(3.1). So far, we have discussed integration of functions on $\mathbb{R}$. We now want to remark that these techniques hold in great generality

Definition   A measurable space is a pair $(X, \mathcal{B})$ where X is a set and $\mathcal{B}$ is a σ-algebra of subsets of X. A set A is called measurable (w.r.t. $\mathcal{B}$) if $A \in \mathcal{B}$.

A positive measure on a msble space is a non-negative set function μ, defined for all sets in $\mathcal{B}$ with values in $\mathbb{R}^+ \cup \{\infty\}$, satisfying

(1) $\mu(\emptyset) = 0$

(2) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for any sequence of disjoint msble sets $\{E_i\}$.

A signed measure satisfies the same axioms with values in $\mathbb{R} \cup \{\pm\infty\}$ with the proviso that at most one of the values $+\infty, -\infty$, is taken.

We may also define complex measures which have values in $\mathbb{C} \cup \{\infty\}$.

For the moment, let's restrict our attention to positive measures.

The tuple $(X, \mathcal{B}, \mu)$ is called a measure space. If $\mu(X) = 1$ it's called a probability space and μ a probability measure.

Examples $\cdot (\mathbb{R}, \mathcal{M}, \lambda)$ is a measure space

- $(\mathbb{R}, \mathcal{B}, \lambda)$ is a measure space
  where $\mathcal{B}$ = the Borel σ-algebra

- Take X = any uncountable set
  $\mathcal{B}$ = set of finite or co-finite sets. Then $\mathcal{B}$ is a σ-algebra. Define

$$\mu(B) = \begin{cases} 0 & \text{if } B \text{ finite} \\ 1 & \text{if } B \text{ co-finite} \end{cases}$$

  This is a (bizarre) measure

- Take $X = \mathbb{Z}$, $\mathcal{B}$ = all subsets of $\mathbb{Z}$ and
  $\mu(B)$ = # elements of B (counting measure)

- $([0,1], \mathcal{M}, \lambda)$ is a probability space.

- Take $X = \prod_{i=1}^{\infty} \{0,1\}$ = all sequences of $0,1$'s

$\mathcal{B}^0$ = set of all rectangles of the form

$$B = B_1 \times B_2 \times \dots \times B_n \times \{0,1\} \times \{0,1\} \times \dots$$

where B_i is any subset of $\{0,1\}$

Let $\nu(B_i) = \begin{cases} \frac{1}{2} & \text{if } B = \{0\} \text{ or } \{1\} \\ 0 & \text{if } B = \emptyset \\ 1 & \text{if } B = \{0,1\} \end{cases}$

$$\mu(B) = \nu(B_1)\nu(B_2)\dots\nu(B_n)$$

Then $\mathcal{B}^0 \subseteq \mathcal{B}$ = Borel σ-algebra and we'll show μ can be extended to all of $\mathcal{B}$

(In fact, a probability measure.)

A measure is called <u>finite</u> if

$$\sup_{A \in \mathcal{B}} |\mu(A)| < \infty$$

and σ-finite if $\exists$ a countable sequence $\{X_n\}$ of disjoint msble sets with $\mu(X_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} X_n$.

(eg Lebesgue measure on $\mathbb{R}$ is σ-finite but not finite)

μ on $\mathbb{R}$ defined by $\mu(A) =$ no of elts of A if A finite
$= \infty$ if A is infinite

is <u>not</u> σ-finite.

~~Proposition~~

<u>Defn</u>. Given a measure space $(X, \mathcal{B}, \mu)$, a set $A \in \mathcal{B}$ is called a <u>null set</u> if $\mu(A) = 0$. The measure space is <u>complete</u> if $\forall$ null set A, $C \subseteq A \Rightarrow C \in \mathcal{B}$.

<u>Proposition</u> If $(X, \mathcal{B}, \mu)$ is a measure space $\exists$ a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that $\mathcal{B} \subseteq \mathcal{B}_0$ and $E \in \mathcal{B} \Rightarrow \mu(E) = \mu_0(E)$.

