# 12 Riemann Integral

A <u>partition</u> of an interval $[a,b]$ is a finite set of numbers $P=\{x_0, x_1, \ldots x_n\}$ with $x_0=a$, $x_n=b$ and $x_0<x_1<\ldots<x_n$

If $f$ is bounded on $[a,b]$, let ~~$M_i=$~~

$$M_i = \sup\{f(x) : x_{i-1} \le x < x_i\}$$
$$m_i = \inf\{f(x) : x_{i-1} \le x < x_i\}$$

The upper and lower Riemann sums

$$\mathcal{U}(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$
$$\mathcal{L}(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

(Clearly these are bounded above by $\|f\|_\infty (b-a)$ and below by $-\|f\|_\infty (b-a)$)

Let $\overline{\int_a^b} f = \inf\limits_{P \text{ a partition of } [a,b]} \mathcal{U}(f,P)$

$$\underline{\int_a^b} f = \sup_P \mathcal{L}(f,P)$$

These both exist and we say $f$ is Riemann integrable if $\overline{\int_a^b} f = \underline{\int_a^b} f$ and denote the common value by $\int_a^b f$.

- Of course you know all about the Riemann integral
  - $\int_a^b c\,dx = c(b-a)$
  - $|\int_a^b f| \le \int_a^b |f|$
  - it's linear
  - $\int f \geq 0$ if $f \geq 0$ on $(a,b)$

etc.

By the defn of sup and inf plus the fact that if $P_0$ is a <u>refinement</u> of $P_1$ then

$$\mathcal{U}(f,P_0) \leq \mathcal{U}(f,P_1) \quad \text{and} \quad \mathcal{L}(f,P_0) \geq \mathcal{L}(f,P_1)$$

we get Riemann's criterion

- $f$ is Riemann integrable iff $\forall \varepsilon > 0 \ \exists P$ with
$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon.$$

<u>Corollary 1</u> Every continuous $f^n$ on a closed bdd interval is Riemann integrable

<u>Corollary 2</u> If $f : [a,b] \to \mathbb{R}$ is monotone increasing and $f(b) < \infty$ then $f$ is Riemann integ.

N.B. $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ defined on $[0,1]$, is <u>not</u> Riemann integrable.

<u>Theorem</u> (Fundamental Thm of Calculus I)
If $f$ is cts on $[a,b]$, then $\forall x \in [a,b]$

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

<u>Proof</u> (see note P 39)

<u>Thm</u> (FTC II) If $f$ is Riemann integ. on $[a,b]$ and if $F' = f$ on $(a,b)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

~~See the~~ <u>Mean Value Thm for integrals</u>. If $f, g$ cts on $[a,b]$, $g(x) \geq 0$. Then $\exists\ c \in [a,b]$ s.t.

$$\int_a^b f(x) g(x)\,dx = f(c) \int_a^b g(x)\,dx. \quad \text{(By Int. Value Thm)}$$

See the nice use of this theorem on P41 of the note to show that the remainder term in the Taylor series is

$$R_n(a,x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n)}(t)\,dt$$

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## Improper Riemann integrals

If $f:[a,b]\to\mathbb{R}$ is cts on $(a,b]$ but discts at $a$. Then we define $\int_a^b f(x)\,dx = \lim_{c\to a}\int_c^b f(x)\,dx$ if the limit exists.

Similarly if $f$ discts at $b$ $\int_a^b f(x)\,dx = \lim_{c\to b}\int_a^c f(x)\,dx$

Example of $f(x)=\frac{1}{\sqrt{x}}$ on $[0,1]$.

Can likewise define $\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx$ and $\int_{-\infty}^b$ etc.

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## Changes of integrals and limits

It's often tempting (but wrong!) to exchange limits and integrals.

For example, if $f_n(x)\to f(x)$ for all $x\in[a,b]$. does $\int_a^b f_n(x)\,dx \to \int f(x)\,dx$?

eg $f_n(x) = nxe^{-nx^2}$ on $[0,1]$

sliding lump on $[0,\infty)$.

