The Radon-Nikodym Theorem

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The Radon-Nikodym Theorem

Remark: We had previously asked about when, given a measure space $(X, \mathcal{A}, \mu)$, and any measure λ on $\mathcal{A}$, does there exists an $f \in \mathcal{M}^+(X, \mathcal{A})$ with the property that for every $E \in \mathcal{A}$,

$$\lambda(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{A}$.

Fact: If such an f exists then it must be the case that $\lambda \ll \mu$.

Problem: Does the converse hold?

Theorem: [Radon-Nikodym Theorem]
Let λ and μ be σ-finite measures on $(X, \mathcal{A})$. Suppose that λ is absolutely continuous with respect to μ. Then there exists $f \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\mu$$

for every $E \in \mathcal{A}$. Moreover f is uniquely determined μ-almost everywhere.

Example: Let μ be the counting measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. Consider the measure λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ given by

$$\lambda(E) = \begin{cases} 0 & \text{if } E \text{ is countable.} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$$

Then it is easy to see that there is no function $f : X \to \mathbb{R}_+^*$ such that $\lambda = \mu_f$.

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Proof: Case 1) Assume that λ and μ are both finite.

For each $c > 0$ let $\{P(c), N(c)\}$ be a Hahn decomposition for the signed measure $\lambda - c\mu$. Let

$$A_1 = N(c)$$

and for each $k \in \mathbb{N}$, let

$$A_{k+1} = N((k+1)c) \setminus \bigcup_{i=1}^{k} A_i.$$

It follows that $\{A_i\}_{i=1}^{\infty}$ is pairwise disjoint and

$$\bigcup_{i=1}^{k} N(ic) = \bigcup_{i=1}^{k} A_i.$$

Consequently, we have

$$A_k = N(kc) \cap \bigcap_{i=1}^{k-1} P(ic).$$

If $E \in \mathcal{A}$ and $E \subseteq A_k$, then $E \subseteq N(kc)$ and $E \subseteq P((k-1)c)$. As such we have

$$(k-1)c\mu(E) \leq \lambda(E) \leq kc\mu(E). \qquad (*)$$

Cont'd: Next, let

$$B = X \setminus \bigcup_{i=1}^{\infty} A_i = X \setminus \bigcup_{i=1}^{\infty} N(ic) = \bigcap_{i=1}^{\infty} P(ic).$$

Since $B \subseteq P(kc)$ for all $k \in \mathbb{N}$, we get that

$$0 \le kc\mu(B) \le \lambda(B) \le \lambda(X) < \infty$$

for each $k \in \mathbb{N}$. Therefore, $\mu(B) = 0$ and since $\lambda \ll \mu$ we have that $\lambda(B) = 0$, as well.

We now define for each $c > 0$,

$$f_c(x) = \begin{cases} (k-1)c & \text{if } x \in A_k, \\ 0 & \text{if } x \in B. \end{cases}$$

Cont'd:

For each $E \in \mathcal{A}$, we have

$$E = (E \cap B) \cup (\bigcup_{k=1}^{\infty} (E \cap A_k)).$$

Applying

$$(k-1)c\mu(E) \leq \lambda(E) \leq kc\mu(E) \qquad (*)$$

to each of the component pieces above, we have that

$$\int_E f_c \, d\mu \leq \lambda(E) \leq \int_E (f_c + c) \, d\mu = \int_E f_c \, d\mu + c\mu(X).$$

Now for each $n \in \mathbb{N}$, let

$$g_n = f_{\frac{1}{2^n}}.$$

We get

$$\int_E g_n \, d\mu \leq \lambda(E) \leq \int_E g_n \, d\mu + \frac{\mu(X)}{2^n} \qquad (**).$$

Cont'd: If we let $m \geq n$ then $(**)$ tells us that

$$\int_E g_n \, d\mu \leq \lambda(E) \leq \int_E g_m \, d\mu + \frac{\mu(X)}{2^m} \quad \text{and} \quad \int_E g_m \, d\mu \leq \lambda(E) \leq \int_E g_n \, d\mu + \frac{\mu(X)}{2^n}.$$

Combining these two give us that

$$\left| \int_E g_n \, d\mu - \int_E g_m \, d\mu \right| \leq \frac{\mu(X)}{2^n}$$

for each $E \in \mathcal{A}$. In particular this holds for $E_1 = \{x \in X \mid g_n - g_m \geq 0\}$ and $E_2 = \{x \in X \mid g_n - g_m < 0\}$. This allows us to deduce that

$$\int_X |g_n - g_m| \, d\mu \leq \frac{2\mu(X)}{2^n} = \frac{\mu(X)}{2^{n-1}}$$

and hence that $\{g_n\}_{n=1}^{\infty}$ is Cauchy in $L_1(X, \mathcal{A}, \mu)$.

