $n-1$ sets. Then

$$A \cap (\bigcup_{i=1}^n E_i) \cap E_n = A \cap E_n$$

and $$A \cap (\bigcup_{i=1}^n E_i) \cap E_n^c = A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)$$

So, since E_n is msble

$$\begin{aligned}\lambda^*(A \cap \bigcup_{i=1}^n E_i) &= \lambda^*(A \cap E_n) + \lambda^*(A \cap (\bigcup_{i=1}^{n-1} E_i)) \\ &= \lambda^*(A \cap E_n) + \sum_{i=1}^{n-1} \lambda^*(A \cap E_i) \quad \text{inductive hypothesis} \\ &= \sum_{i=1}^n \lambda^*(A \cap E_i)\end{aligned}$$

$\square$

Note that if $\mathcal{M}$ is any algebra of sets and $A_1, A_2 \in \mathcal{M}$ then A_1^c and $A_2^c \in \mathcal{M}$ so $A_1^c \cup A_2^c \in \mathcal{M}$ and so $A_1 \cap A_2 \in \mathcal{M}$

Lemma. If $\{E_i\}$ is a countable collection of msble sets, then $\bigcup_{i=1}^\infty E_i$ is msble.

Proof Taking $F_1 = E_1$
$\vdots$
$$F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$$

we have $F_n \cap F_m = \emptyset$ for $n \neq m$ and $\bigcup_{i=1}^\infty F_i = \bigcup_{i=1}^\infty E_i$

Thus we can assume wlog that the E_i's are disjoint.

Let $G_n = \bigcup_{i=1}^{n} E_i$. By the previous lemma G_n is msble. Let $E = \bigcup_{i=1}^{\infty} E_i$, so that

$$G_n^c \subseteq E^c.$$

Let $\varepsilon > 0$. Choose n large enough that

$$\lambda^*(A \cap G_n) = \sum_{i=1}^{n} \lambda^*(A \cap E_i) \geq \sum_{i=1}^{\infty} \lambda^*(A \cap E_i) - \varepsilon$$
$$\geq \lambda^*(A \cap E) - \varepsilon.$$

Then

$$\begin{aligned}\lambda^*(A) &= \lambda^*(A \cap G_n) + \lambda^*(A \cap G_n^c) \\ &\geq \lambda^*(A \cap G_n) + \lambda^*(A \cap E^c) \\ &\geq \lambda^*(A \cap E) + \lambda^*(A \cap E^c) - \varepsilon\end{aligned}$$

Since this holds $\forall \varepsilon$, we have equality and E is msble.

This completes the proof that $\mathcal{M}$ is a σ-algebra.

(1.5) Properties of measurable sets. The measure of a measurable set

__Lemma__. The interval (a, ∞) is msble

__Proof__ Let A be a set; $A_1 = A \cap (a, \infty)$
$A_2 = A \cap (-\infty, a]$

We want to show that $\lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$

If $\lambda^*(A) = \infty$, we've finished

Let $\{I_n\}$ be a collection of open intervals s.t. $\sum_{n=1}^{\infty} \lambda(I_n) < \lambda^*(A) + \varepsilon$ and $A \subseteq \bigcup_{n=1}^{\infty} I_n$

Let $\left.\begin{array}{l} I_n' = I_n \cap (a, \infty) \\ I_n'' = I_n \cap (-\infty, a) \end{array}\right\}$ both are intervals

Thus $\lambda(I_n) = \lambda(I_n') + \lambda(I_n'') = \lambda^*(I_n') + \lambda^*(I_n'')$

$A_1 \subseteq \bigcup_{n\geq 1}^{\infty} I_n'$, so $\lambda^*(A_1) \leq \lambda^*(\bigcup I_n') \leq \sum_{n\geq 1}^{\infty} \lambda^*(I_n')$

and likewise $A_2 \subseteq \bigcup_{n\geq 1}^{\infty} I_n''$, so $\lambda^*(A_2) \leq \lambda^*(\bigcup I_n'') \leq \sum_{n\geq 1}^{\infty} \lambda^*(I_n'')$

It follows that $\lambda^*(A_1) + \lambda^*(A_2) \leq \sum (\lambda^*(I_n') + \lambda^*(I_n''))$
$\leq \sum \lambda(I_n) \leq \lambda^*(A) + \varepsilon$

Since this holds for all $\varepsilon > 0$, we are finished $\square$

Definition

We define, for $E \in \mathcal{M}$ $\quad \lambda(E) = \lambda^*(E)$

This is the <u>measure</u> of a measurable set.

