University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Tutorial 8

1. Consider independent realisations $\{x_1, x_2, ... x_n\}$ of a continuous uniform random variable $X \sim U[a.b]$ where $a < b$ and both parameters are unknown.

   i) Write down the density function of *X*.

   ii) Hence show that the likelihood function associated with the sample $\{x_1, x_2, ... x_n\}$ is $L(\{x_1, x_2, ... x_n\} \mid a, b) = \left(\frac{1}{b-a}\right)^n$.

   iii) Show that $\frac{\partial}{\partial a}\ell(\{x_1, x_2, ... x_n\} \mid a, b)$ and $\frac{\partial}{\partial b}\ell(\{x_1, x_2, ... x_n\} \mid a, b)$ are never zero where $\ell(\{x_1, x_2, ... x_n\} \mid a, b)$ is the loglikelihood function.

   iv) In your own words, clearly explain why the maximum likelihood estimate of *b* is $\hat{b} = \max\{x_1, x_2, ... x_n\}$.

   v) Find the maximum likelihood estimate of *a.* Justify your answer.

   vi) In your own words, explain why

   $$P(\max\{x_1, x_2, ... x_n\} < z) = \left(\frac{z-a}{b-a}\right)\left(\frac{z-a}{b-a}\right) ... \left(\frac{z-a}{b-a}\right) = \left(\frac{z-a}{b-a}\right)^n.$$

   vii) Hence show that the probability density function of $Z = \max\{x_1, x_2, ... x_n\}$ is $f(z) = \frac{n(z-a)^{n-1}}{(b-a)^n}$ (for $z \in [a, b]$.)

   viii) Hence show that $E(Z) = \left(\frac{a+bn}{n+1}\right)$.

   ix) Show that the maximum likelihood estimate $\hat{b} = \max\{x_1, x_2, ... x_n\}$. is biased.

   x) Show that, as the sample size $n \to \infty$, the estimate is asymptotically unbiased. That is, show that, as $n \to \infty$, $bias(\hat{b}, b) \to 0$.

Note: Maximum likelihood estimators are not necessarily unbiased (as is the case here) but they are generally asymptotically unbiased for large sample sizes. That is, the bias tends to zero as the sample size tends to infinity.