Cont'd:
Assume that $g_n \to f$ in $L_1(X, \mathcal{A}, \mu)$. Since $g_n \in \mathcal{M}^+(X, \mathcal{A})$ we can also assume that $f \in \mathcal{M}^+(X, \mathcal{A})$. Moreover, for any $E \in \mathcal{A}$ we have

$$|\int_E g_n \, d\mu - \int_E f \, d\mu| \leq \int_E |g_n - f| \, d\mu \leq \|g_n - f\|_1 \to 0.$$

It then follows from $(**)$ that

$$\lambda(E) = \lim_{n \to \infty} \int_E g_n \, d\mu = \int_E f \, d\mu.$$

Cont'd:
Suppose that $f, h \in \mathcal{M}^+(X, \mathcal{A})$ are such that

$$\int_E f \, d\mu = \lambda(E) = \int_E h \, d\mu$$

for all $E \in \mathcal{A}$.

Let $E_1 = \{x \in X | f(x) > h(x)\}$ and $E_2 = \{x \in X | f(x) < h(x)\}$.
Since

$$\int_{E_1} f - h \, d\mu = \int_{E_1} f \, d\mu - \int_{E_1} h \, d\mu = \lambda(E_1) - \lambda(E_1) = 0$$

and

$$\int_{E_2} f - h \, d\mu = \int_{E_2} f \, d\mu - \int_{E_2} h \, d\mu = \lambda(E_2) - \lambda(E_2) = 0$$

we have that $\mu(E_1) = \mu(E_2) = 0$ and hence that $f = h$ μ-a.e.

Case 2: Assume that λ and μ are σ-finite.

Let $\{X_n\} \subseteq \mathcal{A}$ be an increasing sequence such that $X = \bigcup\limits_{n=1}^{\infty} X_n$, $\lambda(X_n) < \infty$ and $\mu(X_n) < \infty$.

For each $n \in \mathbb{N}$, we get a function $f_n \in \mathcal{M}^+(X, \mathcal{A})$ such that $f_{n_{|X_n^c}} \equiv 0$, and if $E \in \mathcal{A}$ with $E \subseteq X_n$, then

$$\lambda(E) = \int_E f_n \, d\mu.$$

If $m \geq n$, then $X_n \subseteq X_m$, and by our previous uniqueness result, $f_n = f_m$ μ-a.e. Let

$$F_n = \sup\{f_1, f_2, \ldots, f_n\}.$$

Then $\{F_n\}$ is an increasing sequence of positive measurable functions with $F_n = f_n$ μ-a.e and $F_n(x) = 0$ for all $x \in X_n^c$. Let

$$f = \lim_{n \to \infty} F_n.$$

Cont'd: If $E \in \mathcal{A}$, then

$$\lambda(E \cap X_n) = \int_E f_n \, d\mu = \int_E F_n \, d\mu.$$

Given that $E \cap X_n \nearrow E$, continuity from below and the Monotone Convergence Theorem shows us that

$$\lambda(E) = \lim_{n \to \infty} \lambda(E \cap X_n) = \lim_{n \to \infty} \int_E F_n \, d\mu = \int_E f \, d\mu.$$

The uniqueness of f is determined as in the finite case.

Remark: A close look at the proof of the RNT shows that the process used to construct the function f resembles differentiation.

Example: Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $F'(x) > 0$. Then F is strictly increasing.

Let μ_F be the Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ generated by F. Then μ_F is σ-finite and $\mu_F \ll m$. Moreover, the Fundamental Theorem of Calculus shows that

$$\mu_F((a,b]) = F(b) - F(a) = \int_a^b F'(x)\,dx = \int_{[a,b]} F'\,dm = \int_{(a,b]} F'\,dm.$$

From here we can deduce that if $E \in \mathcal{B}(\mathbb{R})$, then

$$\mu_F(E) = \int_E F'\,dm.$$

In particular, the function we would have obtained in via the Radon-Nikodym Theorem is m-a.e equal to F'.

The Radon-Nikodym Theorem

Definition: The function f whose existence was established in the Radon-Nikodym Theorem is called the Radon-Nikodym dervivative of λ with respect to μ and it is denoted by $\frac{d\lambda}{d\mu}$.

Remark:

1) $\frac{d\lambda}{d\mu}$ is integrable if and only if λ is a finite measure.

2) In the case that λ is σ- finite, with $X = \bigcup\limits_{n=1}^{\infty} X_n$ and each X_n being such that $\lambda(X_n) < \infty$, we have that

$$\lambda(X_n) = \int_{X_n} \frac{d\lambda}{d\mu}\, d\mu$$

so $\frac{d\lambda}{d\mu}$ must be finite μ-a.e on X_n. As such, we may assume that $\frac{d\lambda}{d\mu}$ is actually finite everywhere on X.