Another bad example $f_{n,j}(x) = (\cos(n!\pi x))^j$

if $x$ is rational $\lim_{n,j\to\infty} f_{n,j}(x) = 1$

if $x$ irrational $\lim_{n,j\to\infty} f_{n,j}(x) = 0$

so $f_{n,j}$ converges to a fn. which is not Riemann integrable, even though the $f_{n,j}$ are analytic.

## 13 More on Uniform Convergence

Basically, we need uniform convergence of $f_n$ to make this conclusion.

Lemma. If $f_n \xrightarrow{(unif)} f$ on a closed bdd interval $[a,b]$ and $g_n \to g$ (unif) on $[a,b]$ then $f_n g_n \xrightarrow{(unif)} fg$ on $[a,b]$.

Theorem (Weierstrass M-test). S'pose $\{f_n\}$ is a sequence of fns on $X \subseteq \mathbb{R}$ with $|f_n(x)| < M_n$ $\forall x \in X$, where $\sum_{n=1}^{\infty} M_n < \infty$.

Then the sequence $\sum_{n=1}^{\infty} f_n(x)$ is unif convergent

Proof. Let $S_N(x) = \sum_{n=1}^{N} f_n(x)$. Then for $N \geq M$

$$|S_N(x) - S_M(x)| = \left|\sum_{n=M+1}^{N} f_n(x)\right|$$

$$\leq \sum_{n=M+1}^{N} |f_n(x)| \leq \sum_{n=M+1}^{N} M_n. \quad (\forall x)$$

Since $\sum_{n=1}^{\infty} M_n < \infty$, we can choose K so large that $M, N \geq K \Rightarrow \sum_{n=M+1}^{N} M_n < \varepsilon$.

Thus $S_N(x)$ converges uniformly

eg $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1}$ is continuous

**Theorem** If $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R$ and $0 < r < R$ then the series converges uniformly on $[-r, r]$.

(N.B. They don't converge on $(-R, R)$ even the G.P. $\sum_{n=1}^{\infty} x^n \to \frac{1}{1-x}$ not uniform on $(-1, 1)$

**Theorem** (Dini's Test) Let $f_n$ (continuous) converge monotonically to $f$ (continuous) on $[a, b]$.

Then $f_n \to f$ uniformly on $[a, b]$

**Proof** Wlog $f = 0$ and $f_n$ decreases monotonically to $0$ i.e. $\forall x$, $f_n(x) \searrow 0$ as $n \to \infty$.

Let $M_n = \sup\{f_n(x) : x \in [a, b]\}$. Since $f_{n+1} \leq f_n$ $M_{n+1} \leq M_n$. Want to show $M_n \to 0$.

Suppose not. Then $\exists \delta > 0$ s.t. $M_n \geq \delta$ $\forall n$. Thus $\exists x_n$ s.t. $f_n(x_n) \geq \delta$

$\{x_n\} \subseteq [a, b]$ which is compact so by Bolzano-W. it has a convergent subsequence $\{x_{n_k}\}$; $x_{n_k} \to z \in [a, b]$ as $k \to \infty$. But our assumption is that $f_n(z) \to 0$ as $n \to \infty$. So $\exists N$ s.t. $n \geq N \Rightarrow f_n(z) < \frac{\delta}{2}$ Choosing $n \geq N$, since $f_n$ is cts. at $z$ we can choose $\eta$ so that $|x - z| < \eta \Rightarrow |f_n(x) - f_n(z)| < \frac{\delta}{2}$

Then $|x - z| < \eta \Rightarrow f_n(x) < \delta \Rightarrow f_{n_k}(x) < \delta$ for all $k$ with $n_k \geq n$. Now choose $|x_{n_k} - z| < \delta$ and we have a contrad. $\square$

## Weierstrass Approximation Theorem

"Every cts. fn on a compact set can be uniformly approximated by a polynomial!"