<u>Proof</u> Not given. The basic idea is to define $\mathcal{B}_0$ by saying $E \in \mathcal{B}_0 \iff E = B \cup C$ where $B \in \mathcal{B}$ and $C \subseteq A$, A a null set of $(X, \mathcal{B}, \mu)$. ....

<u>Theorem</u> (Carathéodory Extension Theorem)

If μ is a σ-finite measure on a ring of sets R, there is a unique measure $\bar{\mu}$ on the σ-~~ring~~ algebra generated by R such that for $E \in R$

$$\bar{\mu}(E) = \mu(E)$$ and the measure $\bar{\mu}$ is σ-finite.

Furthermore, the σ-algebra S can be extended to a class $\overline{S}$ and a set function $\bar{\mu}$ which is a complete measure on $\overline{S}$

<u>Proof</u> See Photocopy of a couple of pages from Halmos.

(2.2) <u>Measurable Functions – Integration.</u>

A (real-valued) function f on (X, B) is called measurable if $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, $\{x : f(x) > \alpha\}$ is measurable. As in the case of $\mathbb{R}$, this is equivalent to replacing "$>$" by $\geq$, $<$, $\leq$ ...

Similarly to before, if f, g are msbl, so are $f+c$, cf, $f+g$, $f-g$, fg. If $\{f_n\}$ are measurable, so is $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$.

As before, we define <u>simple functions</u> to be those measurable functions of the form $f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$ where the $\{E_i\}$ are disjoint measurable sets

If μ is a complete measure and $f=g$ a.e. then g is msble.

We define the integral of a simple function in the same way as for the Lebesgue case. The theorems we proved for Lebesgue integration continue to hold (Fatou, MCT, BCT, DCT)

§ 13. EXTENSION, COMPLETION, AND APPROXIMATION

Can we always extend a measure on a ring to the generated σ-ring? The answer to this question is essentially contained in the results of the preceding sections; it is formally summarized in the following theorem.

Theorem A. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *then there is a unique measure $\bar{\mu}$ on the σ-ring* $\mathbf{S}(\mathbf{R})$ *such that, for E in* $\mathbf{R}$, $\bar{\mu}(E) = \mu(E)$; *the measure $\bar{\mu}$ is σ-finite.*

The measure $\bar{\mu}$ is called the **extension** of μ; except when it is likely to lead to confusion, we shall write $\mu(E)$ instead of $\bar{\mu}(E)$ even for sets E in $\mathbf{S}(\mathbf{R})$.

Proof. The existence of $\bar{\mu}$ (even without the restriction of σ-finiteness) is proved by 11.C and 12.A. To prove uniqueness, suppose that μ_1 and μ_2 are two measures on $\mathbf{S}(\mathbf{R})$ such that $\mu_1(E) = \mu_2(E)$ whenever $E \in \mathbf{R}$, and let $\mathbf{M}$ be the class of all sets E in $\mathbf{S}(\mathbf{R})$ for which $\mu_1(E) = \mu_2(E)$. If one of the two measures is finite, and if $\{E_n\}$ is a monotone sequence of sets in $\mathbf{M}$, then, since

$$\mu_i(\lim_n E_n) = \lim_n \mu_i(E_n), \quad i = 1, 2,$$

we have $\lim_n E_n \in \mathbf{M}$. (The full justification of this step in the reasoning makes use of the fact that one of the two numbers $\mu_1(E_n)$ and $\mu_2(E_n)$, and therefore also the other one, is finite for every $n = 1, 2, \cdots$; cf. 9.D and 9.E.) Since this means that $\mathbf{M}$ is a monotone class, and since $\mathbf{M}$ contains $\mathbf{R}$, it follows from 6.B that $\mathbf{M}$ contains $\mathbf{S}(\mathbf{R})$.

