Solving Some Differential Equations

The Fourier transform was invented by Fourier to solve the heat equation (1806)
We show how to solve the heat equation and the Laplace equation in this section.

Heat Equ We look for a function $u(x,t)$
$x \in \mathbb{R}$, $t \in \mathbb{R}+$ which satisfies

$$u_{xx} = u_t$$

Subject to $u(x,0) = f(x)$ $\quad f \in L^1(\mathbb{R})$

$$\left\{ \begin{array}{l} |u(x,t)| \to 0 \text{ as } x \to \pm\infty \\ |u_x(x,t)| \to 0 \text{ as } x \to \pm\infty \end{array} \right\} \quad (*)$$

Take the Fourier transform in the x variable

$$\hat{u}(y,t) = \int_{-\infty}^{\infty} u(x,t) e^{-iyx}\, dx$$

Using the DCT it's easy to see that (not t in integral)

$$\hat{u_t}(y,t) = \int_{-\infty}^{\infty} u_t(x,t)\, e^{-iyx}\, dx$$

Integrating by parts

0 by ~~initial~~ boundary conds (*)

$$\int_{-\infty}^{\infty} u_{xx}(x,t)\, e^{-iyx}\, dx = \cancel{\left[u_x(x,t) e^{-iyx} \right]_{-\infty}^{\infty}} + iy \int_{-\infty}^{\infty} u_x(x,t) e^{-iyx}\, dx$$

0 also by ~~initial~~ boundary (*) conditions

$$= \cancel{\left[u(x,t) \right]_{-\infty}^{\infty}} - y^2 \int_{-\infty}^{\infty} u(x,t)\, e^{-iyx}\, dx$$

$$= -y^2\, \hat{u}(y,t)$$

Thus the heat equation has become an ODE

$$\hat{u}_t(y,t) = -y^2 \hat{u}(y,t)$$

We can solve this to get

$$\hat{u}(y,t) = \hat{u}(y,0)\, e^{-y^2 t}$$

By our initial condition $\hat{u}(y,0) = \hat{f}(y)$

so this gives

$$\hat{u}(y,t) = e^{-y^2 t}\, \hat{f}(y)$$

Now use Fourier inversion

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)\, e^{-y^2 t + iyx}\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\, e^{-y^2 t + iy(x-z)}\, dz\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z) \underbrace{\int_{-\infty}^{\infty} e^{-ty^2 + iy(x-z)}\, dy}\, dz$$

Let's calculate this integral

$$\int_{-\infty}^{\infty} e^{-t\left(y^2 + iy\left(\frac{\cancel{x}-z}{-t}\right)\right)}\, d\cancel{z}y$$

$$= \int_{-\infty}^{\infty} e^{-t\left(\left(y - i\left(\frac{x-z}{t}\right)\right)^2 + \frac{(x-z)^2}{4t^2}\right)}\, d\cancel{z}y$$

We did this before

$$= e^{-\frac{(x-z)^2}{4t}} \int_{-\infty + i\left(\frac{x-z}{2t}\right)}^{\infty + i\left(\frac{x-z}{t}\right)} e^{-tu^2}\, du$$

$$= e^{-\frac{(x-z)^2}{4t}} \int_{-\infty}^{\infty} e^{-tu^2}\, du$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-z)^2}{4t}}$$

Writing $\boxed{K(x-\xi,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z)^2}{4t}}}$

we get

$$u(x,t) = \int_{-\infty}^{\infty} K(x-z,t) f(z)\, dz$$

$$= K(\cdot, t) * f \quad (x)$$

K is called the <u>heat kernel</u>.

