University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Class 2 Preparation Work
SOLUTIONS

1.

i) If $f(q)$ is a valid probability density functin, then $\int_{-\infty}^{\infty} f(q)dq = 1$.

$$\int_1^M \frac{1}{q}dq = \left[\ln(q)\right]_1^M = \ln(M) = 1 \text{ hence } M = e = \exp(1).$$

ii) $F(q) = P(Q \leq q) = \int_1^q \frac{1}{t}dt = \left[\ln(t)\right]_1^q = \ln(q)$.

2. i) $f_Y(y) = \frac{d}{dy} F_Y(y)$.

Trivially, we can see that for $y < 0$, $\frac{d}{dy} F_Y(y) = 0$.

For $y \geq 0$,

$$f_Y(y) = \frac{d}{dy}\left(1 - e^{-\left(\frac{y}{\lambda}\right)^k}\right) = \left(\frac{-ky^{k-1}}{\lambda^k}\right)\left(-e^{-\left(\frac{y}{\lambda}\right)^k}\right) = \frac{ky^{k-1}e^{-\left(\frac{y}{\lambda}\right)^k}}{\lambda^k}$$

hence $$f_Y(y) = \begin{cases} \frac{ky^{k-1}e^{-\left(\frac{y}{\lambda}\right)^k}}{\lambda^k} & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

ii) $F_Y(y) = 1 - e^{-\left(\frac{y}{\lambda}\right)^k}$ so $1 - F_Y(y) = e^{-\left(\frac{y}{\lambda}\right)^k}$.

Taking logarithms of both sides gives $\ln\left(1 - F_Y(y)\right) = -\left(\frac{y}{\lambda}\right)^k$.

So $\lambda\left[-\ln\left(1 - F_Y(y)\right)\right]^{\frac{1}{k}} = y$ hence $F_Y^{-1}(y) = \lambda\left[-\ln\left(1 - y\right)\right]^{\frac{1}{k}}$.

iii) Given realisations $\{u_1, u_2, u_3, u_4, ...\}$ of a $U[0,1]$ variable, we can obtain realisations $\{y_1, y_2, y_3, y_4, ...\}$ of Y by calculating $y_i = \lambda\left[-\ln\left(1 - u_i\right)\right]^{\frac{1}{k}}$.

iv) Given four independent realisations of $U \sim U[0,1]$,

$\{u_1, u_2, u_3, u_4\} = \{0.503,\ 0.114,\ 0.760,\ 0.449\}$, we obtain

$\{y_1, y_2, y_3, y_4\} = \{5.02, 2.09, 7.17, 4.63\}$.

3.

i) $f(y) = \begin{cases} \frac{1}{6} & y \in [4,10] \\ 0 & \text{otherwise} \end{cases}$

ii) $F(y) = \int_{-\infty}^{y} f(t)dt$.

If $y < 4$, $F(y) = \int_{-\infty}^{y} 0dt = 0$.

If $y > 10$, $F(y) = \int_{4}^{10} \frac{1}{6}dt + \int_{10}^{y} 0dt = 1$.

If $y \in [4,10]$, $F(y) = \int_{4}^{y} \frac{1}{6}dt = \frac{y-4}{6}$.

Hence $F(y) = \begin{cases} 0 & y < 4 \\ \frac{y-4}{6} & y \in [4,10] \\ 1 & y > 10 \end{cases}$.

iii) For $y \in [4,10]$, $F(y) = \frac{y-4}{6}$ hence $F^{-1}(y) = 6y + 4$.

This gives that, if u_i is a realisation of a $U[0,1]$ variable, then

$y_i = 6u_i + 4$ is a realisation of Y.

$u_1 = 0.710$ hence $y_1 = 6(0.710) + 4 = 8.260$
$u_2 = 0.119$ hence $y_2 = 6(0.119) + 4 = 4.714$
$u_3 = 0.358$ hence $y_3 = 6(0.358) + 4 = 6.148$
$u_4 = 0.883$ hence $y_4 = 6(0.883) + 4 = 9.298$
$u_5 = 0.504$ hence $y_5 = 6(0.504) + 4 = 7.024$