In the general, not necessarily finite, case we proceed as follows. Let A be any fixed set in $\mathbf{R}$, of finite measure with respect to one of the two measures μ_1 and μ_2. Since $\mathbf{R} \cap A$ is a ring and $\mathbf{S}(\mathbf{R}) \cap A$ is the σ-ring it generates (cf. 5.E), it follows that the reasoning of the preceding paragraph applies to $\mathbf{R} \cap A$ and $\mathbf{S}(\mathbf{R}) \cap A$, and proves that if $E \in \mathbf{S}(\mathbf{R}) \cap A$, then $\mu_1(E) = \mu_2(E)$. Since every E in $\mathbf{S}(\mathbf{R})$ may be covered by a countable, disjoint union of sets of finite measure in $\mathbf{R}$ (with respect to either of the measures μ_1 and μ_2), the proof of the theorem is complete. ∎

The extension procedure employed in the proofs of § 12 yields slightly more than Theorem A states; the given measure μ can actually be extended to a class (the class of all μ^*-measurable sets) which is in general larger than the generated σ-ring. The following theorems show that it is not necessary to make use of the theory of outer measures in order to obtain this slight enlargement of the domain of μ.

Theorem B. *If μ is a measure on a σ-ring* $\mathbf{S}$, *then the class* $\bar{\mathbf{S}}$ *of all sets of the form $E \,\Delta\, N$, where $E \in \mathbf{S}$ and N is a subset of a set of measure zero in* $\mathbf{S}$, *is a σ-ring, and the set function $\bar{\mu}$ defined by $\bar{\mu}(E \,\Delta\, N) = \mu(E)$ is a complete measure on* $\bar{\mathbf{S}}$.

The measure $\bar{\mu}$ is called the **completion** of μ.

Proof. If $E \in \mathbf{S}$, $N \subset A \in \mathbf{S}$, and $\mu(A) = 0$, then the relations

$$E \cup N = (E - A) \,\Delta\, [A \cap (E \cup N)]$$

and

$$E \,\Delta\, N = (E - A) \cup [A \cap (E \,\Delta\, N)]$$

show that the class $\bar{\mathbf{S}}$ may also be described as the class of all sets of the form $E \cup N$, where $E \in \mathbf{S}$ and N is a subset of a set of measure zero in $\mathbf{S}$. Since this implies that the class $\bar{\mathbf{S}}$, which is obviously closed under the formation of symmetric differences, is closed also under the formation of countable unions, it follows that $\bar{\mathbf{S}}$ is a σ-ring. If

$$E_1 \,\Delta\, N_1 = E_2 \,\Delta\, N_2,$$

where $E_i \in \mathbf{S}$ and N_i is a subset of a set of measure zero in $\mathbf{S}$, $i = 1, 2$, then

$$E_1 \,\Delta\, E_2 = N_1 \,\Delta\, N_2,$$

and therefore $\mu(E_1 \,\Delta\, E_2) = 0$. It follows that $\mu(E_1) = \mu(E_2)$, and hence that $\bar{\mu}$ is indeed unambiguously defined by the relations

$$\bar{\mu}(E \,\Delta\, N) = \bar{\mu}(E \cup N) = \mu(E).$$

Using the union (instead of the symmetric difference) representation of sets in $\bar{\mathbf{S}}$, it is easy to verify that $\bar{\mu}$ is a measure; the completeness of $\bar{\mu}$ is an immediate consequence of the fact that $\bar{\mathbf{S}}$ contains all subsets of sets of measure zero in $\mathbf{S}$. ∎

The following theorem establishes the connection between the general concept of completion and the particular complete extension obtained by using outer measures.

Theorem C. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *and if* μ^* *is the outer measure induced by* μ, *then the completion of the extension of* μ *to* $\mathbf{S}(\mathbf{R})$ *is identical with* μ^* *on the class of all* μ^**-measurable sets.*

Proof. Let us denote the class of all μ^*-measurable sets by $\mathbf{S}^*$ and the domain of the completion $\bar{\mu}$ of μ by $\bar{\mathbf{S}}$. Since μ^* on $\mathbf{S}^*$ is a complete measure, it follows that $\bar{\mathbf{S}}$ is contained in $\mathbf{S}^*$ and that $\bar{\mu}$ and μ^* coincide on $\bar{\mathbf{S}}$. All that we have left to prove is that $\mathbf{S}^*$ is contained in $\bar{\mathbf{S}}$; in view of the σ-finiteness of μ^* on $\mathbf{S}^*$ (cf. 12.E) it is sufficient to prove that if $E \in \mathbf{S}^*$ and $\mu^*(E) < \infty$, then $E \in \bar{\mathbf{S}}$.