Cont'd:

3) Let $(X, \mathcal{A}, \lambda) = (\mathbb{R}, \mathbf{M}(\mathbb{R}), m)$ be the ususal Lebesgue measure space and let μ be defined on $(\mathbb{R}, \mathbf{M}(\mathbb{R}))$ to be the restriction of the counting measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ to $(\mathbb{R}, \mathbf{M}(\mathbb{R}))$.

   Then $m \ll \mu$, since $\mu(E) = 0$ implies $E = \emptyset$. However, there is no $f \in \mathcal{M}^{+}(\mathbb{R}, \mathbf{M}(\mathbb{R}))$ such that

$$m(E) = \int_E f \, d\mu.$$

   This shows that the Radon-Nikodym Theorem can fail if μ is not σ-finite.

   Note: It is an exercise to show that the Radon-Nikodym Theorem can be extended to the case where λ is arbitrary.

4) Let μ, λ, ν be σ-finite measures on $(X, \mathcal{A})$ with $\lambda \ll \mu$ and $\nu \ll \mu$.

(i) If $c > 0$, then $c\lambda \ll \mu$ and

$$\frac{d(c\lambda)}{d\mu} = c\frac{d\lambda}{d\mu}.$$

(ii) We have $(\lambda + \nu) \ll \mu$ and

$$\frac{d(\lambda + \nu)}{d\mu} = \frac{d\lambda}{d\mu} + \frac{d\nu}{d\mu}.$$

(iii) If $\nu \ll \lambda$ and $\lambda \ll \mu$, then $\nu \ll \mu$ and

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$

(iv) If $\mu \ll \lambda$ and $\lambda \ll \mu$, then

$$\frac{d\nu}{d\mu} = \frac{1}{\frac{d\mu}{d\nu}}.$$

Note: All of the equalities above apply almost everywhere.

5) If λ is a signed measure and μ is a σ-finite measure with $\lambda \ll \mu$, then $\lambda^+ \ll \mu$ and $\lambda^- \ll \mu$. In this case, we let

$$\frac{d\lambda}{d\mu} \stackrel{def}{=} \frac{d\lambda^+}{d\mu} - \frac{d\lambda^-}{d\mu}.$$

Theorem: [Lebesgue Decomposition Theorem]
Let λ and μ be σ-finite measures on $(X, \mathcal{A})$. Then there exists two measures λ_1 and λ_2 on $(X, \mathcal{A})$ such that $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \perp \mu$ and $\lambda_2 \ll \mu$. Moreover, these measures are unique.

Proof: Let

$$\nu = \lambda + \mu.$$

Then ν is σ-finite, $\lambda \ll \nu$ and $\mu \ll \nu$.

It follows that there are functions $f, g \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\nu \quad \text{and} \quad \mu(E) = \int_E g \, d\nu.$$

for every $E \in \mathcal{A}$.
Let

$$A = \{x \in X | \; g(x) = 0\} \quad \text{and} \quad B = \{x \in X | \; g(x) > 0\}.$$

Then $\{A, B\}$ is a partition of X. Let

$$\lambda_1(E) = \lambda(E \cap A) \quad \text{and} \quad \lambda_1(E) = \lambda(E \cap B)$$

for every $E \in \mathcal{A}$. Clearly $\lambda = \lambda_1 + \lambda_2$.

Lebesgue Decompostion Theorem

Cont'd: Since

$$\mu(A) = \mu(E) = \int_E g \, d\nu$$

we have that $\lambda_1 \perp \mu$.

If $\mu(E) = 0$, then

$$\int_E g \, d\nu = 0$$

so $g(x) = 0$ for ν-almost every in E.

It follows that $\nu(E \cap B) = 0$ and hence that

$$\lambda_2(E) = \lambda(E \cap B) = 0$$

since $\lambda \ll \nu$. That is $\lambda_2 \ll \mu$.

Cont'd: To see that λ_1 and λ_2 are unique we first assume that both λ and μ are finite. Assume also that we can find λ_1 and λ_2 and ν_1 and ν_2 with

$$\lambda = \lambda_1 + \lambda_2,\ \lambda_1 \perp \mu \textit{ and } \lambda_2 \ll \mu,$$

and

$$\lambda = \nu_1 + \nu_2,\ \nu_1 \perp \mu \textit{ and } \nu_2 \ll \mu.$$

Then

$$\gamma = \lambda_1 - \nu_1 = \nu_2 - \lambda_2$$

is such that $\gamma \perp \mu$ and $\gamma \ll \mu$ and hence $\gamma = 0$.

The case where λ and μ are σ-finite is left as an exercise.