Converse to Egoroff's Theorem

Let $\{f_k\}$ be a sequence of msble fns with $f_k \to f$ in the sense of Egoroff. Then

$$f_k \to f \quad \text{pwise a.e.}$$

— 1 —

This means that pwise a.e. convergence is equivalent to "uniform off a set of arbitrarily small measure.

Lusin's Theorem

Theorem (Lusin) Let f be a f.v. a.e. msble fn. defined on a measurable set E with $\lambda(E) < \infty$

Given $\varepsilon > 0$ there is a continuous function g so that

$$|f(x) - g(x)| < \varepsilon \quad \text{except on a set of}$$

measure less than ε.

Similarly, there is a simple function h so that

$$\lambda(\{x : |f(x) - h(x)| \geq \varepsilon\}) < \varepsilon.$$

Proof We do the second part first. Suppose $f \geq 0$. For each $n \in \mathbb{N}$ and for $1 \leq k \leq 2^n \cdot n$, let

$$\left.\begin{aligned} A_{n,k} &= \{x \in X : \tfrac{k-1}{2^n} \leq f(x) < \tfrac{k}{2^n}\} \\ B_n &= \{x \in X : f(x) \geq n\} \end{aligned}\right\}$$ Clearly both measurable sets

Define

$$s_n(x) = \sum_{k=1}^{2^n \cdot n} \frac{k}{2^n} \chi_{A_{n,k}}(x) + n\chi_{B_n}(x)$$

Then $\left| f(x) - s_n(x) \right| < \frac{1}{2^n}$ for all $x \in B_n^c$, so

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \text{a.e. } x \in X.$$

If f is not ≥ 0 everywhere, apply the result to $\{x : f(x) \geq 0\}$ and on $\{x : f(x) < 0\}$ to $-f(x)$ to see $\exists$ a sequence of simple functions f_n with $f_n \to f$ pwise a.e. Now apply Egoroff to get the result.

Now for the first statement.

Let $\varepsilon > 0$. Choose an open set U with $E \subseteq U$ and $\lambda(U) \leq \lambda(E) + \frac{\varepsilon}{4}$.

If $f = \chi_A$ for a measurable set $A \subseteq E$ (with $\lambda(A) < \infty$) there is a compact set $F \subseteq A$ with $\lambda(A \cap F^c) < \frac{\varepsilon}{2}$

By the Tietze Extension theorem $\exists$ a continuous function g with $g \equiv 1$ on F and $g \equiv 0$ outside U

Thus $\lambda\{x : f(x) = g(x)\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Next, use the first part of the proof to choose a simple function $h = \sum_{j=1}^{n} a_j \chi_{A_j}$ with $|h(x) - f(x)| \geq \varepsilon$ on a set of measure $< \varepsilon$.

For each $j = 1 \ldots n$, choose a continuous function g_j so that $|\chi_{A_j} - g_j| \geq \frac{\varepsilon}{n}$ on a set F_n of measure $< \frac{\varepsilon}{n}$

It follows that $\left| f - \sum_{j=1}^{n} a_j g_j \right| < \varepsilon$ on $E \setminus \bigcup F_n$.

The measure of the complement of this set is $\lambda(\bigcup F_n) \leq n \cdot \frac{\varepsilon}{n} = \varepsilon$

Littlewood's Three Principles

(1) A measurable set is "almost" open

(2) A measurable function is "almost" continuous

(3) Pwise a.e. convergence is "almost" uniform.

Convergence in measure

This is another type of convergence of msble fns

Definition. Let $\{f_n\}$ be a sequence of f.v.a.e. msble. fns. We say that $f_n \to f$ in measure if

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \lambda(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

We say that $\{f_n\}$ is Cauchy in measure if $\forall \varepsilon, \delta > 0 \; \exists N$ s.t. $n, k \geq N \implies$

