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(Riesz - Fischer)

Theorem. The spaces L^p are complete ~~metric~~ normed vector spaces i.e. Banach spaces

Proof We show every absolutely convergent series is convergent. By the Proposition above, this will show completeness

So suppose $\sum_{n=1}^{\infty} F_n$ is absolutely convergent in L^p and that $\sum_{n=1}^{\infty} \|F_n\|_p < M$.

The F_n's are equivalence classes of functions - for each n pick a function $f_n \in F_n$

Set $g_n = \sum_{k=1}^{n} |f_k|$ and $g = \sum_{k=1}^{\infty} |f_k|$

Then g_n are measurable fns, $g_n \to g$ monotonically and so g is measurable and by the MCT

$$\int g^p = \lim_{n\to\infty} \int g_n^p = \lim_{n\to\infty} \|g_n\|_p^p$$

Now

$$\|g_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p = M$$

so ~~$\|g_n\|$~~ $\int g^p \leq M^p < \infty$. Thus $g^p < \infty$ a.e. and $g < \infty$ ae

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The set $A = \{ x \in \mathbb{R} : g(x) < \infty \}$

is measurable and $\lambda(A^c) = 0$.

On A, $\sum_{k=1}^{\infty} f_k$ is absolutely convergent and

convergent on $\mathbb{R}$. Let its pwise sum be f

f is msble (as f_k, A are msble) and

$$|f| \le \sum_{k=1}^{\infty} |f_k \chi_A| \le \sum_{k=1}^{\infty} |f_k| \chi_A = g\chi_A$$

Thus $\int |f|^p \le \int g^p \chi_A \le \int g^p$ and so $f \in \mathcal{L}^p$

Also $g\chi_A \in \mathcal{L}^p$. Let F be the equivalence

class of f in L^p.

Then for any $n \in \mathbb{N}$

$$\| F - \sum_{n=1}^{n} F_k \|_p^p = \| f - \sum_{k=1}^{n} f_k \|_p^p$$

$$= \int | f - \sum_{k=1}^{n} f_k |^p \le \int | f - \sum_{k=1}^{n} f_k \chi_A |^p$$

Now $\lim_{n\to\infty} | f - \sum_{k=1}^{n} f_k |^p = 0$ pointwise a.e. and

$| f - \sum^{n} f_k |^p \le (2g\chi_A)^p$, which is integrable

Thus by DCT

$$\lim_{n\to\infty} \int | f - \sum_{k=1}^{n} f_k \chi_A |^p = 0$$

Thus $\lim_{n\to\infty} \| F - \sum_{k=1}^{n} F_k \|_p^p = 0$ and so

$$\lim_{n\to\infty} \| F - \sum_{k=1}^{n} F_k \|_p = 0$$

Thus $\sum_{k=1}^{\infty} F_k$ is convergent and L^p is complete $\square$

We showed that the L^p spaces are complete – hence they are all Banach spaces. Amongst them, only L^2 is a Hilbert space

Let's describe an orthonormal basis for $L^2(\mathbb{R})$

<u>Defn</u>. The DE

$$y'' - 2xy' + 2ny = 0 \qquad n = 0, 1, 2, 3 \ldots$$

have polynomial solutions $H_n(x)$ called <u>Hermite Polynomials</u>

<u>Theorem</u> (Rodriguez' formula)

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

<u>Theorem</u> An orthogonal basis for $L^2(\mathbb{R})$ is given by the <u>Hermite functions</u>

$$h_n(x) = e^{-x^2/2} H_n(x)$$

They satisfy

$$\int_{-\infty}^{\infty} h_n(x) h_m(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{nm}$$

The Fourier transform of h_n is

$$\hat{h}_n(y) = (-i)^n \sqrt{2\pi}\, h_n(y).$$

6.3 Bounded Linear Operators

Let $(V, \|\cdot\|)$ be a Banach space. A linear map $V \to V$ is called a bounded linear operator if there is a $K \geq 0$ such that

$$\|Bv\| \leq K \|v\|$$

for all $v \in V$.

