University of Technology Sydney
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Mathematical Statistics (37262) –
Tutorial 11

1. The times $x_1, x_2, ..., x_n$ between successive faults recorded by a maintenance system are assumed to be independent draws from an exponential distribution with unknown rate parameter λ faults per day.

   i) Given that the probability density function of $X \sim \exp(\lambda)$ is
   $$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \in [0, \infty) \\ 0 & \text{otherwise} \end{cases},$$
   find the likelihood of the sample $x_1, x_2, ..., x_n$, $L(x_1, x_2, ..., x_n \mid \lambda)$.

   Prior to any observations being made, the belief around the value of λ is described a Gamma variable, $\lambda \sim Gamma(4, 2)$.

   ii) In your own words, explain what it means for a chosen prior distribution to be a conjugate prior for a given likelihood function.

   The times between the next ten faults (in days) are recorded. These are 0.34, 0.71, 1.11, 0.14, 0.77, 0.33, 0.42, 0.20, 0.83 and 0.15.

   iii) Show that the posterior distribution describing the belief around the value of λ is also described by a Gamma variable. State its parameters.

   iv) Compare the mean and variance of the belief around the value of λ prior to these ten observations and after taking them into account. In your own words, describe how the belief changed with the new observations.

   Notes:
   The probability density function of $Y \sim Gamma(\alpha, \beta)$ is given by
   $$f(y) = \begin{cases} \dfrac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} & y \in [0, \infty) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \alpha > 0, \beta > 0$$
   where $\Gamma(k) = (k-1)!$ if k is a positive integer.

   The mean and variance of Y are given by $E(Y) = \dfrac{\alpha}{\beta}$ and $Var(Y) = \dfrac{\alpha}{\beta^2}$ respectively.

2. A Pareto variable, $\theta \sim Pareto(m,\omega)$ has density function

$$f(\theta)=\begin{cases}\dfrac{\omega m^{\omega}}{\theta^{\omega+1}} & \theta\in[m,\infty)\\ 0 & \text{otherwise}\end{cases} \quad \text{for } m>0,\omega>0.$$

Assume that $m>0$ is known but $\omega>0$ is uncertain.
Initially, the prior belief about the value of ω is described by a Gamma variable, $\omega \sim Gamma(\alpha,\beta)$.

Show that the gamma distribution is a conjugate prior in this case and hence find the parameters of the posterior distribution.

3. Independent samples $x_1,x_2,...,x_n$ are drawn from a uniform random variable $X \sim U[0,\theta]$ where the value of $\theta>0$ is not known.

i) Write down the probability density function of $X \sim U[0,\theta]$.

ii) Hence find the likelihood of the sample $x_1,x_2,...,x_n$, $L(x_1,x_2,...,x_n|\theta)$.

Prior to any observations being made, the belief around the value of θ is described a Pareto variable, $\theta \sim Pareto(m,\alpha)$.

$$\text{That is, } f(\theta)=\begin{cases}\dfrac{\alpha m^{\alpha}}{\theta^{\alpha+1}} & \theta\in[m,\infty)\\ 0 & \text{otherwise}\end{cases} \quad \text{for } m>0,\alpha>0.$$

iii) Show that the posterior distribution describing the belief around the value of θ is also described by a Pareto variable, $\theta\,|\,\boldsymbol{X} \sim Pareto(M,\alpha+n)$ where $M=\max\{x_1,x_2,...,x_n,m\}$.
Hint: Think carefully about when the density function of X is non-zero.

iv) Is the Pareto distribution a conjugate prior for this likelihood function? Justify your answer.