1 The real numbers

CH1 INTRO

Are a complete Archimedean field

$(+, \geq, \cancel{\text{[illegible]}})$

A set $B \subseteq \mathbb{R}$ is <u>bounded above</u> if $\exists \beta_0 \in \mathbb{R}$ s.t. $b \leq \beta_0 \quad \forall b \in B$. β_0 is an <u>upper bound</u> of B.

We say β_0 is a <u>least upper bound</u> for B if $\beta_0 \leq \beta$ whenever β is an upper bound. ~~We also write~~

ETS that ~~there is~~ a l.u.b. is unique. We write

$$\beta_0 = \sup_{b \in B}(B) \quad \text{or} \quad \sup(B).$$

(u.b. β_0 need not belong to B eg $(0,1)$)

<u>Completeness axiom is</u>

Every set ~~of~~ $\mathbb{R}$ which is bounded above has a l.u.b

N.B. $\mathbb{Q}$ is an totally ordered field but is not complete

eg $\{q : q \leq \pi\}$ has no lub in $\mathbb{Q}$.

Similarly $\quad glb(B) = \inf B$

2 Cauchy sequences in $\mathbb{R}$

~~Convergence~~ If $\{x_n\}$ is a sequence of real numbers, we say $x_n \to x \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N_0$ s.t. $n \geq N_0$ $\Rightarrow |x_n - x| < \varepsilon$.

Cauchy ~~If~~ $\{x_n\}$ ~~is~~, a sequence, is called Cauchy if $\forall \varepsilon > 0 \quad \exists N_0$ s.t. $n, m \geq N_0 \Rightarrow |x_n - x_m| < \varepsilon$.

[Clearly every convergent sequence is Cauchy!]

In $\mathbb{Q}$, not every Cauchy sequence ~~conv~~ has a limit.

However, in $\mathbb{R}$, every Cauchy sequence has a limit

This is equivalent to the l.u.b. Property (and is therefore referred to as completeness).

Comments on how this follows

(1) Every increasing sequence $x_0 \le x_1 \le x_2 \le \dots x_n \le x_{n+1} \le \dots$ either (a) Converges to $x_\infty = \sup\{x_n\}$ if the sequence is bounded or

(b) "Diverges to ∞" if not i.e. $\forall M \, \exists N_0$ s.t. $n \ge N_0 \Rightarrow x_n \ge M$.

In case (a) $x_\infty = \lim_{n\to\infty} x_n$.

(2) If $\{x_n\}$ is any bounded sequence let

$$\limsup_{n\to\infty} \{x_n\} = \lim_{n\to\infty} \sup_{m \ge n} \{x_m\}$$

increasing sequence ∴ converges

$$\left[\liminf \{x_n\} = \lim_{n\to\infty} \inf_{m\ge n} \{x_m\} \right]$$

~~If $\{x_n\}$ is Cauchy~~

(3) If $\{x_n\}$ is bounded then $\exists$ a subsequence x_{n_k} s.t. $x_{n_k} \to \limsup\{x_n\}$

(4) If $\{x_n\}$ is Cauchy ~~then~~ and a subsequence $\{x_{n_k}\}$ has a limit x_∞, then $x_n \to x_\infty$ as well.

Therefore $\{x_n\}$ Cauchy $\Rightarrow \{x_n\}$ has a limit (in $\mathbb{R}$)

i.e. $\mathbb{R}$ is complete.

3. Topological approach to limits

If $x \in \mathbb{R}$ we say that a neighbourhood of x is a set containing an open interval $(x-\varepsilon, x+\varepsilon)$.

An open set is a set which contains a nbhd of each one of its points. A set is closed if its complement is open

Proposition $x_n \to x$ can equivalently be defined by saying ~~"each nbhd of x containing elements of $\{x_n : n \geq N\}$"~~ the sequence $\{x_n\}$ eventually lies in any given nbhd of x.

~~We say that~~

If $B \subseteq \mathbb{R}$ then $b_\infty \in \mathbb{R}$ is a limit point of B if every neighbourhood of b_∞ has non-empty intersection with B.