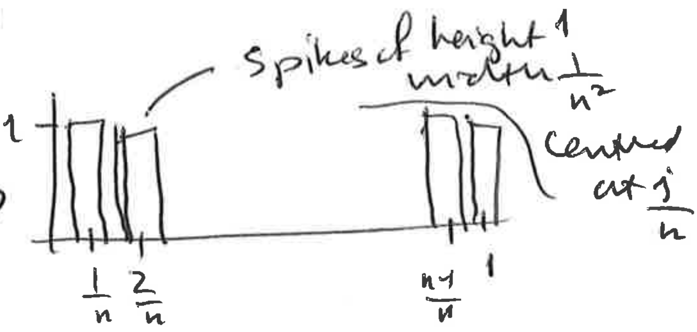
$$\lambda(\{x : |f_n - f_k(x)| \geq \varepsilon\}) < \delta.$$

- Clearly $f_n \to f$ pwise a.e. $\Rightarrow$ $f_n \to f$ in measure

(use Egoroff's Theorem)

- Converse not true

Let $f_n(x) =$



Given $0 < \varepsilon < 1$ $\quad \lambda\{x : f_n(x) > \varepsilon\} = n \cdot \frac{1}{n^2} = \frac{1}{n} \to 0$

So $f_n(x) \to 0$ in measure, but $f_n \not\to 0$ pwise ae

However there's a completeness result for convergence in measure which is quite useful (It's called convergence in probability in prob. Theory)

<u>Thm</u> Let $\{f_k\}$ be a sequence of msble. fns, Cauchy in measure. Then there is a function f s.t. $f_k \to f$ in measure.

<u>Proof</u>

We first prove a Lemma

<u>Lemma</u> Let $\{f_n\}$ be a sequence of measurable functions Cauchy in measure. Then some subsequence is Cauchy in the Egoroff sense.

<u>Proof</u>. $\forall k \ \exists n(k) : n, m \geq n(k)$

$$\Rightarrow \lambda\left(\left\{x : |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$

Chose an increasing sequence $n_1, n_2, n_3 \ldots$ so that

$$\lambda(E_k) = \lambda\left(\left\{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$

Now $\forall x \in \left(\bigcup_{k=i}^{\infty} E_k\right)^c$, we have for $k, j \geq i$ (say $k > j$)

$$|f_{n_k}(x) - f_{n_j}(x)| \leq \sum_{m=j}^{k} |f_{n_m}(x) - f_{n_{m+1}}(x)| < \sum_{m=j}^{k} \frac{1}{2^m}$$

$$\leq \frac{1}{2^j} \leq \frac{1}{2^i}.$$

Furthermore

$$\lambda\left(\bigcup_{k=i}^{\infty} E_k\right) \leq \sum_{k=i}^{\infty} \frac{1}{2^k} = \frac{1}{2^i}$$

Thus f_{n_k} is Cauchy in the Egoroff sense i.e. pwise a.e. $\square$ Lemma proved!

Now define $f(x) = \lim_{k\to\infty} f_{n_k}(x)$

Then f is measurable.

We claim $f_n \to f$ in measure. To see this consider

call this set A

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \{x : |f_n(x) - f_{n_k}(x)| \geq \tfrac{\varepsilon}{2}\}$$

$$\cup \{x : |f_{n_k} - f(x)| \geq \tfrac{\varepsilon}{2}\}$$

call this set B

Now f_n is Cauchy in measure so, for n, n_k large enough, $\lambda(A)$ is small. But $f_{n_k} \to f$ in the sense of Egoroff, so $\lambda(B)$ is small if n_k is large enough.

Thus $\lambda\{x : |f_n(x) - f(x)| \geq \varepsilon\}$ is small if n is large enough. $\square$

(33) $\boxed{\text{The Lebesgue Integral}}$

If $f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is a <u>simple function</u> with $E_1, \ldots, E_n$ being disjoint measurable sets and $a_1 \ldots a_n \in \mathbb{C}$, define

$$\int f \, d\lambda = \int f(x)\, dx = \sum_{i=1}^{n} a_i \lambda(E_i)$$

Further, define, for a msble set E

$$\int_E f \, d\lambda = \int f(x) \chi_E(x)\, dx \quad \left(= \sum_{i=1}^{n} a_i \lambda(E \cap E_i) \right)$$

<u>Basic properties of this integral</u>

If f, g are simple functions and $\alpha, \beta \in \mathbb{C}$

then

- $\int (\alpha f + \beta g)\, d\lambda = \alpha \int f \, d\lambda + \beta \int g \, d\lambda$
- If f is real-valued and $f \geq 0$ a.e. then $\int f \, d\lambda \geq 0$
- ~~the~~ Note that $|f+g|$, $|f|$ and $|g|$ are all simple functions and

$$\int |f+g| \, d\lambda \leq \int |f| \, d\lambda + \int |g| \, d\lambda$$

(so $\|f\|_1 = \int |f| \, d\lambda$ is a norm on the vector space of all simple functions.)

