University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) – Tutorial 10

1. Consider applying the Metropolis-Hastings algorithm to generate 10 realisations of $X \sim Geo(0.7)$.

   Starting at $x_0 = 4$, 10 proposed moves (for $x_1, x_2, ..., x_{10}$ respectively) are drawn from a symmetric distribution which proposes moves to states $1,\ 2, -1,\ 1,\ 5,\ 0,\ 3,\ 2,\ 3, -1$.

   The proposed move at timestep $i \in \{1, 2, ..., 10\}$ is only accepted if the acceptance probability $A_i > u_i$ where $u_1, u_2, ..., u_{10}$ are independent realisations of a $U[0,1]$ variable.

   Here, $u_1, u_2, ..., u_{10}$ are, in order
   0.503 0.111 0.859 0.330 0.474 0.004 0.217 0.712 0.557 0.486

   i) Find the values of $x_1, x_2, ..., x_{10}$, clearly stating all of your working and calculating each acceptance probability.

   ii) Why would using a normal random variable $Y \sim N(\mu, \sigma^2)$ not be suitable as a proposal distribution regardless of the choice of μ and σ^2?

   Consider now applying the Metropolis-Hastings algorithm to generate realisations of a discrete uniform random variable with domain $\{1, 2, 3\}$ i.e. a variable which takes the values 1, 2 and 3, each with equal probability.

   iii) For a proposal distribution which was a discrete uniform random variable with domain $\{0, 1, 2, 3, 4\}$, what would the expected acceptance rate be? Justify your answer.

   Hints:

   A proposed move from state x to state x_p is accepted by the Metropolis-Hastings algorithm with probability $A_i = \min\left\{1, \left(\frac{P(x_p)}{P(x)} \frac{q_{j,i}}{q_{i,j}}\right)\right\}$ where:

   $q_{i,j}$ is the probability that the proposal distribution proposes moving from state i to state j and $P(x)$ is the probability mass that $X = x$.

   The probability mass function of $X \sim Geo(p)$ is given by

$$P(X = n) = \begin{cases} p(1-p)^{n-1} & n \in \{1, 2, 3, ...\} \\ 0 & \text{otherwise} \end{cases}.$$

2. Consider applying the Metropolis-Hastings algorithm to generate 10 realisations of X, which has probability mass function

$$P(X=k)=\begin{cases}0.1 & k\in\{-1,1,2,4\}\\ 0.2 & k\in\{-3,3,0\}\\ 0 & \text{otherwise}\end{cases}$$

Starting at $x_0 = 4$, 10 proposed moves (for $x_1, x_2, ..., x_{10}$ respectively) are drawn from a symmetric distribution which proposes moves to states

$1,\ 2, -1,\ 1,\ 5,\ 0,\ 3,\ 2,\ 3, -1.$

The proposed move at timestep $i \in \{1,2,...,10\}$ is only accepted if the acceptance probability $A_i > u_i$ where $u_1, u_2, ..., u_{10}$ are independent realisations of a $U[0,1]$ variable.

Here, $u_1, u_2, ..., u_{10}$ are, in order

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i) Find the values of $x_1, x_2, ..., x_{10}$, clearly stating all of your working and calculating each acceptance probability.

ii) Why would using a Poisson variable not be suitable as a proposal distribution regardless of the choice of rate parameter?

3. Draw the state diagram for a possible Markov Chain with states A, B, C, D, E and F such that:

i) The Markov Chain is ergodic.

ii) States A, B and C each have period 3 and states D and E have period 2.

iii) The system has two equilibrium distributions, either being in A or B with equal probability or in C or D with equal probability. Starting in E it will always end up in the A or B equilibrium and starting in F it may end up in either of the two equilibria.

Clearly label any transition arrows with their corresponding transition probabilities.