[ETS that if this is the case $\exists$ a sequence $\{b_k\}$ of elements of B $b_k \to b_\infty$]

B, together with its limit points, is denoted $\overline{B}$, the closure of B.

~~Lemma A set is closed if its complement is open~~

[Lemma B is closed iff $B = \overline{B}$ ie B contains all of its limit points.]

4 <u>Compactness</u> We say B is <u>compact</u> if every sequence $\{b_k\}$ of elts of B has a convergent subsequence with limit in B.

<u>Theorem</u> (Bolzano-Weierstrass) in $\mathbb{R}$,

B is compact $\iff$ B is closed and bounded. (above + below)

Equivalent defn of compactness

"Every open cover of B has a finite subcover"

5 <u>Continuous functions</u> • $f:\mathbb{R}\to\mathbb{R}$ is cts at x if $\forall \varepsilon>0\ \exists\delta>0$ $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

• <u>Equivalent defn.</u> If U is open $f^{-1}(U)$ is open

6 Convergence of functions Let $f_n:\mathbb{R}\to\mathbb{R}$

• We say $f_n \to f$ <u>pwise</u> if

$\forall x\in\mathbb{R}$ $f_n(x)\to f(x)$.

ie. $\forall x\in\mathbb{R}\ \forall\varepsilon>0\ \exists N$ (depending on x and ε)

s.t. $n\geq N \Rightarrow |f_n(x)-f(x)|<\varepsilon$

<u>Thm</u> A cts. fn. takes its max on any compact set

<u>Thm</u> A cts image of a compact set is compact

Cts fns map convergent sequences to convergent sequences

• We say $f_n\to f$ uniformly (on $[a,b]$)

if $\forall\varepsilon>0\ \exists N$ (depending on ε) s.t. $n\geq N$

$\Rightarrow |f_n(x)-f(x)|<\varepsilon \quad \forall x\in\mathbb{R}$

Or stated differently

$\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow$

$$\|f_n - f\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon.$$

<u>Theorem</u> A uniform limit of continuous functions is continuous.

N.B. A pwise limit of cts fns need not be continuous (in fact, can be nowhere cts!)

~~7. Metric Spaces~~

<u>Proof by ε-δ.</u> Suppose f_n are cts ($f_n \to f$ limit)

Show cts at x. S'pose $\varepsilon > 0$

- Choose N such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3}$

for all x.

Now f_N is continuous, so choose $\delta > 0$ s.t. $|x - y| < \delta$

$\Rightarrow |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$

Then for $|x - y| < \delta$

$$|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)|$$
$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$
$$< 3\tfrac{\varepsilon}{3} = \varepsilon. \qquad \text{QED}$$

7. Metric Spaces

(X, d) is called a <u>metric space</u> if

$d : X \to \mathbb{R}_+ \cup \{0\}$ satisfies

(1) $d(x,y) = 0$ iff $x = y$

(2) $d(x,z) \leq d(x,y) + d(y,z)$ $\quad \forall\, x, y, z \in X$

<u>Examples</u>
- $\mathbb{R}$, $d(x,y) = |x-y|$
- $\mathbb{R}^n$ $d_2(\underline{x}, \underline{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$
- $\mathbb{R}^n$ $d_1(\underline{x}, \underline{y}) = \sum |x_i - y_i|$
- $\mathbb{R}^n$ $d_\infty(\underline{x}, \underline{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$.
- $X = C([0,1])$

  $d(f,g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$
- $X = C[0,1]$

  $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$
- $d_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2\, dx \right)^{1/2}$
- $X = \prod_{i=1}^{\infty} \{0,1\}$ $\quad d(x,y) = \sup_i |x_i - y_i|$

Let (X,d) be a metric space

$\{x_i\}$ a sequence in X

We say $x_i \to x$ in X if $\forall \varepsilon > 0\ \exists N$ s.t.

$n \geq N \Rightarrow d(x_i, x) < \varepsilon$.