Definition. We say that $\{f_k\}$, a sequence of simple functions, is Cauchy in mean if

$\forall \varepsilon > 0 \ \exists N$ s.t. $m, n > N \Rightarrow \int |f_k - f_m| \, d\lambda < \varepsilon$

[N.B. If $\{f_k\}$ is Cauchy in mean, then $\lim_{k \to \infty} \int f_k \, d\lambda$ exists.]

Definition of integrable function

Defn. A finite valued almost everywhere function f is said to be integrable if $\exists$ a sequence of integrable simple functions $\{f_k\}$ with $\{f_k\}$ Cauchy in mean and such that $f_k \to f$ in measure.

The integral of f is given by

$$\int f \, d\lambda = \lim_{k \to \infty} \int_E f_k \, d\lambda.$$

We have to show this is well-defined. Specifically, is it independent of the choice of $\{f_k\}$?

Lemma A. Suppose that $\{f_k\}$, $\{g_k\}$ are sequences of integrable simple functions, Cauchy in mean with
$$\begin{cases} f_k \to f & \text{in measure} \\ g_k \to f & \text{in measure} \end{cases}$$

Then $\displaystyle\lim_{k\to\infty} \int_E g_k \, d\lambda = \lim_{k\to\infty} \int f_k \, d\lambda$.

Proof. Suppose first f, $\{f_k\}$, $\{g_k\}$ are all supported on E with $\lambda(E) < \infty$. Let $\varepsilon > 0$. Since $\{f_k\}$ and $\{g_k\}$ are both Cauchy in mean, choose k_1 so large that $k \geq k_1 \Rightarrow$

$$\int |f_k - f_{k_1}| \, d\lambda < \varepsilon \quad \text{and} \quad \int |g_k - g_{k_1}| \, d\lambda < \varepsilon$$

Now $\exists\, k_0 \geq k_1$ such that $k \geq k_0 \Rightarrow$

$\lambda(E_k') < \frac{\varepsilon}{2}$ and $\lambda(E_k'') < \frac{\varepsilon}{2}$, where

$$E_k' = \{x \in E : |f(x) - f_k(x)| \geq \varepsilon\}$$

$$E_k'' = \{x \in E : |f(x) - g_k(x)| \geq \varepsilon\}$$

Then for $k \geq k_0$, $\lambda(E_k' \cup E_k'') < \varepsilon$, and

$$|f_k(x) - g_k(x)| < \varepsilon \quad \text{for } x \in (\underbrace{E_k' \cup E_k''}_{E_k})^c$$

We may assume that $k \geq k_0 \Rightarrow$

$$\int_{E_k} |f_{k_1}| \, d\lambda < \varepsilon \quad \text{and} \quad \int_{E_k} |g_{k_1}| \, d\lambda < \varepsilon.$$

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Thus for $k \geq k_0$

$$\left|\int_E (f_k - g_k)\,d\lambda\right| \leq \int_{E \setminus E_k} |f_k - g_k|\,d\lambda + \int_{E_k} |f_k - f_{k_1}|\,d\lambda$$

$$+ \int_{E_k} |g_k - g_{k_1}|\,d\lambda + \int_{E_k} |f_{k_1}|\,d\lambda + \int_{E_k} |g_{k_1}|\,d\lambda$$

$$\leq \varepsilon\,\lambda(E) + \varepsilon + \varepsilon + \varepsilon + \varepsilon = \varepsilon(\lambda(E) + 4)$$

Thus $\lim_k \int_E f_k\,d\lambda = \lim_k \int_E g_k\,d\lambda$.

