Ch3 MEASURABLE FUNCTIONS

(3.1) Let f be a function $f : \mathrm{Dom}(f) \to \mathbb{R} \cup \{-\infty, \infty\}$, where $\mathrm{Dom}(f) \subseteq \mathbb{R}$

(an extended real valued function)

Then f is measurable if $\mathrm{Dom}(f)$ is msble

and $\forall \alpha \in \mathbb{R}$ $\{x : f(x) > \alpha\}$ is msble

Example · $\chi_E(x)$ is msble if E is msble

- If $E_1, E_2 \ldots E_n$ are msble and $a_1, \ldots, a_n \in \mathbb{R}$

  then $f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is msble

  (called a simple function or step function)

Proposition Let f be an extended real-valued function with $\mathrm{Dom}(f)$ msble. TFAE

(1) $\forall$ real numbers α $\{x : f(x) > \alpha\}$ is msble

(2) - - - - - - $\{x : f(x) \geq \alpha\}$ - - -

(3) - - - - - - $\{x : f(x) < \alpha\}$ - - -

(4) - - - - - - $\{x : f(x) \leq \alpha\}$ - - -

Proof (1) $\Leftrightarrow$ (4) since $\{x : f(x) > \alpha\} = \{x : f(x) \leq \alpha\}^c \cap \mathrm{Dom}(f)$

(2) $\Leftrightarrow$ (3) by the same argument

(1) $\Rightarrow$ (2) since $\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$

(2) $\Rightarrow$ (1) same argument with Unions $\square$

Comments

- These conditions imply that $\{x : f(x) = \alpha\}$ is msble, but that is not a sufficient condition.

- If f is continuous and has measurable domain, then f is measurable

  ($\{x : f(x) > \alpha\} = f^{-1}(a, \infty)$ is open, therefore msble)

Theorem A If c is constant and f, g are measurable then $f+c$, cf, $f+g$, $f-g$ and fg are all msble.

Theorem B Let $\{f_n\}$ be a sequence of extended real-valued measurable functions with $\mathrm{Dom}(f_n) = \mathrm{Dom}\, f_0$ $\forall n$

Then the following are all measurable

(i) $\sup\{f_1, \dots, f_n\}$

(ii) $\inf\{f_1 \dots f_n\}$

(iii) $\sup_n \{f_n\}$

(iv) $\inf_n \{f_n\}$

(v) $\limsup_n f_n = \inf_n (\sup_{k \geq n} f_k)$

(vi) $\liminf_n f_n = \sup_n (\inf_{k \geq} f_k)$

Proof Let $h = \sup_n f_n$

Then $\{x : h(x) < \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_i(x) < \alpha\}$

This is measurable. Hence (i) and (iii)

Similarly (ii) and (iv). (v) and (vi) ~~now~~ follow. Δ

Definition. Let P be a property of the values $f(x)$ of a function f.

We say the P holds almost everywhere (a.e) if $\{x : (f(x) \text{ is } P) \text{ is false}\}$ is a set of measure zero

Theorem If f is measurable and $f = g$ a.e. then g is measurable.

(3.2) Convergence of Measurable Functions

Defn. Let $\{f_k\}$ be a sequence of measurable finite-valued a.e. functions, and f a f vae fn.

- $f_k \to f$ pwise ae

  means for a.e. x $f_k(x) \to f(x)$

- $f_k \to f$ unif a.e.

  means $\forall \varepsilon > 0$ $\exists N$ s.t. $n > N \implies$

  $$|f_n(x) - f(x)| < \varepsilon \quad \text{for a.e. } x \in \mathbb{R}$$

Egoroff's Theorem

Theorem. Suppose E is a measurable set with $\lambda(E) < \infty$ and let $\{f_k\}$ be a sequence of msble functions converging pwise ae. to a function f.

Then $\forall \varepsilon > 0$ $\exists$ a subset $F \subseteq E$, with $\lambda(F) < \varepsilon$ such that $f_k \to f$ uniformly on $E \setminus F$

Proof By deleting a set of measure zero, we can assume $f_k \to f$ everywhere

Let $$E_n^m = \bigcap_{i=1}^{\infty} \{x : |f_i(x) - f(x)| < \tfrac{1}{m}\}$$

So that $E_1^m \subseteq \dots \subseteq E_n^m \subseteq E_{n+1}^m \subseteq E$

and $\bigcup_{n=1}^{\infty} E_n^m = E$ for any m

Thus $\lim_{n\to\infty} \lambda(E \setminus E_n^m) = 0$ for any m.

Thus $\forall m$ $\exists N_0(m)$ s.t. $n \geq N_0(m)$

$$\Rightarrow \lambda(E - E_n^m) < \frac{\varepsilon}{2^m}.$$

Now let $F = \bigcup_{m=1}^{\infty} \left(E \setminus E^m_{N_0(m)}\right)$

Then F is a measurable subset of E and

$$\lambda(F) \leq \sum_{m=1}^{\infty} \lambda\left(E \setminus E^m_{N_0(m)}\right) < \varepsilon \sum \frac{1}{2^m} = \varepsilon.$$

We need to show $f_n \to f$ uniformly on $E \setminus F$

Now $E - F = E \cap F^c = E \cap \left(\bigcup_{m=1}^{\infty} E \cap \left(E^m_{N_0(m)}\right)^c\right)^c$

$$= E \cap \left(\bigcap_{m=1}^{\infty} E^c \cup E^m_{N_0(m)}\right)$$

$$= E \cap E^c \cup \bigcap_{m=1}^{\infty} E^m_{N_0(m)} = \bigcap_{m=1}^{\infty} E^n_{N_0(m)}$$

Thus, for all $x \in E \setminus F$, for $n > N_0(m)$ we have

$$|f_n(x) - f(x)| < \frac{1}{m}$$

Thus $f_n \to f$ unif on $E - F$ $\square$

<u>Defn</u>. Let $\{f_n\}$ be a sequence of f.-v.-a.e. fns. We say $f_n \to f$ <u>in the sense of Egoroff</u> if $\forall \varepsilon > 0$ $\exists F$ msble $\subseteq E$ with $\lambda(F) < \varepsilon$ and $f_n \to f$ unif on $E \setminus F$

Converse to Egoroff's Theorem

Let $\{f_k\}$ be a sequence of msble fns with $f_k \to f$ in the sense of Egoroff. Then

$$f_k \to f \quad \text{pwise a.e.}$$

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This means that pwise a.e. convergence is equivalent to "uniform off a set of arbitrarily small measure.

Lusin's Theorem