University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Quiz 3
SOLUTIONS

Question 1.

i) The first moment is $m_1 = \frac{9.6 + 4.9 + 15.9 + 21.7 + 14.6}{5} = 13.34$.

Setting this equal to $E(R) = \sigma\sqrt{\frac{\pi}{2}}$ gives $\hat{\sigma}_{MM} = 13.34\sqrt{\frac{2}{\pi}}$

ii) Write down the likelihood function

$$L(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma) = \prod_{i=1}^{5} \frac{r_i}{\sigma^2} e^{-\frac{r_i^2}{2\sigma^2}} = \left(\frac{1}{\sigma^{10}}\right) e^{-\frac{\sum_{i=1}^{5} r_i^2}{2\sigma^2}} \prod_{i=1}^{5} r_i$$

For the sample $\{r_1, r_2, r_3, r_4, r_5\}$, we have that $\sum_{i=1}^{5} r_i^2 = 1053.03$ and

$\prod_{i=1}^{5} r_i \approx 236961.1$.

iii) $\ell(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma) = \ln\left(L(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma)\right)$

$$\ell(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma) = \ln\left(\prod_{i=1}^{5} r_i\right) - 10\ln(\sigma) - \frac{\sum_{i=1}^{5} r_i^2}{2\sigma^2}$$

$$\approx 12.38 - 10\ln(\sigma) - \frac{1053.03}{2\sigma^2}.$$

iv) $\frac{\partial}{\partial \sigma} \ell(\{r_1, r_2, r_3, r_4, r_5\} \mid \sigma) = \frac{-10}{\sigma} + \frac{1053.03}{\sigma^3}$.

v) Setting $\frac{-10}{\sigma} + \frac{1053.03}{\sigma^3} = 0$, we obtain $10\sigma^2 = 1053.03$ hence $\hat{\sigma}_{MLE} \approx \sqrt{105.303} \approx 10.26$.

(15 Marks)

Question 2.

In all cases, the acceptance probability from state y_i to y_p is

$$A_{i+1} = \min\left\{1, \frac{e^{-y_p^2}\left[3+\sin(5y_p)+\sin(2y_p)+\sin(7y_p)\right]}{e^{-y_i^2}\left[3+\sin(5y_i)+\sin(2y_i)+\sin(7y_i)\right]}\right\}$$

State	Proposal	Acceptance probability	$A_j > u_{10+j}$?	Accept move?
$y_0 = 0.400$	$y_p = 0.101$	$A_1 = \min\{1,\ 1.015\} = 1$	$1 > 0.578$	Yes
$y_1 = 0.101$	$y_p = 0.502$	$A_2 = \min\{1,\ 0.737\} = 0.737$	$0.737 > 0.076$	Yes
$y_2 = 0.502$	$y_p = 0.851$	$A_3 = \min\{1,\ 0.737\} = 0.425$	$0.425 < 0.437$	No
$y_3 = 0.502$	$y_p = 0.223$	$A_4 = \min\{1,\ 1.603\} = 1$	$1 > 0.833$	Yes
$y_4 = 0.223$	$y_p = 1.007$	$A_5 = \min\{1,\ 0.261\} = 0.261$	$0.261 < 0.622$	No
$y_5 = 0.223$	$y_p = 0.521$	$A_6 = \min\{1,\ 0.585\} = 0.585$	$0.585 > 0.303$	Yes
$y_6 = 0.521$				

(10 Marks)

Question 3.

i) $L(x_1, x_2, ..., x_n | p) = \prod_{i=1}^{n} (1-p)^{k_i - 1} p = p^n (1-p)^{\sum_{i=1}^{n}(k_i - 1)}$

ii) The first Heads was observed after two flips, hence $x_1 = 2$ The second Heads was observed one flip later, so $x_2 = 1$. We therefore have that $\{x_1, x_2, ... x_{10}\} = \{2, 1, 1, 3, 2, 1, 2, 1, 2, 1\}$.

iii) $P \sim U[0,1]$ has density function $f(p) = \begin{cases} 1 & p \in [0,1] \\ 0 & \text{otherwise} \end{cases}$. As this is a constant (on the interval $[0,1]$), the posterior is simply 1 multiplied by the likelihood, i.e. is equal to the likelihood.

$f(p | x_1, x_2, ..., x_n) = p^n (1-p)^{\sum_{i=1}^{n}(k_i - 1)}$

iv) The expectation of $P \sim U[0,1]$ is 0.5. The observed dataset shows 10 Heads and only 6 Tails, so the posterior distribution should have a larger expectation than 0.5 (somewhere between 0.5 and 10/16.)

v) $f(p | x_1, x_2, ..., x_n) \propto \left(p^{10}(1-p)^6\right)\left(\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} p^2 (1-p)^2\right)$.

$f(p | x_1, x_2, ..., x_n) \propto p^{12}(1-p)^8$. This is a $Beta(13,9)$ variable. As using a Beta prior gives a Beta posterior, this is a conjugate prior.

vi) For the prior, $P \sim Beta(3,3)$, the expectation is $\frac{3}{3+3} = \frac{1}{2}$.

For the posterior $P | \boldsymbol{X} \sim Beta(13,9)$, the expectation is

$\frac{13}{13+9} = \frac{13}{22}$.

(15 Marks)