<u>Laplace Equation</u>

$$\Delta u = u_{xx} + u_{yy} = 0 \qquad x \in \mathbb{R},\ y > 0$$

- $u(x,0) = f(x)$
- $\lim_{y \to \infty} u(x,y) < \infty$ all x,
- u_{xx} integrable
- Conditions $*$ from heat equation holds

We use the same techniques as above

Let $\hat{u}(\xi, y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-ix\xi}\, dx$

So the Laplace equation gives (as above

$$\frac{d^2\hat{u}}{dy^2} - \xi^2 \hat{u} = 0$$

which again is an ODE with solution

$$\hat{u}(\xi, y) = A(\xi)\, e^{-|\xi| y} + B(\xi)\, e^{|\xi| y} \qquad - (**)$$

132

I claim that the condition

$$\lim_{y\to\infty} u(x,y) < \infty \text{ implies } B(\xi) = 0 \text{ a.e.}$$

To see this note that $e^{|\xi| y} \geq 1 + |\xi| y$

so that if B is non-zero on a set of positive measure

$$\int B(\xi) e^{|\xi| y} \cos(x\xi)\, d\xi \geq \int_{-\infty}^{\infty} B(\xi) \cos(x\xi)\, d\xi + y \int_{-\infty}^{\infty} B(\xi) |\xi| \cos(x\xi)\, d\xi$$

and

$$\int B(\xi) e^{|\xi| y} \sin(x\xi)\, d\xi \geq \int_{-\infty}^{\infty} B(\xi) \sin x\xi\, d\xi + y \int_{-\infty}^{\infty} B(\xi) |\xi| \sin(x\xi)\, d\xi$$

So

$$\left|\int B(\xi) e^{|\xi| y + i\xi x} d\xi\right|^2 \geq g_1(x) + y^2 g_2(x)$$

$$g_1(x) = \left(\int_{-\infty}^{\infty} B(\xi) \cos(\xi x)\, d\xi\right)^2 + \left(\int_{-\infty}^{\infty} B(\xi) \sin(x\xi)\, d\xi\right)^2$$

$$g_2(x) = \left(\int_{-\infty}^{\infty} B(\xi) |\xi| \cos(\xi x)\, d\xi\right)^2 + \left(\int_{-\infty}^{\infty} B(\xi) |\xi| \sin(\xi x)\, d\xi\right)^2$$

If B is nonzero on a set of positive measure, both g_1 and g_2 are positive for ~~[illegible]~~ all x and hence

$\left|\int B(\xi) e^{|\xi| y + i\xi x} d\xi\right|^2$ is unbounded in y as $y \to \infty$

We conclude the $B = 0$ a.e. and $(**)$ gives

$$\hat{u}(\xi, y) = A(\xi) e^{-|\xi| y}$$

Since $\hat{u}(\xi, 0) = \hat{f}(\xi)$, we see $A(\xi) = \hat{f}(\xi)$.

Corollary The Fourier transform is a bounded linear map from the dense subspace $L^1 \cap L^2(\mathbb{R})$ of L^2 into L^2 and for each $f \in L^2(\mathbb{R})$

$$\|\hat{f}\|_2 = \sqrt{2\pi}\,\|f\|_2 .$$

Thus it can be uniquely extended to a map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

— | —

Remark. Similarly, $L^1 \cap L^p$ is dense in L^p for any $1 \le p \le \infty$. Since L^q is the dual space of L^p $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, it is not hard to show that the Fourier Transform maps L^p to L^q for $1 \le p < \infty$.

(Use the identity $\int \hat{f} g = \int f \hat{g}$)

L^2 and L^p Fourier Transforms

The Fourier transform was defined on L^1 only. However, there are functions in L^2 which are not in L^1 – eg $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ \frac{1}{x} & x \geq 1 \end{cases}$

Lemma $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in L^2

Proof. The Hermite functions are in L^1 and also in L^2. Alternatively, use the result from Assignment 1 that step functions (clearly in L^1) are dense in L^2 – in fact they're dense in L^p. $\square$

