University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Quiz 3
Time allowed: 75 minutes
Question paper MUST NOT BE REMOVED from the room
This is an open-book assessment.
You may use textbooks, subject notes and online resources as required. You are not permitted to communicate with other students or to post requests for solutions to online sources

Question 1.

A Rayleigh variable $R \sim Rayleigh(\sigma)$ has probability density function

$$f(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}.$$

Five independent realisations of $R \sim Rayleigh(\sigma)$ are recorded (to 1 decimal place): $\{r_1, r_2, r_3, r_4, r_5\} = \{9.6,\ 4.9,\ 15.9,\ 21.7,\ 14.6\}$

i) Given that $E(R) = \sigma\sqrt{\dfrac{\pi}{2}}$, apply the method of moments using the first moment to obtain an estimate $\hat{\sigma}_{MM}$ of σ.

ii) Write down the likelihood function $L(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma)$ for the sample $\{r_1, r_2, r_3, r_4, r_5\}$.

iii) Showing all of your working, hence show that the log-likelihood function is $\ell(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma) \approx 12.38 - 10\ln(\sigma) - \dfrac{1053.03}{2\sigma^2}$.

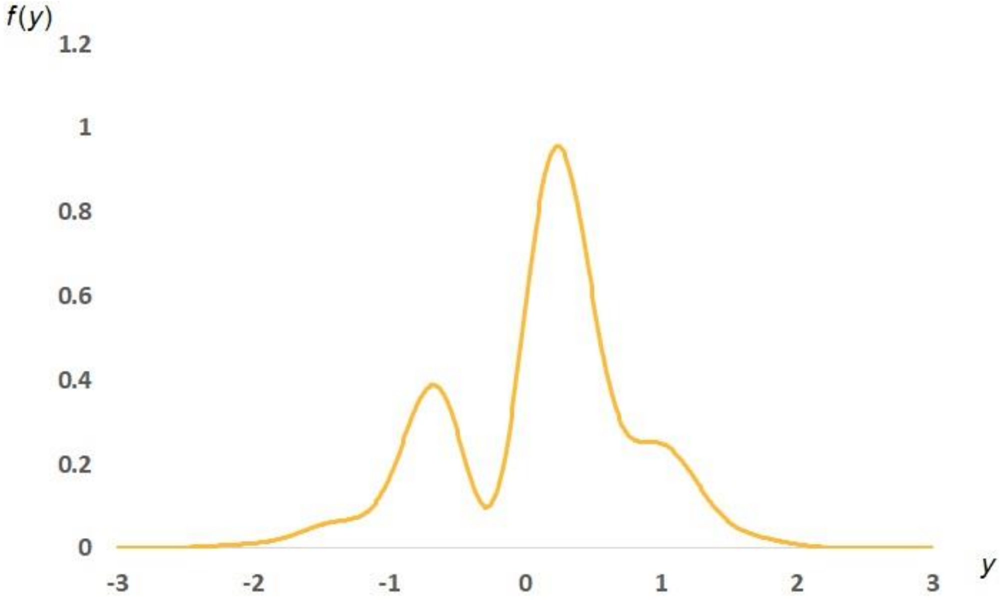
iv) Calculate $\dfrac{\partial}{\partial \sigma} \ell(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma)$.

v) Showing all of your working, hence show that maximum likelihood estimation gives an estimate of σ, $\hat{\sigma}_{MLE} \approx 10.26$.

(15 Marks)

Question 2.

Consider sampling realisations of Y from a probability density function $f(y) = Ke^{-y^2}\left[3+\sin(5y)+\sin(2y)+\sin(7y)\right]$ for some normalising constant K (i.e. such that $f(y) = \dfrac{e^{-y^2}\left[3+\sin(5y)+\sin(2y)+\sin(7y)\right]}{\int_{-\infty}^{\infty} e^{-t^2}\left[3+\sin(5t)+\sin(2t)+\sin(7t)\right]dt}$



Starting at $y_0 = 0.400$, six proposed moves (for $y_1, y_2, ..., y_6$ respectively) are drawn, proposing moves to states

0.101, 0.502 0.851, 0.223, 1007, 0.521.

The proposed move at timestep $i \in \{1,2,...,6\}$ is only accepted if the acceptance probability $A_i > u_i$ where $u_1, u_2, ..., u_6$ are independent realisations of a $U[0,1]$ variable.

Here, $u_1, u_2, ..., u_6$ are, in order,

0.578   0.076   0.437   0.883   0.622   0.303

i) Assume that the proposal values are independent realisations of a distribution such that $g(y_p \mid y) = g(y \mid y_p)$ where y_p and y are, respectively, the proposed and current states. Find the values of $y_1, y_2, ..., y_6$, clearly stating all of your working and calculating each acceptance probability.

(10 Marks)

Question 3.

A (possibly biased) coin which lands Heads with probability p and lands Tails with probability $1-p$ is flipped repeatedly and the number of additional flips required until it next lands Heads is noted as a realisation the random variable X. This gives that $X \sim Geo(p)$ Let the realisations of X be $x_1, x_2, x_3, \ldots$

i) Given that the probability mass function of $X \sim Geo(p)$ is

$$P(X = k) = \begin{cases} (1-p)^{k-1}p & k \in \{1,2,3,\ldots\} \\ 0 & \text{otherwise} \end{cases},$$

find the likelihood of the sample $x_1, x_2, \ldots, x_n$, $L(x_1, x_2, \ldots, x_n | p)$.

Prior to any observations being made, a prior distribution $f(p)$ is used to describe the belief about the value of p

Now, the coin is flipped repeatedly until it has landed Heads ten times. The sequence of observed outcomes is, in order:

Tails, Heads, Heads, Heads, Tails, Tails, Heads, Tails,
Heads, Heads, Tails, Heads, Heads, Tails, Heads, Heads

The first Heads was observed after two flips, hence $x_1 = 2$ etc.

ii) Write down the values of $x_2, x_3, \ldots x_{10}$.

iii) If a uniform prior, $P \sim U[0,1]$ is used, write down the posterior distribution.

iv) Using this prior along with the likelihood of the observations $x_1, x_2, \ldots, x_{10}$, would the expected value of the prior distribution or the expected value of the posterior distribution be larger? Justify your answer. (Note, no calculation is required.)

v) Now, instead of the uniform distribution, a Beta prior is used, $P \sim Beta(3,3)$.
Show that this is a conjugate prior for the likelihood function and find the posterior distribution, $f(p | x_1, x_2, \ldots, x_n)$ including the values of any parameters needed to define it exactly.

vi) Calculate the expected value of p from both the Beta prior distribution and from the resulting posterior distribution. Comment on the result.

Notes:

The probability density function of $Y \sim Beta(\alpha, \beta)$ is given by

$$g(y) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} & y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \alpha > 0, \beta > 0$$

where $\Gamma(k) = (k-1)!$ if k is a positive integer.

The mean of Y is given by $E(Y) = \dfrac{\alpha}{\alpha + \beta}$.

(15 Marks)