We have shown that $\mathcal{M}$, equipped with λ, is a σ-alg. satisfies properties (i) - (iv) of (1.2). ~~where~~

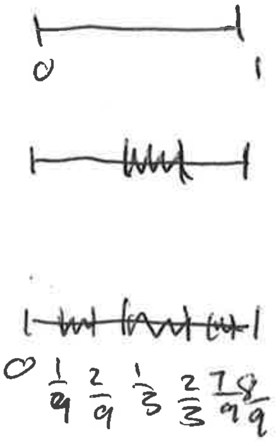
(1.6) <u>The Borel σ-algebra and sets of measure 0.</u>

The Borel σ-algebra is the smallest σ-algebra $\mathcal{B}$ containing the open intervals. It's slightly smaller than the σ-algebra $\mathcal{M}$, which also contains all sets of (outer) measure zero - some of these are not in $\mathcal{B}$.

We proved above that all sets of outer measure zero are measurable.

Clearly all finite sets and all countable sets have measure zero.

The Cantor set is an example of an uncountable set of measure zero.



C_0
$\rotatebox{90}{$\subseteq$}$
$C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$ $\qquad \lambda(C_1) = \frac{2}{3}$
$\rotatebox{90}{$\subseteq$}$
C_2 $\qquad \lambda(C_2) = \left(\frac{2}{3}\right)^2$
$\rotatebox{90}{$\subseteq$}$
$\vdots$
$\rotatebox{90}{$\subseteq$}$
C_n $\qquad \lambda(C_n) = \left(\frac{2}{3}\right)^n$

Let $C = \bigcap_{n=1}^{\infty} C_n$. Then $\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = 0$.

To see that C is uncountable, use the triadic rational expansion of numbers in $[0,1]$

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{where } x_n \in \{0,1,2\}$$

It's easy to see that $C = \{x : x_i = 0 \text{ or } 2 \text{ for all } i\}$ and there are uncountably many choices of 0, 2 by Cantor's diagonal proof.

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<u>Remark</u> If A is measurable and $\varepsilon > 0$ there exist an open set C and a closed set B with $B \subseteq A \subseteq C$ and $\lambda(C \setminus B) < \varepsilon$. Hence $\lambda(A) - \varepsilon \leq \lambda(B)$ and $\lambda(C) \leq \lambda(A) + \varepsilon$.

If $\lambda(A) < \infty$, B can be taken to be bounded

(1.7) Measure Spaces

Definition. A set X is called a measurable space if it is equipped with a σ-algebra of subsets $\mathcal{M}$

$(X, \mathcal{M})$ is a measurable space

$(X, \mathcal{M}, \lambda)$ is a measure space if ~~…~~ $\lambda : \mathcal{M} \to \mathbb{R}_+ \cup \{\infty\}$ is a function which is countably additive – if $\{E_i\} \subseteq \mathcal{M}$ are disjoint sets, then

$$\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i).$$

We just showed that $\mathbb{R}$, equipped with the Lebesgue measurable sets and λ, form a measure space.

$\boxed{\text{Other Examples}}$

(i) If $(X, \mathcal{U})$ is a topological space, the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{U}$. (no obvious measure)

(ii) $X = \mathbb{R}$, $\mathcal{M} =$ all sets, $\lambda(A) = \delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$

(iii) $\mathbb{R}^n$ with ~~…~~ Borel σ-algebra and $\lambda(B) = \mathrm{vol}(B)$

(iv) $(\Omega, \mathcal{B}, P)$ a probability space with $\mathcal{B}$ the σ-algebra of "events", P the probability of the event

We'll explore these in more detail later.

Meanwhile, back to $\mathbb{R}$ ....