University of Technology Sydney
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Mathematical Statistics (37262) – Tutorial 4 SOLUTIONS

1.

i) $S = X + Y$ and $D = X - Y$ .hence $X = \frac{S+D}{2}$ and $Y = \frac{S-D}{2}$.

ii) Differentiating both of the above gives the Jacobian

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial X}{\partial S} & \frac{\partial X}{\partial D} \\ \frac{\partial Y}{\partial S} & \frac{\partial Y}{\partial D} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \text{ hence } \left|\det \boldsymbol{J}\right| = \frac{1}{2}.$$

iii) $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$.

$$\begin{aligned} f_{S,D}(s,d) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(s+d)^2}{8}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-d)^2}{8}} \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{s^2+2sd+d^2}{8}} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2-2sd+d^2}{8}} \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{4}} \\ &= \frac{1}{\sqrt{2(\sqrt{2})^2\pi}} e^{-\frac{s^2}{2(\sqrt{2})^2}} \frac{1}{\sqrt{2(\sqrt{2})^2\pi}} e^{-\frac{d^2}{2(\sqrt{2})^2}} \end{aligned}$$

$f_{S,D}(s,d) = f_S(s) f_D(d)$ where $f_S(s) = \frac{1}{\sqrt{2(\sqrt{2})^2\pi}} e^{-\frac{s^2}{2(\sqrt{2})^2}}$ (and similar for D)

hence S and D are independent and each $\sim N(0, \sqrt{2}^2)$.

2.

i) The row totals are:

Subject A: 100, Subject B: 50, Subject C: 50.

The column proportions are:

High Distinction: 0.1, Distinction: 0.2, Credit: 0.2, Pass: 0.175, Fail: 0.075.

Hence under the null hypothesis, the expected counts are:

	High Distinction	**Distinction**	**Credit**	**Pass**	**Fail**
Subject A	10	20	20	35	15
Subject B	5	10	10	17.5	7.5
Subject C	5	10	10	17.5	7.5

ii)

$$\frac{(11-10)^2}{10}+\frac{(18-20)^2}{20}+\frac{(23-20)^2}{20}+\frac{(30-35)^2}{35}+\frac{(18-15)^2}{15}+$$
$$\frac{(6-5)^2}{5}+\frac{(7-10)^2}{10}+\frac{(11-10)^2}{10}+\frac{(18-15)^2}{17.5}+\frac{(8-7.5)^2}{7.5}+$$
$$\frac{(3-5)^2}{5}+\frac{(15-10)^2}{10}+\frac{(6-10)^2}{10}+\frac{(22-15)^2}{17.5}+\frac{(4-7.5)^2}{7.5}\approx 11.00$$

iii) The contingency table has 5 rows and 3 columns, hence we have $(5-1)(3-1)=8$ degrees of freedom.

For $Y\sim\chi^2(8)$, $P(Y>11.00)\approx 0.202$ hence with 0.05 significance, we do not reject the hypothesis that the grade distribution and subject taken are independent.

3.

$$\left[\frac{m-Np}{\sqrt{Np(1-p)}}\right]^2 = \frac{(m-Np)^2}{Np(1-p)}.$$

$$\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p} \text{ hence } \left[\frac{m-Np}{\sqrt{Np(1-p)}}\right]^2 = \frac{(m-Np)^2}{Np} + \frac{(m-Np)^2}{N(1-p)}$$

$$\frac{(m-Np)^2}{N(1-p)} = \frac{(Np-m)^2}{N(1-p)} \text{ and hence } \left[\frac{m-Np}{\sqrt{Np(1-p)}}\right]^2 = \frac{(m-Np)^2}{Np} + \frac{(Np-m)^2}{N(1-p)}.$$

$$\frac{(Np-m)^2}{N(1-p)} = \frac{(Np-m+N-N)^2}{N(1-p)} = \frac{(N-m-N(1-p))^2}{N(1-p)}.$$

Putting this together gives

$$\left[\frac{m-Np}{\sqrt{Np(1-p)}}\right]^2 = \frac{(m-Np)^2}{Np} + \frac{(N-m-N(1-p))^2}{N(1-p)}.$$