By 12.C, E has a measurable cover F. Since $\mu^*(F) = \mu(F) = \mu^*(E)$, it follows from the finiteness of $\mu^*(E)$, and the fact that μ^* is a measure on $\mathbf{S}^*$, that $\mu^*(F - E) = 0$. Since $F - E$ also has a measurable cover G, and since

$$\mu(G) = \mu^*(F - E) = 0,$$

the relation

$$E = (F - G) \cup (E \cap G)$$

exhibits E as a union of a set in $\mathbf{S}(\mathbf{R})$ and a set which is a subset of a set of measure zero in $\mathbf{S}(\mathbf{R})$. This shows that $E \in \bar{\mathbf{S}}$, and thus completes the proof of Theorem C. ∎

Loosely speaking, Theorem C says that in the σ-finite case the σ-ring of all μ^*-measurable sets and the generated σ-ring $\mathbf{S}(\mathbf{R})$ are not very different; every μ^*-measurable set suitably modified by a set of measure zero belongs to $\mathbf{S}(\mathbf{R})$.

We conclude this section with a very useful result concerning the relation between a measure on a ring and its extension to the generated σ-ring.

Theorem D. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *then, for every set E of finite measure in* $\mathbf{S}(\mathbf{R})$ *and for every positive number ϵ, there exists a set E_0 in* $\mathbf{R}$ *such that* $\mu(E \,\Delta\, E_0) \leqq \epsilon$.

Proof. The results of §§ 10, 11, and 12, together with Theorem A, imply that

$$\mu(E) = \inf\{\textstyle\sum_{i=1}^{\infty} \mu(E_i) \colon E \subset \bigcup_{i=1}^{\infty} E_i, \quad E_i \in \mathbf{R}, \quad i = 1, 2, \cdots\}.$$

Consequently there exists a sequence $\{E_i\}$ of sets in $\mathbf{R}$ such that

$$E \subset \textstyle\bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \mu(\bigcup_{i=1}^{\infty} E_i) \leqq \mu(E) + \dfrac{\epsilon}{2}.$$

Since

$$\lim_n \mu(\textstyle\bigcup_{i=1}^{n} E_i) = \mu(\bigcup_{i=1}^{\infty} E_i),$$

there exists a positive integer n such that if

$$E_0 = \textstyle\bigcup_{i=1}^{n} E_i,$$

then

$$\mu(\textstyle\bigcup_{i=1}^{\infty} E_i) \leqq \mu(E_0) + \dfrac{\epsilon}{2}.$$

Clearly $E_0 \in \mathbf{R}$; since

$$\mu(E - E_0) \leqq \mu(\textstyle\bigcup_{i=1}^{\infty} E_i - E_0) = \mu(\bigcup_{i=1}^{\infty} E_i) - \mu(E_0) \leqq \dfrac{\epsilon}{2}$$

and

$$\mu(E_0 - E) \leqq \mu(\textstyle\bigcup_{i=1}^{\infty} E_i - E) = \mu(\bigcup_{i=1}^{\infty} E_i) - \mu(E) \leqq \dfrac{\epsilon}{2},$$

the proof of the theorem is complete. ∎

(1) Let μ be a finite, non negative, and finitely additive set function defined on a ring $\mathbf{R}$. The function μ^* defined by the procedure of § 10 is still an outer measure, and, therefore, the $\bar{\mu}$ of 11.C may still be formed, but it is no longer necessarily true that $\bar{\mu}$ is an extension of μ; (cf. 10.2, 10.4e, and 11.4).

