University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Class 6 Preparation Work
SOLUTIONS

1. i) We know that, for $Y \sim Poi(\lambda)$, $E(Y) = \lambda$, hence the first moment of the distribution is $m_1 = \lambda$.

The first sample moment is $s_1 = \bar{y} = \dfrac{y_1 + y_2 + ... + y_n}{n}$.

Applying the method of moments and matching the first moment, we obtain $m_1 = s_1$ and hence $\widehat{\lambda}_{MM} = \dfrac{y_1 + y_2 + ... + y_n}{n}$.

ii) $$L(\{y_1, y_2, ..., y_n\} \mid \lambda) = \prod_{i=1}^{n}\left(\frac{e^{-\lambda}\lambda^{y_i}}{y_i!}\right) = \frac{\prod_{i=1}^{n}\left(e^{-\lambda}\lambda^{y_i}\right)}{\prod_{i=1}^{n} y_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}.$$

iii) $$\ell(\{y_1, y_2, ..., y_n\} \mid \lambda) = \ln(L(\{y_1, y_2, ..., y_n\} \mid \lambda))$$
$$= -n\lambda + \ln(\lambda)\sum_{i=1}^{n} y_i - \ln\left(\prod_{i=1}^{n} y_i!\right).$$

iv) We maximise the loglikelihood (and hence also the likelihood) but finding when the derivative with respect to λ is equal to zero.

$$\frac{\partial}{\partial\lambda}\ell(\{y_1, y_2, ..., y_n\} \mid \lambda) = -n + \frac{1}{\lambda}\sum_{i=1}^{n} y_i .$$

Solving $-n + \dfrac{1}{\lambda}\sum_{i=1}^{n} y_i = 0$ gives $\widehat{\lambda}_{MLE} = \dfrac{1}{n}\sum_{i=1}^{n} y_i$ and hence $\widehat{\lambda}_{MLE} = \widehat{\lambda}_{MM}$.

2.

i) $s_1 = \bar{z} = \dfrac{1+4+3+3+3}{5} = \dfrac{14}{5} = 2.8$

and

$s_2 = \dfrac{1^2+4^2+3^2+3^2+3^2}{5} = \dfrac{44}{5} = 8.8$.

ii) For $Z \sim Bin(n,p)$, we have

$E(Z) = np$ and $E(Z^2) = Var(Z) + E(Z)^2 = np(1-p) + (np)^2$

Applying the method of moments and matching the first two moments, we obtain $np = 2.8$ and $np(1-p) + (np)^2 = 8.8$.

Solving these gives $2.8(1-p) + 2.8^2 = 8.8$ hence $\dfrac{8.8-2.8^2}{2.8} = 1-p$, so

$\dfrac{8.8-2.8^2}{2.8} = 1-p$.

This gives $\hat{p}_{MM} = 1 - \dfrac{8.8}{2.8} + 2.8 = \dfrac{23}{35}$ and hence $\hat{n}_{MM} = 2.8\left(\dfrac{35}{23}\right) = \dfrac{98}{23}$.

iii) $$L(\{z_1,z_2,....,z_5\}\mid p)=\prod_{i=1}^{5}\left(\frac{5!}{z_i!(5-z_i)!}p^{z_i}(1-p)^{(5-z_i)}\right)$$ hence

$$L(\{z_1,z_2,....,z_5\}\mid p)=\prod_{i=1}^{5}\left(\frac{5!}{z_i!(5-z_i)!}\right)p^{\sum_{i=1}^{5}z_i}(1-p)^{\sum_{i=1}^{5}(5-z_1)}.$$

This gives

$$\ell(\{z_1,z_2,....,z_5\}\mid p)=\sum_{i=1}^{5}z_i\ln(p)+\sum_{i=1}^{5}(5-z_1)\ln(1-p)+\ln\left(\prod_{i=1}^{5}\left(\frac{5!}{z_i!(5-z_i)!}\right)\right)$$

. Maximising this, we solve $\frac{\partial}{\partial p}\ell(\{z_1,z_2,....,z_5\}\mid p)=0$.

$$\frac{\partial}{\partial p}\ell(\{z_1,z_2,....,z_5\}\mid p)=\frac{1}{p}\sum_{i=1}^{5}z_i-\frac{1}{1-p}\sum_{i=1}^{5}(5-z_1)$$ hence we solve

$$(1-p)\sum_{i=1}^{5}z_i-p\sum_{i=1}^{5}(5-z_1)=0$$, which gives

$$\sum_{i=1}^{5}z_i-p\sum_{i=1}^{5}z_i-p\sum_{i=1}^{5}(5-z_1)=\sum_{i=1}^{5}z_i-p\sum_{i=1}^{5}z_i-p\left(25-\sum_{i=1}^{5}z_i\right)=0.$$

$$\sum_{i=1}^{5}z_i-25p=0$$ hence $$\widehat{p}_{MLE}=\frac{1}{25}\sum_{i=1}^{5}z_i=\frac{14}{25}=0.56.$$

iv)

$$L(\{z_1,z_2,....,z_5\}\mid n)=\prod_{i=1}^{5}\left(\frac{n!}{z_i!(n-z_i)!}0.5^{z_i}(0.5)^{(n-z_i)}\right)=\prod_{i=1}^{5}\left(\frac{n!}{z_i!(n-z_i)!}0.5^{n}\right)$$

This cannot readily be maximised by a derivative since the factorial function is only defined for integer values hence is differentiable nowhere.

We have seen an average of $\frac{14}{5}=2.8$ on the binomial trials. Given that we now know that $p=0.5$, this implies that n should be close to 5.6.

We therefore compare $n=5$ and $n=6$ and select whichever gives a larger likelihood.

$$L(\{z_1,z_2,....,z_5\}\mid n=5)=\prod_{i=1}^{5}\left(\frac{5!}{z_i!(5-z_i)!}0.5^{5}\right)\approx 0.000745$$

$$L(\{z_1,z_2,....,z_5\}\mid n=6)=\prod_{i=1}^{5}\left(\frac{6!}{z_i!(6-z_i)!}0.5^{6}\right)\approx 0.000671$$

hence we conclude that $\widehat{n}_{MLE}=5$.