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In general, there is a k-dimensional

If f is a radial function on $\mathbb{R}^n$ and f satisfies $\int_0^\infty f(r)\sqrt{r}\,dr < \infty$, and if $k > -\frac{1}{2}$, the kth Hankel Transform is

$$\hat{f}(e) = \int_0^\infty f(r)\, r J_k(re)\, dr$$

and the inversion is the same integral.

$$f(r) = \int_0^\infty \hat{f}(e)\, e J_k(re)\, de.$$

Fourier Series

Let $I = [-\pi, \pi]$ and we consider 2π-periodic functions on I.

Since $\lambda(I) = 2\pi$, Hölder's inequality shows that $\|f\|_1 \leq \sqrt{2\pi}\,\|f\|_2$, so that if $f \in L^2$, then $f \in L^1$ (Indeed, if $p_1 \leq p_2$ then $L^{p_1} \supseteq L^{p_2}$) and $\|f\|_{p_1} \leq k\|f\|_{p_2}$; $k = (2\pi)^{1/p_2}$

Defn If $f \in L^1(I)$, we define the nth Fourier coefficient $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$

The Fourier series of f is the infinite sum

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

(This is like the inverse Fourier Transform)

There are a lot of similarities with the Fourier transform:

$f \mapsto \hat{f}$ is linear, injective, $\hat{f}_a = e^{-ina}\hat{f}(n)$

- If f is continuously differentiable $\widehat{f'}(n) = in\hat{f}(n)$
- The Riemann-Lebesgue Lemma holds: $\lim_{|n|\to\infty} |\hat{f}(n)| = 0$

We define $\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx\right)^{1/2}$

In fact, the Riesz-Fischer Theorem says that if $f \in L^2(I)$ then

$$\|f\|_2^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2$$

This says that $\wedge$ is a ~~bounded trans~~ an isometry from $L^2(I)$ to $\ell^2(\mathbb{Z})$.

Dirichlet and Fejér Kernels

If $S_N f(x) = \sum_{|n| \le N} \hat{f}(n) e^{inx}$ is the partial sum of the Fourier series of f, then

$$S_N f = D_N * f$$

where $D_N(x) = \sum_{|n| \le N} e^{inx} = \frac{\sin (N+\frac{1}{2})x}{\sin \frac{1}{2}x}$

[see Assignment 2]

D_N is called the Dirichlet kernel

Note: $\int_{-\pi}^{\pi} D_N(x) = 2\pi$.

<u>Theorem</u>. (Dirichlet) If $f \in L^1(I)$ and $f'(x_0)$ exists, then

$$\lim_{N\to\infty} (S_N f)(x_0) = f(x_0)$$

<u>Proof</u>.

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi}\left\{ \int_{-\pi}^{\pi} D_N(y) f(x_0-y)\,dy - \frac{f(x_0)}{2\pi}\int_{-\pi}^{\pi} D_N(y)\,dy \right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\left(f(x_0-y) - f(x_0)\right) dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sin\left(N+\tfrac{1}{2}\right)y\right) y\, g(y)\,dy, \text{ where}$$

where $g(y) = \dfrac{f(x_0-y) - f(x_0)}{y} \cdot \dfrac{y}{\sin y/2}$

Now $\lim_{y\to 0} g(y) = 2 f'(x_0)$

Now define g on $[-2\pi, 2\pi]$ by making it zero on the intervals $(-2\pi, \pi]$ and $[\pi, 2\pi]$.

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) g(y)\,dy = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) g(y)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin\left((2N+1)y\right) g(2y)\,dy$$

$$= \frac{1}{2\pi}\left[\hat{h}(2N+1) - \hat{h}(2N-1)\right]$$

where $h(x) = 2g(2x)$

The Riemann-Lebesgue lemma tells us

$$\frac{1}{2i}\left[\hat{h}(2N+1) - \hat{h}(-2N-1)\right] \to 0 \quad \text{as } N \to \infty$$

Thus $(S_N f)(x_0) \to f(x_0)$ $\square$

So if f is diff'ble, then $S_N f \to f$ everywhere.

In fact, this theorem can be extended to show that if f is piecewise diff'ble at x_0

$$(S_N f)(x_0) \to \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right].$$

Sadly, this doesn't work for continuous functions!

But happily we have the Fejér kernel

<u>Defn</u> Let

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin(x/2)}$$

$$= \frac{\sin^2\left(\frac{Nx}{2}\right)}{N\sin^2\left(\frac{x}{2}\right)} \quad \text{(Calculation!)}$$

Then

- $F_N(x) \geq 0 \quad \forall x$
- $\int_{-\pi}^{\pi} F_N(x)\,dx = 2\pi$
- $\forall\, 0<\delta<\pi$ $\lim_{N\to\infty} \int_{\delta<|x|<\pi} F_N(x)\,dx = 0$.
- Let $T_N f = \frac{1}{N} \sum_{n=0}^{N-1} S_n f$

Then $T_N f = F_N * f$.

F_N is called a summability kernel.

Theorem (Fejér) If f is continuous and periodic, then

$$T_N f \to f \quad \text{uniformly on } \mathbb{R}$$

Proof As above

$$(T_N f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\left(f(x-y)-f(x)\right]dy$$

$$= \frac{1}{2\pi}\int_{|y|<\delta} F_N(y)\left[f(x-y)-f(x)\right]dy$$

$$+\frac{1}{2\pi}\int_{\delta<y<\pi} F_N(y)\left(f(x-y)-f(x)\right]dy$$

Since f is cts. and periodic, it is uniformly cts on I

Since $|f(x-y)-f(x)| \le 2\sup|f(y)| = C$ (say)

$$\frac{1}{2\pi}\int_{\delta<|y|<\pi} F_N(y)\left[\cancel{f(x)-f(y)}\ f(x-y)-f(x)\right]dy \le \frac{C}{2\pi}\int_{\delta<|y|<\pi} F_N(y)\,dy \to 0 \text{ as } N\to\infty$$

Choose N so large that the RHS is $< \frac{\varepsilon}{2}$.

Now estimate the other integral

$$\left|\frac{1}{2\pi}\int_{|y|<\delta} F_N(y)\left(f(x-y)-f(x)\right]dy\right|$$

$$\le \sup\{|f(x-y)-f(x)| : |y|<\delta\}\ \underbrace{\frac{1}{2\pi}\int F_N(y)\,dy}_{\le 1}$$

Since f is unif cts, one can choose δ so that $|y|<\delta \Rightarrow |f(x-y)-f(x)| < \frac{\varepsilon}{2}$

Hence

$$|T_N f(x)-f(x)| < \varepsilon \quad \text{for } N \text{ large enough}$$

$\forall x\in I$.

$\square$