(2) If $\bar{\mu}$ is the extension of the measure μ on the ring $\mathbf{R}$ described in § 8, then, for any countable set E, $E \in \mathbf{S}(\mathbf{R})$ and $\bar{\mu}(E) = 0$.

(3) The uniqueness assertion of Theorem A is not true if the class $\mathbf{R}$ is not a ring. (Hint: let $X = \{a,b,c,d\}$ be a space of four points and define the measures μ_1 and μ_2 on the class of all subsets of X by

$$\mu_1(\{a\}) = \mu_1(\{d\}) = \mu_2(\{b\}) = \mu_2(\{c\}) = 1,$$

$$\mu_1(\{b\}) = \mu_1(\{c\}) = \mu_2(\{a\}) = \mu_2(\{d\}) = 2.)$$

(4) Is Theorem A true for semirings instead of rings?

(5) Let $\mathbf{R}$ be a ring of subsets of a countable set X, with the property that every non empty set in $\mathbf{R}$ is infinite and such that $\mathbf{S}(\mathbf{R})$ is the class of all subsets of X; (cf. 9.7). If, for every subset E of X, $\mu_1(E)$ is the number of points in E and $\mu_2(E) = 2\mu_1(E)$, then μ_2 and μ_1 agree on $\mathbf{R}$ but not on $\mathbf{S}(\mathbf{R})$. In other words,

The Hahn Decomposition Theorem. Signed measures

Let $(X, \mathcal{B})$ be a measure space, ν a signed measure. A set $A \in \mathcal{B}$ is called <u>positive</u> if $\forall$ msble subsets E of A $\nu(E) \geq 0$

[similarly, <u>negative</u>]

A set which is both positive and negative is called a <u>null set</u>.

Note, we could have a set which is of measure 0 but not null!

eg, let $\nu(E) = \int_E \sin x \, dx$.

$\nu[0, 2\pi] = 0$ but E is not null since $\nu[0,\pi] = 1$
$\nu(\pi, 2\pi] = -1$

<u>Theorem</u> (Hahn Decomposition Thm)

Let ν be a signed measure on $(X, \mathcal{B})$. Then $\exists$ a positive set A and a negative set B s.t. $X = A \cup B$ and $A \cap B = \phi$.

<u>Lemma A</u> Let $\{E_i\}$ be a collection of positive sets. Then $E = \bigcup_{i=1}^{n} E_i$ is positive.

<u>Proof</u> Let $A \subseteq E$ and set $A_n = A \cap E_n \cap E_{n-1}^c \cap \dots \cap E_1^c$

Then $A_n \subseteq E$ so $\nu(A_n) \geq 0$. But A_n and A_k are disjoint for $n \neq k$, and ν is countably additive. Thus

$$A = \bigcup_{n=1}^{\infty} A_n \Rightarrow \nu(A) = \sum \nu(A_n) \geq 0 \qquad \square$$

~~Lemma~~ Let E be a msble set with $0 < \nu(E) < \infty$

Then there's a positive set $A \subseteq E$ with $\nu(A) > 0$

<u>Proof</u> If E is positive we are finished

Thus, suppose E contains sets with negative measure.

Let n_1 be the smallest integer s.t. $\exists$ a measurable set $E_1 \subseteq E$ with $\nu(E_1) < -\frac{1}{n_1}$

Inductively, let n_k be the smallest positive integer for which $\exists$ a msble set E_k s.t.

$$E_k \subseteq E \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$$

and $\nu(E_k) < \frac{-1}{n_k}$.

Let $A = E \setminus \bigcup_{k=1}^{\infty} E_k$, so that

$E = A \cup \left(\bigcup_{k=1}^{\infty} E_k\right)$ is a disjoint union

Then $\nu(E) = \nu(E) + \sum_{k=1}^{\infty} \nu(E_k)$

Since $\nu(E) < \infty$, $\sum_{k=1}^{\infty} \nu(E_k)$ must be absolutely convergent. However $|\nu(E_k)| \geq \frac{1}{n_k}$ so we see that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ converges.

