University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Tutorial 11
SOLUTIONS

1. i) $L(x_1, x_2, ..., x_n | \lambda) = (\lambda e^{-\lambda x_1})(\lambda e^{-\lambda x_2})...(\lambda e^{-\lambda x_n}) = \lambda^n e^{-\lambda(x_1 + ... + x_n)}$

ii) A conjugate prior is one such that, when combined with the likelihood function, gives a posterior distribution of the same form/family, albeit with different parameter values.

iii) $L(x_1, x_2, ..., x_n | \lambda) = \lambda^n e^{-\lambda(x_1 + ... + x_n)}$. $f(\lambda) = \dfrac{2^4 \lambda^3 e^{-2\lambda}}{\Gamma(4)}$.

The posterior is proportional to the prior multiplied by the likelihood hence $f(\lambda | x_1, x_2, ..., x_n) \propto \lambda^n e^{-\lambda(x_1 + ... + x_n)} \times \dfrac{2^4 \lambda^3 e^{-2\lambda}}{\Gamma(4)}$.

$f(\lambda | x_1, x_2, ..., x_n) \propto \lambda^{n+3} e^{-\lambda(2 + x_1 + ... + x_n)}$ hence

$\lambda \sim Gamma(4 + n,\ 2 + x_1 + ... + x_n)$.

Here, $\lambda \sim Gamma(14, 7)$

iv) The prior mean was 2 and variance 1. The additional observations seemed to confirm the prior belief, keeping the posterior mean at 2 but shrinking the posterior variance to $\frac{2}{7}$, suggesting increased belief that the rate is close to 2.

2.

Let a set of observations be $\boldsymbol{\Theta} = (\theta_1, \theta_2, ...\theta_n)$.

The likelihood function is then $L(\boldsymbol{\Theta} \mid \omega) = \frac{\omega m^{\omega}}{\theta_1^{\omega+1}} \frac{\omega m^{\omega}}{\theta_2^{\omega+1}} ... \frac{\omega m^{\omega}}{\theta_n^{\omega+1}}$ (assuming all $\theta_i \geq m$, else the likelihood is zero.)

$L(\boldsymbol{\Theta} \mid \omega) = \frac{\omega^n m^{n\omega}}{\prod_{i=1}^{n} \theta_i^{\omega+1}}$.Combined with the prior density

$$f(\omega) = \begin{cases} \frac{\beta^{\alpha} \omega^{\alpha-1} e^{-\beta\omega}}{\Gamma(\alpha)} & \omega \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

, we obtain a posterior density of

$$f(\omega \mid \boldsymbol{\Theta}) = L(\boldsymbol{\Theta} \mid \omega) f(\omega) = \frac{\omega^n m^{n\omega}}{\prod_{i=1}^{n} \theta_i^{\omega+1}} \frac{\beta^{\alpha} \omega^{\alpha-1} e^{-\beta\omega}}{\boldsymbol{\Gamma}(\alpha)} \propto \omega^{\alpha+n-1} e^{-\beta\omega} \frac{m^{n\omega}}{\prod_{i=1}^{n} \theta_i^{\omega+1}}$$

$$\propto \omega^{\alpha+n-1} e^{-\beta\omega} e^{n\omega \ln(m)} e^{-(\omega+1)\sum_{i=1}^{n} \ln(\theta_i)} \propto \omega^{\alpha+n-1} e^{-\omega(\beta - n\ln(m) + \sum_{i=1}^{n} \ln(\theta_i))} e^{-\sum_{i=1}^{n} \ln(\theta_i))}$$

Note now that the term $e^{-\sum_{i=1}^{n} \ln(\theta_i))}$ can be considered constant with respect to ω hence

$$f(\omega \mid \boldsymbol{\Theta}) \propto \omega^{\alpha+n-1} e^{-\omega(\beta - n\ln(m) + \sum_{i=1}^{n} \ln(\theta_i))}$$

This tells us that

$$f(\omega \mid \boldsymbol{\Theta}) = L(\boldsymbol{\Theta} \mid \omega) f(\omega) \propto \omega^{\alpha+n-1} \exp\left(-\omega\left(\beta + \sum_{i=-1}^{n} \ln(\theta_i) - n\ln(m)\right)\right)$$ or,

simplifying this,

$$f(\omega \mid \boldsymbol{\Theta}) \propto \omega^{\alpha+n-1} \exp\left(-\omega\left(\beta + \sum_{i=-1}^{n} \ln\left(\frac{\theta_i}{m}\right)\right)\right)$$

This gives that the posterior density is a Gamma variable

$$Gamma\left(\alpha + n, \beta + \sum_{i=-1}^{n} \ln\left(\frac{\theta_i}{m}\right)\right).$$

3.

i) $X \sim U[0,\theta]$ therefore $f(x) = \begin{cases} \frac{1}{\theta} & x \in [0,\theta] \\ 0 & \text{otherwise} \end{cases}$

ii) $$L(x_1, x_2, ..., x_n | \theta) = \begin{cases} \left[\frac{1}{\theta}\right]\left[\frac{1}{\theta}\right]...\left[\frac{1}{\theta}\right] = \frac{1}{\theta^n} & x_1, x_2, ..., x_n \le \theta \\ 0 & \text{otherwise} \end{cases}.$$

$$L(x_1, x_2, ..., x_n | \theta) = \begin{cases} \frac{1}{\theta^n} & x_1, x_2, ..., x_n \le \theta \\ 0 & \text{otherwise} \end{cases}$$

iii) $f(\theta | x_1, x_2, ..., x_n) \propto L(x_1, x_2, ..., x_n | \theta) f(\theta)$ so

$f(\theta | x_1, x_2, ..., x_n) \propto \left[\frac{1}{\theta^n}\right]\left[\frac{\alpha m^\alpha}{\theta^{\alpha+1}}\right] \propto \frac{1}{\theta^{n+\alpha+1}}$ for $x_1, x_2, ..., x_n, m \le \theta$.

Since we need all x_i to be no greater than θ and also need m to be no greater than θ, we have that $M = \max\{x_1, x_2, ..., x_n, m\}$ is the lower bound on θ.

This gives that the posterior distribution for θ is a $Pareto(M, \alpha + n)$ variable where $M = \max\{x_1, x_2, ..., x_n, m\}$.

iv) Yes. If the prior and posterior distributions are the same type of variable (in this case, both Pareto distributed) then the prior distribution is a conjugate prior for the likelihood function.