$\boxed{\text{BOOK 2}}$

FUNCTIONAL

ANALYSIS

Ch 6 NORMED LINEAR SPACES

<u>Defn</u>. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

A <u>norm</u> on V is a function $f \mapsto \|f\|$ defined for $f \in V$, $0 \le \|f\| < \infty$ such that

(i) $\|f\| \geq 0 \quad \forall f \in V$

(ii) $\|f\| = 0 \iff f = 0$

(iii) $\|f+g\| \leq \|f\| + \|g\| \qquad \forall f, g \in V$

(iv) $\|\lambda f\| = |\lambda| \, \|f\|$ for all scalars λ $\forall f \in V$.

We call $(V, \|\cdot\|)$ a <u>normed vector space</u> (or normed linear space)

Note that if $\|\cdot\|$ is a norm on V, then

$$d(f,g) = \|f-g\| \text{ defines a metric on } V$$

so that it becomes a metric space.

<u>Examples</u>

(1) $\mathbb{R}^n$, with the norm $\|\underline{x}\|_2 = (x_1^2 + \dots + x_n^2)^{1/2}$.
(Euclidean norm)

or $\|\underline{x}\|_1 = (|x_1| + \dots + |x_n|)$ or $\|\underline{x}\|_\infty = \max(|x_1|, \dots, |x_n|)$

(2) Let X, μ be a measure space. Then for $1 \le p \le \infty$ $L^p(X, \mu)$ is a normed vector space (modulo comments on P. 63 that $\|f\|_p = 0 \Rightarrow f = 0$ a.e.)

(3) For $1 \le p \le \infty$, let $\ell^p(\mathbb{N})$ be the set of all sequences $\underline{x} = (x_1, x_2, \dots)$ so that

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty \quad \left(\text{for } p<\infty - \text{for } p=\infty \;\; \|\underline{x}\|_\infty = \sup_{1\le i<\infty} |x_i|\right)$$

(4) $C([0,1])$, the continuous functions on $[0,1]$ with norm $\|\cdot\|_\infty$

(5) $C(\mathbb{R})$ with $\|\cdot\|_\infty$

$C^\infty(\mathbb{R})$ .. ..

$C_0(\mathbb{R})$ – the cts functions which vanish at ∞.

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In a normed vector space, we can consider Cauchy sequences, convergence and completeness as we did in Ch 1.

Thus
- $f_n \to f$ means $\|f_n - f\| \to 0$ as $n \to \infty$
- $\{f_n\}$ is Cauchy if $\forall\, \varepsilon > 0\ \exists N$ s.t. $m,n > N \Rightarrow \|f_n - f_m\| < \varepsilon$
- $(V, \|\cdot\|)$ is <u>complete</u> if every Cauchy sequence has a limit in V.

All the above examples are complete except $(C^\infty(\mathbb{R}), \|\cdot\|_\infty)$!

Banach Spaces

Defn. A complete normed vector space is called a Banach space.

Thus $L^p(X,\mu)$ $(1 \le p \le \infty)$, $C(\mathbb{R})$, $C_0(\mathbb{R})$, ℓ^p $(1 \le p \le \infty)$ are all examples of Banach spaces.

Defn. A series $\sum_{n=1}^{\infty} v_n$ in a normed vector space is convergent iff the sequence of partial sums $\sum_{n=1}^{N} v_n = S_N$ is a convergent sequence.

If $S_N \to S$, we write $\sum_{n=1}^{\infty} v_n = S$.

A series $\sum v_n$ converges absolutely if $\sum \|v_n\|$ converges (in $\mathbb{R}$)

Proposition 1 If $(V, \|\cdot\|)$ is a Banach space, every abs. cgnt sequence ~~seq~~ is convergent.

Proof $\|S_m - S_n\| = \|\sum_{n+1}^{m} v_i\| \le \sum_{n+1}^{m} \|v_i\| \to 0$ as $n, m \to \infty$

Thus S_m is Cauchy. Since V is a Banach space $S_m \to S$ for some $S \in V$. $\square$

<u>Proposition</u> Suppose $(V, \|\cdot\|)$ is a normed vector space s.t. every absolutely convergent series converges. Then $(V, \|\cdot\|)$ is a Banach space

<u>Proof</u>. If $\{v_n\}$ is a Cauchy sequence

Let N_1 be s.t. $n, m \geq N_1 \Rightarrow \|v_m - v_n\| < \frac{1}{2}$

Choose $n_1 = N_1$

Now choose $N_2 \in \mathbb{N}$, $N_2 \geq N_1$ st.