The sets $B(x,\varepsilon) = \{ y : d(x,y) < \varepsilon \}$ $\leq$

are called the (open) balls of X.
(closed)

We say a set in X is open if it contains a nbhd of each point

- NB ~~We say~~ $x_n \to x$ iff given any nbhd of x $\{x_n\}_{n \geq N}$ eventually lies inside that nbhd.

- We define (as above) Cauchy sequence in X.

X is called <u>complete</u> if every Cauchy sequence converges to a point in X.

- $B \subseteq X$ is <u>closed</u> if all its limit points lie in B

- $B \subseteq X$ is <u>compact</u> if each sequence $\{b_k\}$ of elements of B ~~converges to~~ has a convergent subsequence ($\Rightarrow B$ closed)

~~c#~~

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8 Uniform continuity

Defn $f: X \to \mathbb{R}$ is unif. cts. if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Defn. f is sequentially unif cts if $x_n - y_n \to 0 \Rightarrow f(y_n) - f(x_n) \to 0$

Thm. f seq. unif cts $\Leftrightarrow$ f unif cts

Propn A cts fn. on a compact set is unif cts

Proof f not unif cts

choose r s.t. $\forall \delta > 0 \ \exists x, y \in [a,b]$ with

$|x-y| < \delta$ and $|f(x) - f(y)| > r$

for $n \in \mathbb{N}$ choose $x_n, y_n \in$ ~~[a,b]~~ X

$|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > r$

By B-W $\{x_n\}$ has a cgnt subsequence

$x_{n_k} \to x$ (say).

$x_{n_k} - y_{n_k} \to 0$ but $|f(x_{n_k}) - f(y_{n_k})| > r$

~~but~~ $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$.

Contrad. Thus f is unif cts. $\square$

Thm (Intermediate Value Thm) ~~Intermed~~ f cts on $[a,b]$ and

$f(a).f(b) < 0$ ~~[illegible]~~ $\Rightarrow \exists \ c \in [a,b]$ s.t. $f(c) = 0$

Proof Suppose $f(a) < 0$, $f(b) > 0$

$A = \{x \in [a,b] : f(x) < 0\}$

$c = \text{l.u.b.}(A)$ Then $f(c) = 0$.

(intermediate value theorem)
Corollary If f is continuous on $[a,b]$ and $f(a) \neq f(b)$ and M lies between $f(a)$ and $f(b)$, then $\exists\, c$ between a and b with $f(c) = M$

Definition f is monotone $\left\{ \begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix} \right\}$ on ~~a set~~ $B \subseteq \mathbb{R}$ if $x, y \in B$ and $x \leq y \Rightarrow \left\{ \begin{matrix} f(x) \leq f(y) \\ f(x) \geq f(y) \end{matrix} \right\}$

We say strictly monotone increasing or decreasing if $<$ or $>$.

Theorem. f monotone on $(a,b) \Rightarrow f$ is continuous except possibly at countably many points

Proof Wlog f is increasing. Also assume (a,b) is bounded, as, if not we can write it as a countable union of bounded ~~sets~~ intervals, and the discontinuities of f are a countable union of countable sets, therefore countable.

Given $x_0 \in (a,b)$, the l.u.b. axiom implies that

$$f(x_0^-) = \sup\{f(x) : a < x < x_0\}$$
$$f(x_0^+) = \inf\{f(x) : x_0 < x < b\}$$

both exist. And since f is increasing, $f(x_0^-) \leq f(x_0^+)$
If f is discontinuous at x_0, we must have $f(x_0^-) < f(x_0^+)$ so there's a jump discontinuity. Let

$$J(x_0) = \{y : f(x_0^-) \leq y \leq f(x_0^+)\}$$

This is a sub-interval of $(f(a), f(b))$ and so is bounded. The (jump) intervals $J(x_0)$ are clearly disjoint, as f is increasing. So for each n there are only finitely many jump intervals of length $\geq \frac{1}{n}$.
Hence the set of points of discontinuity of f is a countable union of finite sets — therefore countable

