University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Quiz 1
SOLUTIONS

Question 1.

i) $$P(X=k)=\begin{cases}0.2 & k=-1\\ 0.3 & k=0\\ 0.2 & k=1\\ 0.3 & k=10\\ 0 & \text{otherwise}\end{cases}$$

ii) $E(X)=-1(0.2)+0(0.3)+1(0.2)+10(0.3)=3$.

iii) $\{x_1,x_2,...x_5\}=\{10,\ -1,\ 1,\ 0,\ 1\}$

iv) $\int_{-1}^{y}-4t^3dt=\left[-t^4\right]_{-1}^{t}=1-y^4$ hence $F(y)=\begin{cases}0 & y<-1\\ 1-y^4 & y\in[-1,0]\\ 1 & y>0\end{cases}$.

v) $F^{-1}(y)=-\sqrt[4]{1-y}$ hence we can generate $\{y_1,y_2,...y_5\}$, by setting $y_i=-\sqrt[4]{1-u_i}$. Note we use the negative of the fourth root, not the positive, since the range of Y is $[-1,0]$.
Here $\{y_1,y_2,...y_5\}\approx\{-0.711,\ -0.972,\ -0.832,\ -0.932,\ -0.825\}$

vi) Y is never positive, hence $P(XY\geq 0)=P(X\leq 0)=0.5$

(10 Marks)

Question 2.

Let *X, Y* and *Z* be random variables with joint probability mass function

$$P((X,Y,Z))=(x,y,z))=\begin{cases}0.015 & (x,y,z)=(0,0,0)\\ 0.1 & (x,y,z)=(1,0,0)\\ 0.295 & (x,y,z)=(1,1,-1)\\ 0.21 & (x,y,z)=(0,2,-1)\\ 0.16 & (x,y,z)=(-1,2,1)\\ 0.005 & (x,y,z)=(1,2,1)\\ 0.01 & (x,y,z)=(-1,0,0)\\ 0.08 & (x,y,z)=(-1,1,-1)\\ 0.125 & (x,y,z)=(1,3,0)\\ 0 & \text{otherwise}\end{cases}$$

i)

$$\begin{aligned}P(XYZ\le 0)&=1-P(XYZ>0)\\ &=1-P((X,Y,Z))=(1,2,1))-P((X,Y,Z))=(-1,1,-1))\end{aligned}$$

$$P(XYZ\le 0)=1-0.005-0.08=0.915.$$

ii)

$$P(Y=0)=0.015+0.1+0.01=0.125$$
$$P(Y=1)=0.295+0.08=0.375$$
$$P(Y=2)=0.21+0.16+0.005=0.375$$
$$P(Y=3)=0.125$$

This defines $Y\sim Bin(3,0.5)$.

iii) $$P(Y=y\,|\,(X=-1))=\begin{cases}\dfrac{0.01}{0.01+0.08+0.16} & y=0\\ \dfrac{0.08}{0.01+0.08+0.16} & y=1\\ \dfrac{0.16}{0.01+0.08+0.16} & y=2\\ 0 & \text{otherwise}\end{cases}\quad \text{so}$$

$$P(Y=y\,|\,(X=-1))=\begin{cases}0.04 & y=0\\ 0.32 & y=1\\ 0.64 & y=2\\ 0 & \text{otherwise}\end{cases}.$$

iv) $Y\,|\,(X=-1)\sim Bin(2,0.8)$.

(8 Marks)

Question 3.

i) $\dfrac{x_1}{\sqrt{x_2^{\,2}}} \sim t_1$;

ii) $\dfrac{x_1^{\,2}+x_2^{\,2}+x_3^{\,2}+x_4^{\,2}}{x_5^{\,2}+x_6^{\,2}+x_7^{\,2}+x_8^{\,2}} \sim F(4,4)$;

iii) $-x_1+x_2-x_3 \sim N(0,3)$.

(6 Marks)

Question 4.

i) $\dfrac{(61-50)^2}{50}+\dfrac{(19-10)^2}{10}+\dfrac{(14-20)^2}{20}+\dfrac{(6-20)^2}{20}=22.12$.

ii) We have 4 hypothesises proportions, hence 3 degrees of freedom. The critical value for $\chi^2(3)$ is 7.815. Our test statistic is greater than this value, so we reject the null hypothesis of the assumed proportions.

iii) The factor proportions are $p_{adult}=0.8,\ p_{child}=0.2,\ p_{bowl}=0.6,\ p_{arcade}=0.4$ hence our Pearson statistic is
$\dfrac{(61-60)^2}{60}+\dfrac{(19-20)^2}{20}+\dfrac{(14-15)^2}{15}+\dfrac{(6-5)^2}{5}=\dfrac{1}{3}$

iv) This is effectively a 2-by-2 contingency table, hence has $(2-1)(2-1)=1$degree of freedom. The critical value for $\chi^2(1)$ is 3.841. Our test statistic is below this value, so we do not reject the null hypothesis of age and activity type being independent.

Question 5.

i) $\displaystyle\int_0^y 10xy^2dx = 10y^2\int_0^y 10xdx = 10y^2\left[\frac{x^2}{2}\right]_0^y = 5y^4$

Hence $f_Y(y)=\begin{cases}5y^4 & 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}$.

ii) $\displaystyle E(Y)=\int_0^1 y(5y^4)dy=\left[\frac{5}{6}y^6\right]_0^1=\frac{5}{6}$.

iii) $\displaystyle\int_x^1 10xy^2dy = 10x\int_x^1 10y^2dx = 10x\left[\frac{y^3}{3}\right]_x^1 = \frac{10x(1-x^3)}{3}$ hence

Hence $f_X(x)=\begin{cases}\dfrac{10x(1-x^3)}{3} & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$.

iv) $f_{Y|X=x}(y)=\dfrac{f_{X,Y}(x,y)}{f_X(x)}=\begin{cases}\dfrac{3y^2}{1-x^3} & x\le y\le 1\\ 0 & \text{otherwise}\end{cases}$.

(8 Marks)