It follows that $n_k \to \infty$ and, for any $\varepsilon > 0$, we may choose k so large that $\frac{1}{n_k - 1} < \varepsilon$

Since $A \subseteq E \setminus \left(\bigcup_{j=1}^{k} E_j\right)$, A contains no measurable set of measure less than $-\frac{1}{n_{k-1}} > -\varepsilon$

Thus A contains no negative sets $\square$

<u>Proof of the Theorem</u>

Suppose ∞ is the value omitted by ν.

Let $\lambda = \sup_{\substack{A \text{ positive} \\ \text{for } \nu}} \nu(A)$ and choose a sequence of positive sets $\{A_i\}$ s.t. $\lambda = \lim_{i \to \infty} \nu(A_i)$

By Lemma B, A is a positive set, so $\lambda \geq \nu(A)$. But $A \setminus A_i \subseteq A$, so $\nu(A \setminus A_i) \geq 0$

Thus $\nu(A) = \nu(A_i) + \nu(A \setminus A_i) \geq \nu(A_i)$

Hence $\nu(A) \geq \lambda$, so $\nu(A) = \lambda$ and $\lambda < \infty$.

Let $B = A^c$. We want to show that B is a negative set.

Suppose $C \subseteq B$ is a positive set. Then C and A are disjoint and $C \cup A$ is a positive set. Hence

$$\lambda \geq \nu(C \cup A) = \nu(C) + \nu(A) = \nu(C) + \lambda.$$

Thus $\nu(C) = 0$.

Thus B contains no positive sets of positive measure and hence no sets of positive measure. Thus B is a negative set

$\square$

A decomposition $X = A \cup B$, where A is positive and B is negative is called a Hahn decomposition for X. It's not unique, as sets of measure zero can be moved from A to B.

Given a Hahn decomposition of X, define

$$\nu^+(E) = \nu(A \cap E)$$
$$\nu^-(E) = -\nu(B \cap E)$$

Then $\nu = \nu^+ - \nu^-$. This is called the Jordan decomposition of ν.

Definition. Two measures ν_1, ν_2 are called mutually singular if $\exists$ disjoint measurable sets A, B with $X = A \cup B$ and $\nu_1(A) = \nu_2(B) = 0$

We write $\nu_1 \perp \nu_2$ to denote mutual singularity.

We proved above

Propn Let ν be a signed measure on the measurable space $(X, \mathcal{B})$. There are two mutually singular measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$

Moreover, there is only one such pair of mutually singular measures.

We call μ^+ and μ^- the positive and negative part of μ.

Either μ^+ or μ^- must be finite.

$|\mu|(E) = (\mu^+ + \mu^-)(E)$ is called the absolute value or total variation of μ.

A set E is positive iff $\mu^-(E) = 0$ and null iff $|\mu|(E) = 0$.

Now we define

$$\int_E f \, d\mu = \int_E f \, d\mu^+ - \int_E f \, d\mu^-$$

and the usual theory of integration works.

The theory can also be extended to complex valued measures, but we don't discuss them in this course.

(Also vector measures!)

The Radon Nikodým Theorem

We return to the setting of a positive measure μ on $(X, \mathcal{B})$.

Given a non-negative integrable fn. on $(X, \mathcal{B}, \mu)$ we can define a measure ν on $(X, \mathcal{B})$ by

$$\nu(E) = \int_E f \, d\mu \qquad (*)$$

[And then $\int g \, d\nu = \int g f \, d\mu$]

Recall from the last section that $\nu \perp \mu$ if $\exists$ two disjoint sets A, B with $\mu(A) = 0$ and $\nu(B) = 0$ and $A \cup B = X$.

We say that $\nu \ll \mu$ (ν is <u>absolutely continuous</u> w.r.t. μ) if for all msble sets F

$$\mu(F) = 0 \Rightarrow \nu(F) = 0$$

We say $\nu \sim \mu$ (ν and μ are <u>mutually absolutely continuous</u>) if $\mu \ll \nu$ and $\nu \ll \mu$.

Notice that if ν is defined by $(*)$ above, we always have $\nu \ll \mu$.

