University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Tutorial 1
SOLUTIONS

1.

i) We need a rule which assigns 1 with probability 0.75 and 0 with probability 0.25. One such rule would be

$$\begin{array}{ll} \text{if } u_i < 0.75 & t_i = 1 \\ \text{if } u_i \geq 0.75 & t_i = 0 \end{array}.$$

Applying this gives
$$\begin{array}{cccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}.$$

ii) A realisation of $S \sim Bin(10, 0.75)$ can be obtained by summing 10 independent realisations of $T \sim Bern(0.75)$. Summing across each of the rows gives

$$\begin{aligned} s_1 &= t_1 + t_2 + ... + t_{10} = 9 \\ s_2 &= t_{11} + t_{12} + ... + t_{20} = 8 \\ s_3 &= t_{21} + t_{22} + ... + t_{30} = 9 \\ s_4 &= t_{31} + t_{32} + ... + t_{40} = 10 \\ s_5 &= t_{41} + t_{42} + ... + t_{50} = 5 \end{aligned}$$

iii) A realisation of $R \sim Geo(0.75)$ can be obtained by counting how many successive realisations of $T \sim Bern(0.75)$ until the next 1 is observed.

Here, we obtain
$$\begin{array}{l} 2,1,1,1,1,1,1,1,1, \\ 1,2,2,1,1,1,1,1, \\ 1,1,2,1,1,1,1,1,1, \\ 1,1,1,1,1,1,1,1,1,1, \\ 4,1,1,1,2,... \end{array}$$

iv) The nth 1 cannot be observed until at least n Bernoulli variables have been observed and there is no theoretical upper limit on the number needed, hence the range of $V \sim NegBin(n, p)$ is $\{n, n+1, n+2, ...\}$.

2.

i) $$P(Y=k)=\begin{cases}0.2 & k=0\\ 0.8 & k=1\\ 0 & \text{otherwise}\end{cases}$$

ii) $X+Y$ has range $\{0,1,2,3,4\}$ and its probability mass function is

$$P(X+Y=k)=\begin{cases}0.2(0.5)^3 & k=0\\ 0.2\dfrac{3!}{k!(3-k)!}0.5^3+0.8\dfrac{3!}{(k-1)!(4-k)!}0.5^3 & k\in\{1,2,3\}\\ 0.8(0.5)^3 & k=4\\ 0 & \text{otherwise}\end{cases}$$

$$P(X+Y=k)=\begin{cases}0.2(0.5)^3 & k=0\\ 0.2(3)(0.5)^3+(0.8)(0.5)^3 & k=1\\ 0.2(3)(0.5)^3+0.8(3)(0.5)^3 & k=2\\ 0.2(0.5)^3+0.8(3)(0.5)^3 & k=3\\ 0.8(0.5)^3 & k=4\\ 0 & \text{otherwise}\end{cases}$$

or $$P(X+Y=k)=\begin{cases}0.025 & k=0\\ 0.175 & k=1\\ 0.375 & k=2\\ 0.325 & k=3\\ 0.1 & k=4\\ 0 & \text{otherwise}\end{cases}$$

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iii) We would therefore have the rule that (for $Z=X+Y$)

$$\begin{array}{ll}\text{if } u_i<0.025 & z_i=0\\ \text{if } 0.025\le u_i<0.2 & z_i=1\\ \text{if } 0.2\le u_i<0.575 & z_i=2\\ \text{if } 0.575\le u_i<0.9 & z_i=3\\ \text{if } 0.9\le u_i & z_i=4\end{array}$$

3. $$E(R)=\sum_{k=1}^{\infty}kP(R=k)=p+2p(1-p)+3p(1-p)^2+4p(1-p)^3+5p(1-p)^4+\ldots$$

We can rewrite this in terms of an infinite series of geometric series.

$$\begin{aligned}E(R)&=p+2p(1-p)+3p(1-p)^2+4p(1-p)^3+5p(1-p)^4+\ldots\\&=p+p(1-p)+p(1-p)^2+p(1-p)^3+p(1-p)^4+\ldots\\&\quad+p(1-p)+p(1-p)^2+p(1-p)^3+p(1-p)^4+\ldots\\&\quad+p(1-p)^2+p(1-p)^3+p(1-p)^4+\ldots\\&\quad+p(1-p)^3+p(1-p)^4+\ldots\\&\quad+p(1-p)^4+\ldots\end{aligned}$$

The first of these is a geometric series with first term p and common ratio $1-p$, hence sums to $\frac{p}{1-(1-p)}=1$.

The second of these is a geometric series with first term $p(1-p)$ and common ratio $1-p$, hence sums to $\frac{p(1-p)}{1-(1-p)}=1-p$.

The third of these is a geometric series with first term $p(1-p)^2$ and common ratio $1-p$, hence sums to $\frac{p(1-p)^2}{1-(1-p)}=(1-p)^2$.

Taking these sums, we obtain a geometric series

$$E(R)=1+(1-p)+(1-p)^2+(1-p)^3+(1-p)^4+\ldots=\frac{1}{p}.$$

Practice Assessable Labwork

i) $$P(X=k)=\begin{cases}0.15 & k=0\\ 0.15 & k=-1\\ 0.05 & k=1\\ 0.65 & k=3\\ 0 & \text{otherwise}\end{cases}$$

ii) $E(X)=0.15(0)+0.15(-1)+0.05(1)+0.65(3)=1.85$

iii)

$$Var(X)=E(X^2)-E(X)^2$$

$$=0.15(0^2)+0.15(-1)^2+0.05(1^2)+0.65(3^2)-1.85^2=2.6275$$

iv) $\{u_1,u_2,...,u_5\}=\{0.054,\ 0.507,\ 0.883,\ 0.248,\ 0.716\}$,

hence $\{x_1,x_2,...,x_5\}=\{0,\ 3,\ 3,-1,\ 3\}$

v) $\bar{X}=\dfrac{0+3+3-1+3}{5}=1.6$

vi) $\{u_1,u_2,...,u_5\}=\{0.054,\ 0.507,\ 0.883,\ 0.248,\ 0.716\}$,
hence $\{y_1,y_2,...,y_5\}=\{0,\ 6,\ 6,-1,\ 6\}$.

vii) $E(Y)=0.15(0)+0.1(-1)+0.75(6)=4.4$.