**University of Technology Sydney**
**School of Mathematical and Physical Sciences**

Mathematical Statistics (37262) –
Class 9 Preparation Work
SOLUTIONS

1.

i)

$$f(x)=\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\beta-1}\exp\left((\alpha-1)\ln x\right)$$

This is of the form $f(x)=a(x)b(\alpha)\exp\left(c(\alpha)d(x)\right)$ where

$$a(x)=(1-x)^{\beta-1}$$
$$b(\alpha)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$c(\alpha)=(\alpha-1)$$
$$d(x)=\ln(x)$$

ii)

$$f(y)=\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\exp\left((\alpha-1)\ln y-\beta y\right)$$

This is of the form

$f(y)=a(y)b(\boldsymbol{\Theta})\exp\left(\boldsymbol{c}(\boldsymbol{\Theta}).\boldsymbol{d}(y)\right)$ where

$$a(y)=1$$
$$b(\boldsymbol{\Theta})=\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right)$$
$$\boldsymbol{c}(\boldsymbol{\Theta})=\begin{pmatrix}\alpha-1 & -\beta\end{pmatrix}$$
$$\boldsymbol{d}(y)=\begin{pmatrix}\ln(y)\\ y\end{pmatrix}$$

2.

i) $$f(s)=\frac{1}{\sqrt{2^{v}}\Gamma\left(\frac{v}{2}\right)}e^{-\frac{s}{2}}s^{\frac{v}{2}-1}=\frac{1}{\sqrt{2^{v}}\Gamma\left(\frac{v}{2}\right)}e^{-\frac{s}{2}}e^{\left(\frac{v}{2}-1\right)\ln(s)}$$

This is of the form $f(s)=a(s)b(v)\exp\left(c(v)d(s)\right)$ where

$$a(s)=e^{-\frac{s}{2}}$$
$$b(v)=\frac{1}{\sqrt{2^{v}}\Gamma\left(\frac{v}{2}\right)}$$
$$c(v)=\left(\frac{v}{2}-1\right)$$
$$d(s)=\ln(s)$$

ii) The natural parameter is therefore $c(v)=\left(\frac{v}{2}-1\right)$.

iii) $$E(d(S))=E(\ln(S))=-\frac{\partial}{\partial c}\ln(b(c))=-\frac{\partial}{\partial c}\ln\left(\frac{1}{\sqrt{2^{2c+2}}\Gamma\left(c+1\right)}\right)$$

$$=-\frac{\partial}{\partial c}\ln\left(\frac{1}{\sqrt{2^{2c+2}}\Gamma\left(c+1\right)}\right)=-\frac{\partial}{\partial c}\ln\left(\frac{1}{\sqrt{2^{2c+2}}}\right)-\frac{\partial}{\partial c}\ln\left(\frac{1}{\Gamma\left(c+1\right)}\right)$$
$$=\frac{\partial}{\partial c}\ln\left(2^{c+1}\right)+\frac{\partial}{\partial c}\ln\left(\Gamma\left(c+1\right)\right)$$
$$=\frac{\partial}{\partial c}\left((c+1)\ln(2)\right)+\frac{\partial}{\partial c}\ln\left(\Gamma\left(c+1\right)\right)$$
$$=\ln(2)+\frac{\Gamma'\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}$$

iv)

$$Var(d(S))=Var(\ln(S))=-\frac{\partial^2}{\partial^2 c}\ln(b(c))$$
$$=\frac{\partial}{\partial c}\left(\ln(2)+\frac{\Gamma'\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\right)=\frac{\Gamma''\left(\frac{v}{2}\right)\Gamma\left(\frac{v}{2}\right)-\Gamma'\left(\frac{v}{2}\right)^2}{\Gamma\left(\frac{v}{2}\right)^2}.$$

3.

i) $\beta_1$ is likely to be negative, since it is probable that taller children will complete the run in less time than shorter children, as the primary driver of height in children will be age. An average 170cm child will be quicker than an average 80cm child, since the 170cm child is likely a teenager and the 80cm child is likely a toddler.

ii) $\beta_2$ is likely to be negative, since it is probable that heavier children will complete the run in less time than lighter children, as the primary driver of body mass in children will be age. An average 50kg child will be quicker than an average 10kg child, since the 50kg child is likely a teenager and the 10kg child is likely a toddler.

iii) In the model with both height and mass is included, it is likely that $\beta_1$ will remain negative but $\beta_2$ will be positive in this model. When comparing two children of the same height, is it likely that the heavier child will be slower than the lighter one.