University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Class 11 Preparation Work

SOLUTIONS

1.

i) $$L(x_1,x_2,...,x_n|\lambda)=\left[\frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\right]\left[\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\right]...\left[\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}\right]=\left[\frac{e^{-n\lambda}\lambda^{\sum_i^n x_i}}{x_1!x_2!...x_n!}\right]$$

ii) Posterior $\propto$ Prior $\times$ Likelihood

$$f(\lambda|x_1,x_2,...,x_n)\propto f(\lambda)\times L(x_1,x_2,...,x_n|\lambda)$$

$$\propto \frac{\beta^\alpha\lambda^{(\alpha-1)}e^{-\beta\lambda}}{\Gamma(\alpha)}\left[\frac{e^{-n\lambda}\lambda^{\sum_i^n x_i}}{x_1!x_2!...x_n!}\right]$$

$$\propto \frac{\beta^\alpha\lambda^{(\sum_i^n x_i+\alpha-1)}e^{-(n+\beta)\lambda}}{\Gamma(\alpha)}\left[\frac{1}{x_1!x_2!...x_n!}\right]$$

$$\propto \lambda^{(\sum_i^n x_i+\alpha-1)}e^{-(n+\beta)\lambda}$$

Hence the posterior for $\lambda \sim Gamma(\sum_i^n x_i+\alpha, n+\beta)$.

As both the prior and posterior are of the same family of variables (both gamma), the prior is a conjugate prior.

iii) The posterior mean is therefore $\dfrac{\sum_i^n x_i+\alpha}{n+\beta}$.

2.

i) The probability density of a normal $N(\theta,1)$ variable is $\frac{1}{\sqrt{2\pi}}e^{-\frac{(y_i-\theta)^2}{2}}$ hence

$$L(y_1,y_2,...,y_n|\theta)=\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{(y_1-\theta)^2}{2}}\right]\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{(y_2-\theta)^2}{2}}\right]...\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{(y_n-\theta)^2}{2}}\right]$$

$$=\left[\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}e^{\frac{-\sum_{i=1}^{n}(y_i-\theta)^2}{2}}\right]$$

ii) Posterior $\propto$ Prior $\times$ Likelihood

$$f(\theta|y_1,y_2,...,y_n)\propto f(\theta)\times L(y_1,y_2,...,y_n|\theta)$$

$$.\qquad\propto\left[10e^{-10\theta}\right]\left[\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}e^{\frac{-\sum_{i=1}^{n}(y_i-\theta)^2}{2}}\right]$$

$$\propto\left[e^{-10\theta}\right]\left[e^{\frac{-\sum_{i=1}^{n}(y_i-\theta)^2}{2}}\right]$$

hence $f(\theta|y_1,y_2,...,y_n)=\dfrac{e^{\frac{-\sum_{i=1}^{n}(y_i-\theta)^2-20\theta}{2}}}{\int_0^\infty e^{\frac{-\sum_{i=1}^{n}(y_i-\theta)^2-20\theta}{2}}d\theta}$.

iii) The Metropolis-Hastings algorithm to generate $\theta_1, \theta_2, ..., \theta_n$

One – Draw an independent realisation z_p from the proposal distribution.

Two – Calculate the acceptance probability . For step *j* (i.e. to generate θ_j), this is equal to $A_j = \min\left\{1, \frac{e^{\frac{-\sum_{i=1}^{n}(y_i - z_p)^2 - 20\theta}{2}}}{e^{\frac{-\sum_{i=1}^{n}(y_i - \theta_j)^2 - 20\theta}{2}}}\right\}$.

Three – If the acceptance probability is 1, then set $\theta_j = z_p$.

If the acceptance probability is < 1 then draw an independent sample u_j from $U \sim U[0,1]$. If $u_j < A_j$ then accept the proposal and set $\theta_j = z_p$. If $u_j \geq A_j$ then do not accept the proposal and set $\theta_j = \theta_{j-1}$.

Four – Repeat from step one until *n* samples have been drawn.

(Note that we never need to calculate the denominator in the pdf of the posterior distribution, as the Metropolis-Hastings algorithm only needs the ratio of the probability density of the proposed state to the probability density of the current state.)

3.

i) $$f(x)=\begin{cases}\dfrac{1}{\theta} & x\in[0,\theta]\\ 0 & \text{otherwise}\end{cases}$$

ii) The likelihood of the sample $x_1, x_2, ..., x_n$ is

$$L(x_1,x_2,...,x_n|\theta)=\left(\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)...\left(\frac{1}{\theta}\right)=\left(\frac{1}{\theta}\right)^n$$

(valid for all $x_i\in[0,\theta]$ and zero otherwise)

Prior to any observations being made, the belief around the value of θ is described a Pareto variable, $\theta\sim Pareto(m,\alpha)$.

That is, $$f(\theta)=\begin{cases}\dfrac{\alpha m^\alpha}{\theta^{\alpha+1}} & \theta\in[m,\infty)\\ 0 & \text{otherwise}\end{cases}$$ for $m>0, \alpha>0$.

iii) $f(\theta\,|\,x_1,x_2,...,x_n)\propto L(x_1,x_2,...,x_n|\theta)f(\theta)$ hence

$$f(\theta\,|\,x_1,x_2,...,x_n)\propto\left(\frac{1}{\theta}\right)^n\frac{\alpha m^\alpha}{\theta^{\alpha+1}}=\frac{\alpha m^\alpha}{\theta^{n+\alpha+1}}.$$

This is valid for $\theta\in[m,\infty)$ (i.e. $\theta\geq m$) and all $x_i\in[0,\theta]$ (i.e. $\theta\geq\max\{x_1,x_2,...x_n\}$

Putting these together gives $\theta\geq\max\{m,x_1,x_2,...x_n\}$.

This is also a Pareto variable

$$f(\theta\,|\,x_1,x_2,...,x_n)=\begin{cases}\dfrac{\alpha m^\alpha}{\theta^{n+\alpha+1}} & \theta\in[\max\{m,x_1,x_2,...x_n\},\infty)\\ 0 & \text{otherwise}\end{cases}$$

so the posterior distribution is a $\theta\sim Pareto(\tilde{m},\tilde{\alpha})$ variable where $\tilde{m}=\max\{m,x_1,x_2,...x_n\}$ and $\tilde{\alpha}=n+\alpha$.

iv) Both the prior distribution and posterior distribution are both Pareto distributions, hence this is a conjugate prior for the likelihood function.