9 Differentiation

As this is probably quite familiar to you, I am going to refer to ~~the~~ Mark's notes. Make sure you understand

- If f is defined on an open set X, and $x \in X$ we say f is differentiable at x if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$

$$\boxed{f' = \frac{df}{dx} \text{ or } Df}$$

- The map $f \mapsto f'$ is linear and satisfies the product rule
- If f is ~~cts~~ diff'ble at x it is continuous at x
- The chain rule $(f \circ g)' = f' \circ g \cdot g'$
- Quotient rule
- Defn. of a local extremum (i.e. max or min)

If f diff'ble at c and there's a local extremum then $f'(c) = 0$

<u>Rolle's Theorem</u> If $[a,b]$ is closed, f is cts on $[a,b]$ and diff'ble on (a,b) and $f(a) = f(b)$ ~~then~~ Then $\exists\ c \in (a,b)$ s.t. $f'(c) = 0$

<u>Mean Value Thm</u> $[a,b]$ closed and bounded interval on $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is cts and diff'ble on (a,b), $\exists\ c \in (a,b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

<u>Corollary</u> $[a,b]$ closed, bdd f cts on $[a,b]$, diff'ble on (a,b)

$\Rightarrow f$ Lipschitz cts on $[a,b]$

i.e. $\exists\, \kappa$ s.t. $|f(x)-f(y)| \le \kappa |x-y| \quad \forall x,y \in [a,b]$

• <u>Cauchy Mean Value Thm</u> f, g cts on $[a,b]$, diff'ble on (a,b) g' doesn't vanish on (a,b) Then $\exists c$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

<u>Corollary</u> L'Hôpital's rule

f, g diff'ble on (a,b), neither g nor g' vanish on (a,b)

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

[Can replace $\lim_{x\to a^+} = 0$ by $\lim_{x\to a} = \infty$]

10 Inverse function Theorem and Convex Functions

Defn $f : A \to B$ is one-one (injective)

if $f(a) = f(a_1) \Rightarrow a = a_1$.

$f : A \to B$ is onto (surjective)

if $\forall\, b \in B \ \exists\, a \in A : f(a) = b$

f is bijective (or a bijection) if it is both injective + surjective (one-one and onto in English!)

If f is injective, then ~~there is an inverse fn~~ f^{-1} defined on ~~the range of~~ $f(A) \subseteq B$ defined by $f(f^{-1}(a)) = a \quad \forall\, a \in A$

[or equiv $f^{-1}(f(a)) = a$]

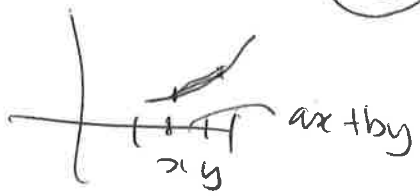
Proposition If f is strictly $\left\{\begin{matrix}\text{increasing} \\ \text{~~monotone~~} \\ \text{decreasing}\end{matrix}\right\}$ and cts on X then f^{-1} exists and is also $\left\{\begin{matrix}\text{increasing} \\ \text{decreasing}\end{matrix}\right\}$ on $f(X)$ and continuous.

Proof (See notes for proof of continuity)

Thm (IFT) If f is diff'ble and injective on an open interval I and if $f'(a) \neq 0$ ~~then~~ then f^{-1} exists and is diff'ble at $f(a)$

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

<u>Defn</u> $f: I \to \mathbb{R}$ is <u>convex</u> on I if $\forall x \in I$ and for $a, b \geq 0$ with $a+b=1$



$$f(ax+by) \leq a f(x) + b f(y)$$

<u>concave</u> if

$$f(ax+by) \geq a f(x) + b f(y).$$

Convex functions are always continuous

<u>Lemma</u> If f is convex on I the LH and RH derivatives

$$f'(x^+) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$f'(x^-) = \lim_{x \to 0^-} \frac{f(x+h) - f(x)}{h}$$

exist. If $x < y$ in I then

$$f'(x^-) \leq f'(x^+) \leq \frac{f(y) - f(x)}{y - x} \leq f'(y^-) \leq f^+(y^+)$$

<u>Proof</u> Just play with the defn. of convexity

<u>Corollary</u>. If f is convex on I (open) f is continuous on I.