**Theorem (W)** If $f$ is continuous on a compact set, then

$\forall \varepsilon > 0$ $\exists$ a polynomial $P$ such that

$$\|f - P\|_\infty < \varepsilon .$$

Proof due to Bernstein

**Lemma**
- $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$ (binomial thm)
- $\sum_{k=1}^{n} k \binom{n}{k} x^k (1-x)^{n-k} = nx$
- $\sum k^2 \binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 + nx$

**Proof** (i) ~~is~~ is the binomial theorem

(ii) $k\binom{n}{k} = n\binom{n-1}{k-1}$

(iii) $k^2 = k(k-1) + k$ $\square$

**Proof of Weierstrass Thm** Wlog $[a,b] = [0,1]$

If $f$ cts on $[0,1]$, let

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

(The Bernstein polys for $f$)

Claim $P_n \to f$ (unif) on $[0,1]$

Let $\varepsilon > 0$. We want to show $\exists N$ s.t. $n \geq N$

$$\Rightarrow \sup_{0 \leq x \leq 1} |P_n(x) - f(x)| < \varepsilon$$

Now $P_n(x) - f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} - f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$

$$= \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x)\right] \binom{n}{k} x^k (1-x)^{n-k}$$

Note that $f$ is unif. cts. ~~Given~~ Choose $\delta > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$

We want to find values of $\frac{k}{n}$ s.t. $|x - \frac{k}{n}| < \delta$

$$P_n(x) - f(x) = \sum_{\{k : |x - \frac{k}{n}|\} < \delta} \left[f\left(\frac{k}{n}\right) - f(x)\right] \binom{n}{k} x^k (1-x)^{n-k}$$

$$+ \sum_{\{k : |x - \frac{k}{n}| > \delta\}} \left[f\left(\frac{k}{n}\right) - f(x)\right] \binom{n}{k} x^k (1-x)^{n-k}$$

$$= S_1 + S_2$$

Since $f$ is unif cts, and $\sum_k \binom{n}{k} x^k (1-x)^{n-k} = 1$

$|S_1| < \frac{\varepsilon}{2}$ since $|x - \frac{k}{n}| < \delta$ for all $k$ in this sum

Now look at $S_2$

$$|S_2| \leq \sum_{|x - \frac{k}{n}| \geq \delta} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq 2 \sup_{x \in [0,1]} |f(x)| \left| \sum_{|x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \right|$$

Since for the $k$'s in the second sum $x - \frac{k}{n} \geq \delta$, we have $(x - \frac{k}{n})^2 \geq \delta^2$, so

$$\sum_{|x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{\delta^2} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{1}{\delta^2} \sum_{k=0}^{n} \left(x^2 - \frac{2xk}{n} + \frac{k^2}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{1}{\delta^2}\left(x^2 - \frac{2x}{n}\cdot nx + \frac{1}{n^2}\left(n(n-1)x^2 + nx\right)\right)$$

$$= \frac{x(1-x)}{n\delta^2} \leq \frac{1}{4n\delta^2}. \quad \text{since } x(1-x) \leq \frac{1}{4} \text{ if } 0 \leq x \leq 1$$

Thus we get

$$\sum_{|x-\frac{k}{n}|\geq\delta} \left[f\left(\tfrac{k}{n}\right) - f(x)\right]\binom{n}{k} x^k(1-x)^{n-k}\Big] \leq \frac{2A}{4n\delta^2}$$

where $A = \|f\|_\infty = \sup_{0\leq x\leq 1} |f(x)|$.

Thus, if we choose $n \geq \frac{2\varepsilon A}{\delta^2}$.

Then $\|P_n - f\| < \varepsilon$. $\square$

<u>Remark</u>. The Stone-Weierstrass theorem ~~says that~~ If $A$ is a subalgebra of $C(X)$ which contains the constant functions and which separates points of $X$ i.e. if $x \neq y$ in $X$ $\exists$ $a$ in $A$ with $a(x) \neq a(y)$. Then $A$ is dense in $C(X)$.

(Let $X$ be a compact Hausdorff space)

<u>Corollary</u> If $\{f_n\}$ is a sequence of Riemann integrable fns, $f_n \to f$ (unif) on $[a,b]$. Then $f$ is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

<u>Proof</u> If the $f_n$ are cts, then $f$ is cts so for $n \geq N$

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon.$$

Completeness of $C(X)$

We now show that $C(X)$ is complete under $\|\cdot\|_\infty$.

Proposition If $f_n$ is a uniformly Cauchy sequence of functions, it converges uniformly to a continuous function

Proof If $\{f_n\}$ is a uniformly ~~convergent~~ Cauchy sequence on an $X \subseteq \mathbb{R}$, then for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence of numbers, so it has a limit $f(x)$. This defines a function $f$. If we show $f_n \to f$ (unif) then our previous theorem allows us to deduce that $f$ is continuous.