$n, m \geq N_1 \Rightarrow \|v_m - v_n\| < \frac{1}{4}$

~~C~~ontinue inductively.

We get $n_1 < n_2 < \ldots$

Then $\{v_{n_k}\}$ is a subsequence of $\{v_n\}$. Look at the ~~[illegible]~~ partial sums $S_k = v_{n_1} + (v_{n_2} - v_{n_1}) + \cdots + (v_{n_k} - v_{n_{k-1}})$

$= v_{n_k}$

Since $\|v_{n_1}\| + \|v_{n_2} - v_{n_1}\| + \|v_{n_3} - v_{n_2}\| + \ldots < \|v_{n_1}\| + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$
$< \infty$

v_{n_k} is absolutely convergent, hence convergent to v, say

Thus v_{n_k} is a subsequence of the Cauchy sequence v_n, such that $v_{n_k} \to v$.

It follows that $v_n \to v$. $\square$

Hilbert Spaces

Defn. If V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ an inner product is a function

$$(\cdot,\cdot) : V \times V \to \mathbb{R} \ (\text{or } \mathbb{C}) \quad \text{such that}$$

(i) $(f,f) \geq 0 \ \forall f \in V$ and $(f,f) = 0 \Leftrightarrow f = 0$

(ii) $(f,g) = \overline{(g,f)}$ (complex conjugate)

(iii) $(af,g) = a(f,g)$ $\quad a \in \mathbb{C}$ (or $\mathbb{R}$)

$$(f, ag) = \bar{a}(f,g)$$

(iv) $(v+w,u) = (v,u) + (w,u) \quad \forall\ u,v,w \in V$

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We call $(V, (\cdot,\cdot))$ an inner product space.

Define $\|f\| = ((f,f))^{1/2}$

Claim $\|f\|$ defines a norm on V

Proof Cauchy – Schwarz inequality says

$$|(f,g)| \leq \|f\| \, \|g\|.$$

Let's prove it. If $g \neq 0$ we put $k = \frac{g}{\|g\|}$ and calculate

$$\cancel{|(f,k)|} \quad 0 \leq \| f - (f,k)k \|^2$$

$$= (f - (f,k)k, \ f - (f,k)k)$$

$$= (f,f) - (f,k)(k,f) - \overline{(f,k)}(f,k) + |(f,k)|^2 (k,k)$$

$$= (f,f) - |(f,k)|^2 \quad \text{since } (k,k)=1$$

$$= \|f\|^2 - |(f,k)|^2$$

So $|(f,g)|^2 = |(f, \|g\|k)|^2$

$$= \|g\|^2 |(f,k)|^2 \leq \|g\|^2 \|f\|^2$$

It follows that $|(f,g)| \leq \|f\|\|g\|$. $\square$

We can now check the $\triangle$ inequality

$$\|f+g\|^2 = (f+g, f+g) = \|f\|^2 + (g,f) + (f,g) + \|g\|^2$$

$$= \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\,(f,g)$$

$$\leq \|f\|^2 + \|g\|^2 + 2|(f,g)|$$

$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|.$$

$$= (\|f\| + \|g\|)^2 \qquad \square$$

<u>Example</u>. On $\mathbb{R}^n$, the dot product $\underline{x}.\underline{y}$ is an inner product which gives the Euclidean norm

On $L^2(X,\mu)$, $(f,g) = \int f \, \overline{g} \, d\mu$

defines an inner product, whose corresponding norm is $\|f\|_2 = \left(\int |f|^2 d\mu\right)^{1/2}$.

<u>Defn</u>. A vectorspace with an inner product $(\cdot,\cdot)$ which is complete w.r.t. the associated norm is called a <u>Hilbert space</u>.