<u>Pf</u> $\lim_{h \to 0^+} f(x+h) - f(x) = \lim_{h \to 0^+} \left(\frac{f(x+h) - f(x)}{h} \right) h = 0$

and $\lim_{h \to 0^-} f(x+h) - f(x) = 0$. The 2 limits exist and are equal, so f is cts. $\square$

<u>Corollary</u> f convex on $(a,b) \Rightarrow$ Lipschitz cts on each closed bdd $[c,d] \subseteq (a,b)$

<u>Proof</u> We have $f'(c+) \le f'(u+) \le \frac{f(v)-f(u)}{v-u} \le f'(v^-) \le f'(d^-)$

So let $M = \max\{f'(c+), f'(d^-)\}$ (*)

<u>Theorem</u>. A convex fn on (a,b) is diff'ble except at an at most countable set of points. The derivative is an increasing function

<u>Pf</u> Left + Right derivatives exist everywhere
Also increasing. Therefore by previous ~~limit~~ theorem they are cts. except ~~at a~~ possibly at a countable set of pts.

If they are cts at x, apply (*) to a sequence $x_n \to x$ to see $f'(x^-) \le f'(x^+) \le f'(x^-)$ so that f is diff'ble at x.

From (*) it also follows that f' is increasing □

<u>Lebesgue's Theorem</u> Every monotone fn is diff'ble almost everywhere.

— | —

- f is twice diff'ble on $(a,b) \Rightarrow f$ is convex on $[a,b]$ iff $f''(x) \ge 0$ $\forall x \in I$

11 Power Series and C^∞ functions

We say that f is in $C^n(I)$ if $f^{(n)}$ exists on all of I i.e. f is n times diff'ble.

If f is infinitely diff'ble ie all derivatives exist, we say f is C^∞ or <u>smooth</u>.

(fave example e^x !)

A <u>power series</u> about x_0 is a fn defined by

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If for example, we use the ratio test, we see this converges when $|x - x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$

If a power series converge ~~for~~ for $|x - x_0| < R$ we call R the <u>radius of convergence</u>

(in the example, ~~is~~ if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell$, then $R = \frac{1}{\ell}$.)

($x_0 = 0$ called MacLaurin series) We'll put $x_0 = 0$ for simplicity

<u>Thm</u> If $\sum a_n x^n$ is a power series with radius of convergence R, then the series

- converges for $|x| < R$
- diverges for $|x| > R$

[Comparison test].

Observation. If $\sum a_n x^n$ has radius of cgce R then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also has radius of cgce R

Proof.

Theorem. If $\sum a_n x^n$ is a power series with radius of cgce R ~~and~~ define $f: (-R,R) \to \mathbb{R}$ by
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
Then f is diff'ble on $(-R,R)$ and
$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof. We need to show that $\sum a_n x^n$ converges ~~uniformly on the interval $(-R,R)$~~ and ~~so does~~ it follows that $\sum n a_n x^{n-1}$ also converges uniformly.

— 1 —

Defn. If f is smooth, ~~on~~ in an open set X and $a \in X$, define
$$T_f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \cdots$$
Called the Taylor series of f at ~~a~~

If the series converges for all $x \in X$ and $|T_f(x) - f(x)| = 0$ $\forall x \in X$ we say f is *analytic* at a.

<u>Theorem</u> (Taylor's Theorem) Let I be an open ~~set~~ interval in $\mathbb{R}$, $n \in \mathbb{N}$

Let $a \in I$ and $x \in I$ with $x \neq a$

There is a point ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

$$+ \frac{f^{(n+1)}(\xi)}{n!} (x-a)^n$$

Write this as $T_n(f)(x) + R_n$ – the nth Taylor polynomial plus a remainder term.

<u>Proof</u> Uses Rolle's Theorem

This can be used to estimate the error in approximating a smooth function by its Taylor series.