Now suppose we have no restriction on the support of f.

Since $\lambda\{x : |f(x) - f_k(x)| \geq \delta\} \to 0$ and f_k is finitely supported, $\lambda\{x : |f(x)| \geq \delta\}$ must be finite. Call it $E(\delta)$.

Apply the ~~previous~~ first part of the proof to $f|_{E(\frac{1}{2^r})}$, $f_k|_{E(\frac{1}{2^r})}$, $g_k|_{E(\frac{1}{2^r})}$ to deduce that

$\forall \varepsilon > 0 \ \exists k_0^{(r)}$ such that $k > k_0 \Longrightarrow$

$$\int_{E(\frac{1}{2^r})} |f_k - g_k|\,d\lambda < \varepsilon$$

Now let $r \to \infty$, to see

$$\int_E |f_k - g_k|\,d\lambda < \varepsilon.$$

$\square$

<u>Lemma B</u> Let f be an a.e. bdd msble fn on a set E of finite measure.

Then $\exists$ a sequence $\{f_k\}$ of simple functions such that $f_k \to f$ in measure and f_k is Cauchy in mean.

Thus f is integrable.

<u>Proof</u> Since f is a.e. bdd $\exists$ N s.t.

$-N \leq f(x) \leq N$ for all $x \in E$.

Now for each integer k, $[-N, N]$ can be divided into 2^k intervals of length $\frac{2N}{2^k}$ by

$$I_{k,i} = \left(-N + (i-1)\frac{2N}{2^k}, \; -N + i\frac{2N}{2^k}\right] \quad i = 1 \dots 2^k$$

Since f is msbl, $f^{-1}(I_{k,i})$ is a msble set $\forall i, k$

Set $E_{k,i} = f^{-1}(I_{k,i})$ and define a function

$$f_k(x) = \inf_{x \in E_{k,i}} [f(x)]$$

Then

(1) $f_k \to f$ in measure (indeed, $f_k \to f$ unif.)
because by defn $|f_k(x) - f(x)| \leq \frac{2N}{2^k}$ $\forall x \in E$

(2) $\{f_k\}$ is Cauchy in mean because if $k, l > k_0$

then $|f_k(x) - f_l(x)| < \frac{2N}{2^{k_0}}$

$\therefore \int_E |f_k(x) - f_l(x)| \, d\lambda < \frac{2N}{2^{k_0}} \lambda(E)$

So, as $k_0 \to \infty$, the RHS $\to 0$

__Observe__ that if f, g are integrable, then $f+g$, $|f|$, $f \vee g$ are all integrable

The basic result for integration of simple functions pass over to integrable functions

__Defn__ If $\{f_n\}$ and f are integrable, we say $f_n \to f$ __in mean__ (or in L^1) if

$$\int |f_n - f| \, d\lambda \longrightarrow 0 \text{ as } n \to \infty.$$

__Proposition__ A. Let $f_n \to f$ in mean. Then $f_n \to f$ in measure

[__Proof__ Let $\varepsilon > 0$ Define $E_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$

Since $\int |f_n - f| \, d\lambda \geq \int \varepsilon \chi_{E_n} \, d\lambda = \varepsilon \lambda(E_n)$

Thus $\lambda(E_n) \to 0$ as $n \to \infty$ for any ε □

__Proposition B__ Let f be an a.e. non-negative integrable fn. Then $\int f \, d\lambda = 0$ iff $f = 0$ a.e.

__Proof__ $\Rightarrow$ If $f = 0$ a.e. define simple fns $f_n \equiv 0$ $\forall n$

$\Leftarrow$ Let f_n be simple and Cauchy in mean converging to f in measure. Since $f \geq 0$ we may (by taking $f_n \vee 0$) assume $f_n \geq 0$

Then $\int f \, d\lambda = 0 \Rightarrow \lim_{n \to \infty} \int f_n \, d\lambda = 0 \Rightarrow \lim \int |f_n| \, d\lambda = 0$

$\Rightarrow f_n \to 0$ in mean $\Rightarrow f_n \to 0$ in measure

But $f_n \to f$ in measure. Therefore $f = 0$ a.e. □

<u>Proposition C</u> S'pose f is integrable, $f > 0$ a.e. ~~around~~ on E a measurable set, and $F \subseteq E$.