The <u>operator norm</u> of B is

$$|||B|||_{op} = \sup_{\|v\| \leq 1} \|Bv\| = \sup_{v \in V} \frac{\|Bv\|}{\|v\|}.$$

In finite dimensional spaces, all linear maps are bounded – in fact, if B is an $n \times n$ matrix

$$|||B|||_{op} = \max_{1 \leq i,j \leq n} |B_{ij}|.$$

It's not too hard to see that if A, B are linear operators then $|||A \circ B||| \leq |||A||| \, |||B|||$.

Also, the set of linear operators on a Banach space is ~~a~~ itself a Banach space.

We'll return to some results on operator theory later.

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Similarly, if V, W are Banach spaces $\boxed{\mathcal{L}(V, W)}$ is the set of linear operators $V \to W$ such that $\exists K$ $\forall v \in V$
$\|Bv\|_W \leq K \|v\|_V$.

Ch 7 The Fourier Transform

If $f \in \mathcal{L}^1(\mathbb{R})$, we define its Fourier transform:

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{-iyx}\, dx$$

Comment: Since $|f(x)e^{-iyx}| = |f(x)|$ this function is integrable.

Basic Properties of the Fourier Transform

(i) $f \mapsto \hat{f}$ is a linear mapping on $\mathcal{L}^1(\mathbb{R})$

$(af+bg)^{\wedge} = a\hat{f} + b\hat{g}$

(ii) If $f \in \mathcal{L}^1$, $\hat{f} \in \mathcal{L}^\infty$ and $\|\hat{f}\|_\infty \le \|f\|_1$

(iii) (Riemann-Lebesgue Lemma)

If $f \in \mathcal{L}^1$ then $\hat{f}(y) \to 0$ as $y \to \pm\infty$

(iv) If $f \in \mathcal{L}^1$ and $\frac{df}{dx} \in \mathcal{L}^1$ then

$$\left(\frac{df}{dx}\right)^{\wedge}(y) = -iy \cdot \hat{f}(y)$$

[Integrate by parts!]

(v) (Fourier Inversion) If $f \in \mathcal{L}^1$ and $\hat{f} \in \mathcal{L}^1$

then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)\, e^{ixy}\, dy$.

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(vi) If $f_a(x) = f(x-a)$ then

$$\hat{f}_a(y) = e^{-iay}\hat{f}(y)$$

(vii) If $g(x) = e^{iax} f(x)$ then

$$\hat{g}(y) = \hat{f}(y-a)$$

(viii) (Dilations) If $(M_b f)(x) = f(bx)$ $b \neq 0$

then $(M_b f)^\wedge(y) = \frac{1}{b}\hat{f}\left(\frac{y}{b}\right)$

<u>Examples</u> • $\chi_{[-1,1]}$

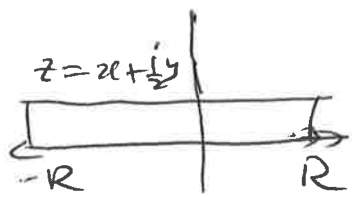
$$\hat{\chi}_{[-1,1]}(y) = \int_{-1}^{1} e^{-iyx}\,dx = \left[\frac{e^{-iyx}}{-iy}\right]_{-1}^{1}$$

$$= \frac{1}{-iy}\left[e^{-iy} - e^{iy}\right] = 2\,\frac{\sin y}{y}.$$

• e^{-x^2} (The Gaussian)

