University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Class 3 Preparation Work

1. Let X and Y be independent random variables, $X \sim \exp(1)$ and $Y \sim \exp(1)$.

   That is, for example, the density function of X is given by

$$f_X(x) = \begin{cases} e^{-x} & x \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}.$$

   Consider the variables defined as $S = X + Y$ and $D = X - Y$.

   i) What are the ranges of S and D? Justify your answers.

   ii) Find statements for X and Y in terms of S and D.

   iii) Hence show that the Jacobian is given by

$$\boldsymbol{J} = \begin{pmatrix} \dfrac{\partial X}{\partial S} & \dfrac{\partial X}{\partial D} \\ \dfrac{\partial Y}{\partial S} & \dfrac{\partial Y}{\partial D} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{pmatrix}.$$

   iv) Hence find the joint density of S and D.

   v) Calculate the marginal density of S and hence show that $S \sim gamma(2,1)$.

   vi) Calculate the marginal density of D and hence show that $D \sim Laplace(1)$..

Note: A Laplace variable $Z \sim Laplace(\lambda)$ has probability density function

$$f_Z(z) = \begin{cases} \dfrac{1}{2}\lambda e^{-\lambda|x|} & z \in (-\infty, \infty) \\ 0 & \text{otherwise} \end{cases}.$$

2. Let X and Y be independent random variables with joint density function

$$f_{X,Y}(x,y)=\begin{cases} 4xy & (x,y)\in[0,1]^2 \\ 0 & \text{otherwise} \end{cases}.$$

i) Calculate $P(0<X<0.1<Y<0.5)$.

ii) Calculate $P(X+Y<1)$.

iii) Calculate $P(X>Y)$.

iv) Find the marginal density of X and the marginal density of Y.

Consider the variables defined as. $R=X$ and $P=XY$.

v) Show that the joint density of R and P is given by

$$f_{P,R}(p,r)=\begin{cases} \dfrac{4p}{r} & 0\le p\le r\le 1 \\ 0 & \text{otherwise} \end{cases}.$$

vi) Hence find the marginal density of R and the marginal density of P.