Then $\int_F f \, d\lambda = 0 \Rightarrow \lambda(F) = 0$.

<u>Proposition D</u> S'pose f is integrable and E has measure 0. Then $\int_E f \, d\lambda = 0$

<u>Proof of C</u> Let $F_0 = \{ x : f(x) > 0 \}$
$F_n = \{ x : f(x) > \frac{1}{n} \}$

Since $f > 0$ on E, $\lambda(E \setminus F_0) = 0$, so we must show that $\lambda(F \cap F_0)$ is measure 0.

$$0 = \int_{F \cap F_n} f \, d\lambda \geq \frac{1}{n} \lambda(F \cap F_n) \geq 0$$

$\therefore \lambda(F \cap F_n) = 0 \quad \forall n$

But $F \cap F_0 \subseteq F \cap F_n \quad \forall n$

$\therefore \quad \lambda(F \cap F_0) \leq \sum \lambda(F \cap F_n) = 0$ $\square$

(3.4) The Big Theorems of Lebesgue Integration

Recall the problems of exchanging limits and integrals, in particular the sliding hump example.

We showed in the last section that if f is ~~integrable~~ a bdd a.e. msble fn on a set E of finite measure, $\exists$ an integrable simple function g with $\int_E |f - g| \, d\lambda < \varepsilon$.

Theorem (Bounded Convergence Theorem)

Let $\{f_k\}$ be a sequence of msble fns on a set E of finite measure.

Suppose $\exists M \in \mathbb{R}$ with $|f_k(x)| \leq M$ $\forall k$, a.e. $x \in E$

If $f(x) = \lim_{k \to \infty} f_k(x)$ a.e. x then

$$\int_E f \, d\lambda = \lim_{k \to \infty} \int f_k \, d\lambda.$$

Proof. Let $\varepsilon > 0$. $\exists N$ and a msble set $A \subseteq E$ with $\lambda(A) < \frac{\varepsilon}{4M}$ s.t. $n \geq N$ and $x \in E \setminus A$

$$\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2\lambda(E)}.$$

Then $\left| \int_E f_n \, d\lambda - \int_E f \, d\lambda \right| = \left| \int_E f_n - f \, d\lambda \right| \leq \int_E |f_n - f| \, d\lambda$

$$= \int_{E \setminus A} |f_n - f| \, d\lambda + \int_A |f_n - f| \, d\lambda$$

$$\leq \frac{\varepsilon}{2} + 2M\lambda(A) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \square$$

Theorem (Fatou's Lemma) Let $\{f_k\}$ be a sequence of non-negative msble fns s.t. $f_k \to f$ a.e. on a msble set E. Then

$$\int_E f \, d\lambda = \liminf_{n \to \infty} \int_E f_n \, d\lambda.$$

Proof of Fatou. By deleting a set of measure zero, we may as well assume that $f_k \to f$ everywhere.

Let h be a bdd. msble function vanishing outside a set $E' \subseteq E$ with $\lambda(E') < \infty$, where $h \leq f$.

Let $h_n(x) = \min\{h(x), f_n(x)\}$ $n = 1, 2, \ldots$

Then $\int_E h\, d\lambda = \int_{E'} h\, d\lambda = \lim_{n\to\infty} \int_{E'} h_n\, d\lambda$

$$\leq \liminf \int_{E'} f_n\, d\lambda$$

Given $\varepsilon > 0$,
~~This~~ may be done for any integrable simple fn $h \leq f$ such that $\int_E |f - h|\, d\lambda < \varepsilon$.

Thus $\left| \int_E f - \int_E h \right| \leq \varepsilon$ ie.

$$\int_E f\, d\lambda \leq \liminf \int_E f_n + \varepsilon$$

Since ε is arbitrary, we see

$$\int_E f\, d\lambda \leq \liminf_{n\to\infty} \int_E f_n\, d\lambda \qquad \square$$

N.B. We didn't assume that f was measurable in the above proof. That's a consequence...