Recall that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-x^2 - ixy}\,dx$$



$$= \int_{-\infty}^{\infty} e^{-(x+\frac{i}{2}y)^2 - \frac{y^2}{4}}\,dx$$

$$= e^{-y^2/4}\int_{-\infty}^{\infty} e^{-(x+\frac{i}{2}y)^2}\,dx$$

$$= e^{-y^2/4}\int_{-\infty+\frac{1}{2}iy}^{\infty+\frac{1}{2}iy} e^{-z^2}\,dz$$ (Contour integral)

$$= e^{-y^2/4}\int_{-\infty}^{\infty} e^{-z^2}\,dz$$

$$= \sqrt{\pi}\,e^{-y^2/4}$$

More on the Fourier Transform

Theorem If $f \in L^1(\mathbb{R})$ then $\hat{f}$ is uniformly continuous

Proof

$$\hat{f}(y+\eta) - \hat{f}(y) = \int_{-\infty}^{\infty} f(x)\left(e^{-ix(y+\eta)} - e^{-ixy}\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-iyx} \left(e^{-ix\eta} - 1\right) dx$$

Thus $|\hat{f}(y+\eta) - \hat{f}(y)| \leq \int_{-\infty}^{\infty} |f(x)|\, |e^{-ix\eta} - 1| \, dx$

Now $|e^{-ix\eta} - 1| \leq 2$ so we get by DCT

$$\lim_{\eta \to 0} |\hat{f}(y+\eta) - \hat{f}(y)| \leq \int_{-\infty}^{\infty} |f(x)| \lim_{\eta \to 0} |e^{-ix\eta} - 1| \, dx = 0$$

The limit on the RHS is independent of y i.e. $\forall \varepsilon > 0$ ~~$\forall \delta > 0$~~ $\exists \delta > 0$ s.t. for $|\eta| < \delta$

$$|\hat{f}(y+\eta) - \hat{f}(y)| < \varepsilon$$

and $\hat{f}$ is uniformly continuous $\square$

Together with the Riemann Lebesgue Lemma we see that for $f \in L^1$

$\hat{f} \in C_0^{u}(\mathbb{R})$ i.e. uniformly continuous and vanishing at ∞.

Proof Denote by $h(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy$

Fubini tells us that

$$\int_{-\infty}^{\infty} |h(x)|\, dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y) g(x-y)|\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y) g(x-y)|\, dx\, dy$$

$$= \|g\|_1 \|f\|_1$$

So $h \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

[which means L^1 is a Banach algebra!]

Proposition If $f, g \in L^1(\mathbb{R})$

$$\widehat{(f * g)}(y) = \hat{f}(y)\, \hat{g}(y)$$

Proof

$$\widehat{(f*g)}(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-w) g(w) e^{-iyx}\, dw\, dx$$

(Fubini) $$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-w) g(w) e^{-iyx}\, dx\, dw$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(w) e^{-iy(u+w)}\, du\, dw$$

$$= \int_{-\infty}^{\infty} f(u) e^{-iyu}\, du \int_{-\infty}^{\infty} g(w) e^{-iyw}\, dw$$

$$= \hat{f}(y)\, \hat{g}(y). \qquad \square$$

<u>Proposition</u> If $f, g \in L^1(\mathbb{R})$

$$\int_{-\infty}^{\infty} \hat{f}(y) g(y)\, dy = \int_{-\infty}^{\infty} f(y) \hat{g}(y)\, dy$$

<u>Proof</u> As we saw on Page 125, $\hat{f} \in L^\infty$ and $\|\hat{f}\|_\infty \leq \|f\|_1$.

Thus $\hat{f}(y) g(y) \in L^1$ and $|\hat{f}(y) g(y)| \leq \|f\|_1 |g(y)|$ which belongs to L^1

Thus
$$\int_{-\infty}^{\infty} \hat{f}(y) g(y)\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-iyx} g(y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y) e^{-iyx}\, dy\, dx$$
(by Fubini)
$$= \int_{-\infty}^{\infty} f(x) \hat{g}(x)\, dx. \qquad \square$$

(Fubini is legal here because $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x) g(y)|\, dx\, dy \leq \|f\|_1 \|g\|_1$.

<u>Definition (Convolution)</u> If $f, g \in L^1$ then define
$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy$$
$$\left(= \int_{-\infty}^{\infty} f(x-y) g(y)\, dy \right)$$

Using Fubini, it follows that $f * g \in L^1$