University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Class 3 Preparation Work
SOLUTIONS

1. i) $x \in [0,\infty)$ and $y \in [0,\infty)$ so the range of S is also $[0,\infty)$ where as the range of D is $(-\infty,\infty)$ (since Y can be larger than X and this difference is unbounded below.

ii) $S = X + Y$ and $D = X - Y$ .hence $X = \frac{S+D}{2}$ and $Y = \frac{S-D}{2}$.

iii) Differentiating both of the above gives the Jacobian

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial X}{\partial S} & \frac{\partial X}{\partial D} \\ \frac{\partial Y}{\partial S} & \frac{\partial Y}{\partial D} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \text{ hence } \left|\det \boldsymbol{J}\right| = \frac{1}{2}.$$

iv) Because both X and Y are non-negative, we have that $S \geq D$. Although D can take any value, positive or negative, it also cannot be less than $-S$ (since even a large negative value for D implies $Y > X > 0$ which would still give $-D = Y - X < Y + X$ i.e. $-D < S$.

Together, these imply that $-\infty \leq -s \leq d \leq s \leq \infty$.

$$f_{S,D}(s,d) = \begin{cases} e^{-\frac{s+d}{2}} e^{-\frac{s-d}{2}} \frac{1}{2} & -\infty \leq -s \leq d \leq s \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

$$f_{S,D}(s,d) = \begin{cases} \frac{1}{2} e^{-s} & -\infty \leq -s \leq d \leq s \leq \infty \\ 0 & \text{otherwise} \end{cases}.$$

v) $f_S(s) = \int_{-s}^{s} \frac{1}{2} e^{-s} dd = \left[\frac{1}{2} d e^{-s}\right]_{-s}^{s} = s e^{-s}$ for $s \in [0,\infty)$ hence

$S \sim gamma(2,1)$

vi) $$f_D(d) = \int_{-\infty}^{-d} \frac{1}{2} e^{-s} ds + \int_{d}^{\infty} \frac{1}{2} e^{-s} ds = \left[-\frac{1}{2} e^{-s} \right]_{-\infty}$$

To calculate the marginal density of D , we consider the two cases that $D \leq 0$ and $D > 0$.

If $D > 0$ then $D < S < \infty$ hence

$$f_D(d) = \int_{d}^{\infty} \frac{1}{2} e^{-s} ds = \left[-\frac{1}{2} e^{-s} \right]_{d}^{\infty} = \frac{1}{2} e^{-d}$$

If $D \leq 0$ then $-D \leq S < \infty$ hence

$$f_D(d) = \int_{-d}^{\infty} \frac{1}{2} e^{-s} ds = \left[-\frac{1}{2} e^{-s} \right]_{d}^{\infty} = \frac{1}{2} e^{d}$$

Together, these give

$$f_D(d) = \begin{cases} \frac{1}{2} e^{-|d|} & d \in (-\infty, \infty) \\ 0 & \text{otherwise} \end{cases}$$

so $D \sim Laplace(1)$.

2. i) $P(0 < X < 0.1 < Y < 0.5) = \int_{0.1}^{0.5}\int_{0}^{0.1} 4xydxdy = \int_{0.1}^{0.5}\left[2x^2y\right]_0^{0.1} dy = \int_{0.1}^{0.5}\left[0.02y\right]dy$.

$$= \int_{0.1}^{0.5}\left[0.02y\right]dy = \left[0.01y^2\right]_{0.1}^{0.5} = 0.0024$$

ii) $P(X+Y<1) = \int_0^1\int_0^{1-y} 4xydxdy = \int_0^1 2y\left[x^2\right]_0^{1-y} dy = \int_0^1 2y(1-y)^2dy$.

$$= \int_0^1 2y(1-y)^2dy = \left[\frac{y^4}{2} - \frac{4y^3}{3} + y^2\right]_0^1 = \frac{1}{6}$$

iii) $P(X > Y) = 0.5$ by symmetry.

iv) X and Y are clearly independent, hence the marginal densities are easily seen to be

$f_X(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$ and $f_Y(y) = \begin{cases} 2y & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

v) Setting .$R = X$ and $P = XY$ gives $X = R$ and $Y = \dfrac{P}{R}$.

The Jacobian is therefore $\begin{pmatrix} 1 & 0 \\ -\dfrac{P}{R^2} & \dfrac{1}{R} \end{pmatrix}$ hence

$$f_{P,R}(p,r) = \begin{cases} 4(r)\left(\dfrac{p}{r}\right)\left(\dfrac{1}{r}\right) & 0 \le p \le r \le 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $$f_{P,R}(p,r) = \begin{cases} \dfrac{4p}{r} & 0 \le p \le r \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that $0 \le Y = \dfrac{P}{R} \le 1$, which means that $0 \le p \le r \le 1$.

vi) $$\int_p^1 \frac{4p}{r}dr = \left[4p\ln(r)\right]_p^1 = -4p\ln(p)$$

so

$$f_P(p) = \begin{cases} -4p\ln(p) & 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_0^r \frac{4p}{r}dp = \left[2\frac{p^2}{r}\right]_0^r = 2r$$

so

$$f_R(r) = \begin{cases} 2r & 0 \le r \le 1 \\ 0 & \text{otherwise} \end{cases}$$