Theorem (Monotone Convergence Theorem)

Let f_n be an increasing sequence of non-neg. msble fns, $f = \lim f_n$ (pwise limit)

Then f is integrable iff $\lim_{n\to\infty} \int f_n\, d\lambda < \infty$ and furthermore

$$\int f\, d\lambda = \lim_{n\to\infty} \int f_n\, d\lambda.$$

Proof. By Fatou, $\int f\,d\lambda \le \liminf_{n\to\infty} \int f_n\,d\lambda$

But, for all n, $f_n \le f$, so $\int f_n d\lambda \le \int f\,d\lambda$

Thus $\limsup \int f_n\,d\lambda \le \int f\,d\lambda \le \liminf \int f_n\,d\lambda$

and hence $\lim \int f_n\,d\lambda = \int f\,d\lambda$ $\square$

Proposition. Let f be a non-negative function, integrable over E.

For all $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall A \subseteq E$ with $\lambda(A) < \delta$

$$\int_A f\,d\lambda < \varepsilon$$

Proof Suppose not. Then there is a sequence A_n of sets such that

$$\int_{A_n} f\,d\lambda \geq \varepsilon \text{ and } \lambda(A_n) < \frac{1}{2^n}.$$

Let $g_n = f \cdot \chi_{A_n}$. Then $g_n \to 0$, except on the set $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$

But $\lambda\left(\bigcup_{i=n}^{\infty} A_i\right) \le \sum_{n}^{\infty} \frac{1}{2^i} = \frac{2}{2^n}$.

Hence $g_n \to 0$ a.e.

Let $f_n = f - g_n$. Then $\{f_n\}$ is a sequence of non-neg. functions. By Fatou's lemma

$$\int f\,d\lambda \le \liminf \int f_n\,d\lambda \le \int f - \limsup \int g_n \le \int f\,d\lambda - \varepsilon$$

This is a contradiction $\square$

<u>Theorem</u> (Dominated Convergence Theorem - DCT)

Let g be a nonnegative integrable function on the msble set E, $\{f_n\}$ a sequence of measurable functions with $|f_n| \leq g$ on E (a.e.)

Let $f(x) = \lim_{n \to \infty} f_n(x)$

Then $\lim_{n\to\infty} \int_E f_n \, d\lambda = \int_E f \, d\lambda$.

<u>Proof</u>. It follows from the assumption that $g - f_n \geq 0$, and hence by Fatou's Lemma

$$\int (g - f) \, d\lambda \leq \liminf_{n \to \infty} \int (g - f_n) \, d\lambda$$

Now $|f| \leq g$ and g is integrable, so by MCT f is integrable and hence

$$\int_E g \, d\lambda - \int_E f \, d\lambda \leq \int_E g \, d\lambda - \limsup \int_E f_n \, d\lambda$$

hence $\int_E f \, d\lambda \geq \limsup \int_E f_n \, d\lambda$

Now repeat the same argument with $g + f_n$ (instead of $g - f_n$) to get

$$\int_E f \, d\lambda \leq \liminf \int_E f_n \, d\lambda$$

and hence

$$\limsup \int_E f_n \, d\lambda \leq \int_E f_n \, d\lambda \leq \liminf \int_E f_n \, d\lambda \leq \limsup \int_E f_n \, d\lambda$$

Thus they're all equal and the limit exists and is equal to $\int f \, d\lambda$. $\square$

Convergence (Summary)

Suppose $\{f_n\}$ is a sequence of measurable functions and f is a measurable function. We have considered the following 4 types of convergence (more to come!!)