<u>Theorem</u> (Radon-Nikodým)

Suppose $(X, \mathcal{B}, \mu)$ is a σ-finite measure space ν a non-negative measure on $(X, \mathcal{B})$ with $\nu \ll \mu$

Then there is an a.e non-negative integrable function f such that for all $E \in \mathcal{B}$

$$\nu(E) = \int_E f \, d\mu .$$

f is uniquely defined a.e. (w.r.t. μ).

<u>Proof</u>. If $\nu(E) = \int f d\mu$ then

$A_\alpha = \{ x : f(x) > \alpha \}$, $B_\alpha = \{ x : f(x) \leq \alpha \}$

form a Hahn decomposition of $\nu - \alpha\mu$.

The idea of the proof is to mimic this construction in the general case.

So for a completely general pair of measures ν, μ we let A_α, B_α be a Hahn decomposition for $\nu - \alpha\mu$, where α is a rational number. Our idea is to define

$f(x) = \inf \{ \alpha : x \in B_\alpha \}$

Claim If $t \geq \alpha$, B_α is negative for $\nu - t\mu$

[Proof Let E be a measurable subset of B_α. Then

$(\nu - t\mu)(E) \leq \nu(E) - \alpha\mu(E) \leq 0$]

Similarly, A_α is positive for $\nu - t\mu$ when $t \leq \alpha$

Let $A_\infty = \bigcap_{\alpha \in \mathbb{Q}} A_\alpha$.

This is a measurable set, and if $E \subseteq A_\infty$ then $E \subseteq A_\alpha$ for all α, so that $(\nu - \alpha\mu)(E) \geq 0$ for all positive rationals α

i.e. $\alpha\mu(E) \leq \nu(E)$ for all positive rational numbers α.

From this it follows that either $\mu(E) = 0$ or $\nu(E) = \infty$

Let $f(x) = \begin{cases} \inf\{\alpha \in \mathbb{Q} : x \in B_\alpha\} & \text{if } x \in \bigcup_\alpha B_\alpha \\ \infty & \text{if } x \notin \bigcup_\alpha B_\alpha \quad (\Rightarrow x \in A_\infty) \end{cases}$

Claim f is a measurable function

[Proof It suffices to show that

$\{x : f(x) \leq t\} = \bigcap_{\beta > t} \bigcup_{\alpha < \beta} B_\alpha$ (RHS is a measurable set)

This is clear, for

$x_0 \in \{x : f(x) \leq t\} \iff \inf\{\alpha : x_0 \in B_\alpha\} \leq t$

$\iff \exists\, \beta_0 \leq t$ with $x_0 \in B_{\beta_0} \iff$ for $\beta > t$, $\beta_0 \leq t$

$\beta_0 \leq \beta$ iff $x_0 \in \bigcup_{\alpha < \beta} B_\alpha$

$\Leftrightarrow x_0 \in \bigcup_{\alpha<\beta} B_\alpha \quad \forall \beta > t \Leftrightarrow x_0 \in \bigcap_{\beta>t} \bigcup_{\alpha<\beta} B_\alpha$]

Claim: If $E = \{x : \alpha < f(x) \le j\}$ then

$$\alpha \mu(E) \le \nu(E) \le j\mu(E)$$

[Proof: E is measurable and $f(x) \le j \ \forall x \in E$

$\Rightarrow \forall \beta > j,\ x \in B_\beta \Rightarrow x \in \bigcup_{\alpha<\beta} B_\alpha$

i.e. $E \subseteq \bigcup_{\alpha<\beta} B_\alpha$

Thus $\nu(E) - \beta\mu(E) \le 0$ (E is negative for $\nu - \beta\mu$)

$\therefore \nu(E) \le j\mu(E)$ (as $\nu(E) \le \beta\mu(E)$ $\forall \beta < j$)

This is the R.H. inequality.

