University of Technology Sydney
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Mathematical Statistics (37262) –
Tutorial 6

1. A broken machine works each time it is turned on with probability $p \in [0,1]$, independent of whether or not it worked on any other occasion.

   A worker who needs to use the machine turns it on repeatedly until it works and he/she records how many times it is turned on before it works. For example, if it works first time it is turned on, the worker records 1. If it does not work first time but works second time, the worker records 2 etc. Let *R* be this value recorded by the worker.

   i) What is the distribution of *R*?

   ii) Given a sample $\{r_1, r_2, ..., r_n\}$ be independent realisations of *R*, show that the loglikelihood function is given by

   $$\ell(\{r_1, r_2, ..., r_n\} \mid p) = n\ln(p) + \left(\sum_{i=1}^{n} r_i - n\right)\ln(1-p).$$

   iii) Hence calculate the maximum likelihood estimate of *p*.

   iv) Show that this value is the same as the one obtained by applying the method of moments using the first sample moment.

2. Let $\{x_1, x_2, ..., x_n\}$ be independent realisations of $X \sim Pareto(m, \alpha)$ where $m > 0, \alpha > 0$ are unknown parameters.

That is X has probability density function $f(x) = \begin{cases} \dfrac{\alpha m^{\alpha}}{x^{\alpha+1}} & x \in [m, \infty) \\ 0 & \text{otherwise} \end{cases}$.

i) Write down the likelihood function $L(\{x_1, x_2, ..., x_n\} \mid m, \alpha)$.

ii) Hence find the loglikelihood function $\ell(\{x_1, x_2, ..., x_n\} \mid m, \alpha)$.

iii) Calculate $\dfrac{\partial \ell}{\partial m}$ and hence show that this is never zero.

iv) Find the maximum likelihood estimate of *m*. Justify your answer.

v) Calculate $\dfrac{\partial \ell}{\partial \alpha}$ and hence show that the maximum likelihood estimate of α is $\hat{\alpha}_{MLE} = \dfrac{n}{\sum_{i=1}^{n} \ln(x_i) - n\ln(\min\{x_1, x_2, ..., x_n\})}$.

3. i) Given five independent realisations of a $U[0,1]$ random variable, $\{0.223,\ 0.750,\ 0.882,\ 0.573,\ 0.250)$ use the Monte Carlo method to estimate the value of $\int_0^1 \ln\left(45 - \left|\arctan(4x-2)\right|\right) dx$.

Note: $\arctan(x)$ represents the inverse tangent function i.e. if $y = \arctan(x)$ then $x = \tan(y)$ and $|x|$ represents the absolute value of *x*.

ii) Clearly explain all the steps you would follow to obtain an estimate of the definite integral $\int_{-\infty}^{M} \dfrac{\ln(x^2+2)}{(x^4+4)} dx$ via the Monte Carlo method. (Assume $M < \infty$)

Note: You do not need to obtain an estimate of the integral. You should, however, clearly state any changes of variable required and derive the resulting equivalent definite integral.