- $f_n \to f$ pwise ae
- $f_n \to f$ in measure
- $f_n \to f$ in the sense of Egoroff
- $f_n \to f$ in mean. (or L^1)

The relationships between these are as follows

$$\begin{tikzcd}[column sep=huge]
\text{pwise a.e.} \arrow[r, Rightarrow, shift right] \arrow[d, Leftrightarrow] & \text{in measure} \arrow[l, Rightarrow, shift right, "/" marking] \arrow[l, Rightarrow, bend right, "\text{(subsequence)}"'] \\
\text{in sense of Egoroff} \arrow[r, Rightarrow, shift left, "/" marking] & \text{in mean} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[u, Rightarrow]
\end{tikzcd}$$

(3.4) L^p-spaces

Defn Let $1 \le p < \infty$. Denote

$$\mathcal{L}^p = \{ f : |f|^p \text{ is integrable} \}$$

If $p = \infty$, let

$$\mathcal{L}^\infty = \{ f : |f| \text{ is a.e. bounded} \}$$

Let $\|f\|_p = \left(\int |f|^p \, d\lambda \right)^{1/p}$ $\quad 1 \le p < \infty$

$$\|f\|_\infty = \text{ess.sup} \, |f(x)|$$

$$= \inf_{X \text{ meas } 0} \; \sup_{x \in \mathbb{R} \setminus X} |f(x)|$$

These spaces are nearly Banach spaces.

Proposition (~~Minkowski's~~ Hölder's Inequality)

Let $f \in L^p$, $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$

Then $\left| \int f g \, d\lambda \right| \le \|f\|_p \|g\|_q$

Proof Note that for $x, y > 0$ $\quad xy \le \frac{1}{p} x^p + \frac{1}{q} y^q$

(Prove this by minimizing $\frac{1}{p} x^p + \frac{1}{q} y^q - xy$!)

Now $\left| \int f g \, d\lambda \right| \le \int |f| |g| \, d\lambda$

so $$\int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \, d\lambda \le \frac{1}{p} \int \frac{|f|^p}{\|f\|_p^p} \, d\lambda + \frac{1}{q} \int \frac{|g|^q}{\|g\|_q^q} \, d\lambda = \frac{1}{p} + \frac{1}{q} = 1 \quad \square$$

Lemma (Minkowski's Inequality). Let $1 \leq p \leq \infty$. If $f, g \in \mathcal{L}^p$, $f+g \in \mathcal{L}^p$ and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof $\int |f+g|^p \leq \int |f+g|^{p-1} |f| + \int |f+g|^{p-1} |g|$

$$\leq \|f\|_p \, \|(f+g)^{p-1}\|_q + \|g\|_p \, \|(f+g)^{p-1}\|_q$$

Now $\|(f+g)^{p-1}\|_q = \left(\int (f+q)^{(p-1)q}\right)^{1/q} = \left(\|f+g\|_p\right)^{p/q}$

since $(p-1)q = p$

and so $\|f+g\|_p^p \leq (\|f\|_p + \|g\|_q)\,(\|f+g\|)^{p/q}$

$\therefore$ $\|f+g\|_p = \|f+g\|_p^{(p-p/q)=1} \leq \|f\|_p + \|g\|_q$ $\square$

Unfortunately, $\|f\|_p$ is not quite a norm since we can have $f \neq 0$ with $\int |f| = 0$ if f is supported on a set of measure zero!

So we define $f \sim g$ if $f = g$ a.e. and take L^p to be the set of all equivalence classes of functions with this relation.

Mostly, we just treat elements of L^p as functions, but we have to remember that our conclusions are up to $=$ a.e.

Then L^p, equipped with norm $\|\cdot\|_p$ is a metric space. We show that it is complete.

<u>Proposition</u> L^p is a complete metric space — a Banach space

<u>Proof</u>. Suppose $\{f_k\}$ is Cauchy in the L^p sense

ie. $\forall \varepsilon > 0 \ \exists N$ s.t. $m, n \geq N \Rightarrow$

$\|f_n - f_m\|_p < \varepsilon$.

Then by Proposition X $|f_n - f_m|^p$ is Cauchy in measure. Hence $\{f_n\}$ is Cauchy in measure.

By Theorem Y $\exists$ a msble function f s.t. $f_n \to f$ in measure.

Since $|f_n|^p \to |f|^p$ in measure and $\int |f_n|^p < \infty$

Fatou's lemma implies that $\int |f|^p = \liminf \int |f_n|^p$

Thus $f \in L^p$. $\square$

<u>Remarks</u> See later in the course

- L^2 is a Hilbert space
- L^q is the dual space of L^p if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. ($L^\infty$ is the dual of L^1 Not conversely!)