Now to show that $\alpha\mu(E) \le \nu(E)$, note that $\forall \alpha$ rational $A_\alpha = B_\alpha^c \subseteq \{x : f(x) > \alpha\}$

Now A_α is positive for $\nu - \alpha\mu$, so

$\nu(E) \geq \alpha\mu(E)$. This is the L.H. inequality]

We now want to prove that $\forall E \in \mathcal{B}$

$$\nu(E) = \int_E f \, d\mu.$$

Firstly, suppose $\mu(E \cap A_\infty) > 0$. We then have $\nu(E \cap A_\infty) = \infty$ and also

$$\int_E f\,d\mu \geq \int_{E \cap A_\infty} f\,d\mu = \infty$$

So in this case, the equality becomes $\infty = \infty$.

Thus, we may suppose $\mu(E \cap A_\infty) = 0$.

Since ν is absolutely continuous w.r.t. μ we also have $\nu(E \cap A_\infty) = 0$

Now for any positive integer N, let

$$E_k = \left\{ x \in E : \frac{k-1}{N} < f(x) \leq \frac{k}{N} \right\}$$

By the above, we have

$$\frac{k-1}{N}\,\mu(E_k) \leq \nu(E_k) \leq \frac{k}{N}\,\mu(E_k)$$

E is the disjoint union of the E_k's and $E \cap A_\infty$.

Thus $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k)$ and

$$\int_E f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu$$

We have

$$\frac{k-1}{N}\,\mu(E_k) \leq \int_{E_k} f\,d\mu \leq \frac{k}{N}\,\mu(E_k)$$

whence $-\frac{1}{N}\mu(E_k) \le \int_{E_k} f\,d\mu \le \frac{1}{N}\mu(E_k)$

But $\mu(E) \le \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$

$\therefore$ $-\frac{1}{N}\mu(E) \le \int_E f\,d\mu - \nu(E) \le \frac{1}{N}\mu(E)$

for all N.

Hence $\int f\,d\mu = \nu(E)$ QED!

<u>Proposition</u> Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, ν a measure on $\mathcal{B}$.

Then $\exists$ $\nu_1 \ll \mu$ and $\nu_0 \perp \mu$ s.t.

$$\nu = \nu_0 + \nu_1$$

Furthermore, ν_0 and ν_1 are unique

<u>Proof</u> Since μ and ν are σ-finite, $\lambda = \mu + \nu$ is also σ-finite. Furthermore μ and ν are both abs. cts. w.r.t. λ

By the R-N theorem $\exists$ non-negative msble functions f, g st

$$\mu(E) = \int_E f\,d\lambda \quad \text{and} \quad \nu(E) = \int_E g\,d\lambda$$

Let $A = \{x : f(x) > 0\}$ and $B = \{x : f(x) = 0$

Then $X = A \cup B$ and $A \cap B = \phi$.

Furthermore $\mu(B)=0$

Let $\nu_0(E)=\nu(E\cap B)$

$$\nu_1(E)=\nu(E\cap A)=\int_{E\cap A} g\,d\lambda$$

Then $\nu_0(A)=0$ so $\nu_0 \perp \mu$

Now $\nu=\nu_0+\nu_1$, so we want to show $\nu_1 \ll \nu$

Suppose $\mu(E)=0=\int_E f\,d\lambda$. Then

$f=0$ λ-a.e. on E, so $\lambda(E\cap A)=0$

Now $\lambda(A\cap E)=0 \Rightarrow \nu(A\cap E)=0$

It follows that $\nu_1(E)=0$. Thus $\nu_1 \ll \mu$ $\square$

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On $\mathbb{R}$, each measure can thus be decomposed as a sum of two parts, one absolutely continuous w.r.t. Lebesgue measure and the other singular w.r.t. Lebesgue measure.

<u>Example of a measure singular w.r.t. Lebesgue</u>

Cantor measure ν has the property that $\nu(C)=1$ where C is the Cantor set. Let

$$\int f\,d\nu=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\int f(x)\chi_{F_1}(x)\chi_{F_2}(x)\cdots\chi_{F_n}(x)\,dx$$

where F_i is the set at the ith stage ~~which~~ of the middle thirds construction. Then $\nu \perp \lambda$.