If $\varepsilon > 0$, choose $N$ s.t. $n, m \geq N \Rightarrow$

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}$$

Since $f_m(x) \to f(x)$ pwise, this $\Rightarrow$ that

$$|f(x) - f_n(x)| \leq \frac{\varepsilon}{2}$$

Hence $f_n \to f$ (unif)

$\square$

Interchange of limits and derivatives

~~Thm~~ Example $f_n(x) = \dfrac{x}{1+nx^2}$ so $f_n \to 0$ even uniformly as $n \to \infty$

But $f_n'(x) = \dfrac{1 - \cancel{2}nx^2}{(1+nx^2)^2}$ so $f_n'(0) = 1$

Theorem. If $I$ is an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$, $\{f_n\}$ a sequence of continuous functions on $I$ which converge pwise to $f$.

Suppose $f_n' \to g$ (unif) on $I$.

Then $f$ is differentiable and $f'(x) = g(x)$.

# CH2 INTEGRATION

## 1 Riemann–Stieltjes integration

The idea is to take a bdd fn $\phi$ and consider sums of the fn

$$\sum f(x_i^*)\left[\phi(x_i) - \phi(x_{i-1})\right] = \sum_P f \Delta \phi$$

**Defn** $f, \phi$ bdd on $[a,b]$

S'pose $\exists A$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$

$$\left| \sum_{k=1}^{n} f(c_k)\left(\phi(x_k) - \phi(x_{k-1})\right) - A \right| < \varepsilon$$

whenever $x_{k-1} \le c_k \le x_k$ and $|x_k - x_{k-1}| < \delta$

Then $f$ is RS integrable and

$$RS \int_a^b f(x)\, d\phi(x) = A$$

or $\int_a^b f\, d\phi = A$

If $\phi(x) = x$, this becomes the Riemann integral

If $\phi$ = constant $\int f\, d\phi = 0$

If $f = c$ $\quad \int_a^b c\, d\phi = c\left[\phi(b) - \phi(a)\right]$

**Thm** S'pose $f$ is cts on $[a,b]$ and $\phi$ is a step fn

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{(x_{k-1}, x_k)}(x)$$

Then the RS integral exists and

$$\int f(x)\, d\phi(x) = \sum_{k=1}^{n} f(x_k)\left[\phi(x_k^+) - \phi(x_k^-)\right]$$

Example

$f(x) = x^2 \qquad \phi(x) = [x]$

$$\int_0^n x^2 \, d[x] = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Theorem S'pose $f$ is ~~cts~~ and $\phi$ diff'ble on $(a,b)$

$\phi'$ Riemann integ. on $[a,b]$

Then $RS\int_a^b f \, d\phi = \int_a^b f(x)\phi'(x)dx$

where the RHS is a regular Riemann integral

Proof Notes

Eg $\displaystyle\int_0^1 x^2 \, d(x^2) = \int_0^1 x^2 \, 2x \, dx = 2\int_0^1 x^3 dx$

$= 2 \times \frac{1}{4} = \frac{1}{2}.$

Integration by parts looks nice

Theorem If $f, \phi$ are bdd with no common discontinuities on $[a,b]$ and $\int_a^b f d\phi$ exists.

Then $\int_a^b \phi \, df$ also exists and

$$\int_a^b \phi(x) \, df(x) = f(b)\phi(b) - f(a)\phi(a) - \int_a^b f(x) \, d\phi(x)$$

$$\left[ \int_a^b \phi \, df = [f\phi]_a^b - \int f \, d\phi \right]$$

~~[illegible]~~

<u>Thm</u> If $\phi$ is cts on $[a,b]$ and $\phi$ is monotone increasing on $[a,b]$ then $\int_a^b f\,d\phi$ exists.

Call $F(x) = \int_a^x f\,d\phi$

Then

(1) F is continuous at any pt where $\phi$ is cts.

(2) F is diff'ble at any pt. where $\phi$ is diff'ble

and $F'(x) = f(x)\phi'(x)$.