Theorem (Plancherel Theorem) Let $f \in L^1 \cap L^2(\mathbb{R})$

Then $\|\hat{f}\|_2 = (2\pi)^{1/2} \|f\|_2$

Proof Let $g(x) = f * \check{f}(x) = \int f(y-x) f(y)\,dy$

$\check{f}(x) = f(-x)$

$\cancel{= \int\int f(y-x) f(y)\,dy\,dx = \|f\|_1^2}$

$$g(0) = \int |f(y)|^2\,dy = \|f\|_2^2$$

Then $\hat{g}(t) = \hat{f}(t)\,(\check{f})^\wedge(t) = \hat{f}(t)\,\overline{\hat{f}(t)} = |\hat{f}(t)|^2$

Since $\hat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{-ixt}\,dx$ and $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(t) e^{ixt}\,dt$

and $g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt$, we get

$$\|f\|_2^2 = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt = \frac{1}{2\pi}\|\hat{f}\|_2^2 \quad \square$$

Fourier Transform on $\mathbb{R}^n$

If $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is defined by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot y} \, dx$$

$x, y \in \mathbb{R}^n$ and $x \cdot y$ the dot product.

The usual basic properties hold as for $\mathbb{R}$.

Theorem. Let $\hat{f}$ be the Fourier transform of $f \in L^1(\mathbb{R}^n)$. If $\hat{f}$ is integrable, then

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(y) e^{ix \cdot y} \, dy$$

Theorem (Riemann-Lebesgue Lemma) Let $f \in L^1(\mathbb{R}^n)$. Then $\lim_{|y| \to \infty} |\hat{f}(\hat{y})| = 0$.

Fourier transforms of radial functions are interesting. If R is a rotation matrix (orthogonal matrix) $Rx \cdot y = x \cdot R^{-1} y$. It follows that $(f \circ R)^\wedge(x) = \hat{f} \circ R(y)$.

~~where for R~~

Thus radial functions have radial Fourier transforms.

Looking at $\mathbb{R}^2$

If $f(x,y) = f\left(\sqrt{x^2+y^2}\right)$, where f is a function $\mathbb{R}_+ \to \mathbb{C}$.

$$\hat{F}(\xi,\eta) = \int_{\mathbb{R}^2} f\left(\sqrt{x^2+y^2}\right) e^{-ix\xi - iy\eta}\, dx\, dy$$

Put $x = r\cos\theta$, $\xi = \rho\cos\phi$

$y = r\sin\theta$, $\eta = \rho\sin\phi$

so $$\hat{F}(\rho,\phi) = \int_0^\infty \int_0^{2\pi} f(r) e^{-ir\rho\cos(\theta-\phi)}\, r\,dr\,d\theta$$

Change of variables $\theta \to \theta + \phi$

$$= \int_0^\infty \boxed{\int_0^{2\pi} f(r) e^{-ir\rho\cos\theta}\, d\theta}\, r\,dr$$

is independent of ϕ.

The function in the box $\int_0^{2\pi} e^{-ir\rho\cos\theta}\, d\theta$

is called $J_0(r\rho)$, the zeroth order Bessel function and

$$\hat{F}(\rho) = 2\pi\int_0^\infty f(r) J_0(r\rho)\, r\,dr.$$

This is called the Hankel transform: if f is radial its <u>Hankel transform</u> is

$$\hat{f}(\rho) = \int_{\mathbb{R}_+} f(r) J_0(r\rho)\, r\,dr$$

There is also an inverse Hankel Transform

$$f(r) = \int_0^\infty \hat{f}(\rho)\, \rho\, J_0(r\rho)\, d\rho$$

In general, there is a k-dimensional

If f is a radial function on $\mathbb{R}^n$ and f satisfies $\int_0^\infty f(r)\sqrt{r}\,dr < \infty$, and if $k > -\frac{1}{2}$, the kth Hankel Transform is

$$\hat{f}(\rho) = \int_0^\infty f(r)\, r J_k(r\rho)\, dr$$

and the inversion is the same integral.

$$f(r) = \int_0^\infty \hat{f}(\rho)\, \rho J_k(r\rho)\, d\rho .$$

Fourier Series