You can "do" a lot of geometry in Hilbert space eg the parallelogram law holds

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2) \qquad \square$$

Useful for testing whether a Banach space is a Hilbert space!

Hilbert spaces are "like" $\mathbb{R}^n$ in many ways. They have orthonormal bases!

<u>Theorem</u> Every Hilbert space H has a basis $\{e_n\}$, which is orthonormal in the sense that

$$(e_n, e_m) = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

We say that H is <u>separable</u> if the basis is countable.

<u>Example</u>. The unit ball in an infinite dimensional Hilbert space is not compact.

Take the orthonormal basis $\{e_i\}$. Then $\|e_i\|=1$ so they all belong to the unit ball, but for $i \neq j$ $\|e_i - e_j\| = \sqrt{2}$ so there is no convergent subsequence.

<u>Theorem</u> If K is an orthonormal set in a Hilbert space H, TFAE

(a) The closed linear subspace spanned by K is H

(b) K is an orthonormal basis

(c) Parseval's formula holds

$$\|x\|^2 = \sum_{y \in K} |(x,y)|^2$$

If these conditions hold, we say K is <u>complete</u>.

<u>Then hence</u> ~~Theorem~~. $x = \sum_{y \in K} (x,y)\, y \quad \left(\text{or } \sum_{n=1}^{\infty} (x,e_n) e_n \right)$

We say that H_1, H_2 are isometrically isomorphic if $\exists$ a linear map $T : H_1 \to H_2$ which is bijective, with $\|Tx\|_2^2 = \|x\|_1^2 \quad \forall x \in H_1$

norm of H_2 — norm of H_1

<u>Proof</u>. The idea is to define $T : H_1 \to H_2$ by $Te_i = f_i$ where e_i is an onb for H_1,
f_i .... H_2

It follows from Parseval that $\|Tx\|_2 = \|x\|_1$ for all $x \in H_1$.

Lemma (Projection Lemma)

Let M be a closed convex set in H

For all $x_0 \in H$ $\exists!$ $y_0 \in M$ s.t.

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|.$$

Proof Not given

Theorem (Projection Theorem). Let M be a closed linear subspace of H. Any $x \in H$ can be written $x = y_0 + z_0$ $\quad y_0 \in M$, $z_0 \in M^\perp$

y_0 and z_0 are uniquely determined

Proof. If $x \in M$, take $y_0 = x$, $z_0 = 0$.

If $x \notin M$, let y_0 be s.t. $\|x - y_0\| = \inf_{y \in M} \|x - y_0\|$ (which is unique by Proj^n Thm)

Take $y \in M$ and λ a scalar, so that $y_0 + \lambda y \in M$ — (M is a subspace!). Then

$$\|x - y_0\|^2 \leq \|x - y_0 - \lambda y\|^2$$
$$= \|x - y_0\|^2 - 2\,\mathrm{Re}(\lambda (y, x - y_0)) + |\lambda|^2 \|y\|^2$$

Thus

$$-2\,\mathrm{Re}(\lambda (y, x - y_0)) + |\lambda|^2 \|y\|^2 \geq 0$$

Taking $\lambda = \varepsilon > 0$ and letting $\varepsilon \to 0$ we see

$$\mathrm{Re}(\lambda (y, x - y_0)) \leq 0$$

Now take $\lambda = i\varepsilon$ with $\varepsilon > 0$ we get

$$\mathrm{Im}\,(y, x - y_0) \leq 0 .$$

But these inequalities also hold for $-y$ $(\in M)$

So we get $(y, x-y_0) = 0 \ \forall\, y \in M$.

This shows that $z_0 = x - y_0 \in M^\perp$

For uniqueness, if $x = y_1 + z_1$, $y_1 \in M$, $z_1 \in M^\perp$

then $y_0 - y_1 = z_1 - z_0$ is in both M and $M^\perp$

Since it's a subspace. Thus $y_0 - y_1 = z_1 - z_0 = 0$ □

In the above lemma, the point y is called

the projection of x onto M and the operator

$Px = y$ is the <u>projection operator</u>

<u>Theorem</u> Let H be Hilbert V a closed linear subspace of H.

Then $H = V \oplus V^\perp$

is a decomposition of H.