University of Technology Sydney
School of Mathematical and Physical Sciences

Mathematical Statistics (37262) –
Tutorial 8
SOLUTIONS

1.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a.b] \\ 0 & \text{otherwise} \end{cases}.$$

i) $L(\{x_1, x_2, \ldots x_n\} \mid a, b) = \left(\dfrac{1}{b-a}\right)^n$.

ii) $\ell(\{x_1, x_2, \ldots x_n\} \mid a, b) = n \ln\left(\dfrac{1}{b-a}\right) = -n\ln(b-a)$.

$\dfrac{\partial}{\partial a}\ell(\{x_1, x_2, \ldots x_n\} \mid a, b) = -\dfrac{n}{b-a}$ and $\dfrac{\partial}{\partial b}\ell(\{x_1, x_2, \ldots x_n\} \mid a, b) = \dfrac{n}{b-a}$.

We note that neither of these derivatives is ever zero. We therefore maximise the likelihood with

$\hat{a} = \min\{x_1, x_2, \ldots x_n\}$ and $\hat{b} = \max\{x_1, x_2, \ldots x_n\}$.

iii) We now consider the cumulative density of $\max\{x_1, x_2, \ldots x_n\}$.

$P(\max\{x_1, x_2, \ldots x_n\} < z) = P(x_1 < z)P(x_2 < z)\ldots P(x_n < z)$.

$$P(\max\{x_1, x_2, \ldots x_n\} < z) = \left(\frac{z-a}{b-a}\right)\left(\frac{z-a}{b-a}\right)\ldots\left(\frac{z-a}{b-a}\right) = \left(\frac{z-a}{b-a}\right)^n$$

So the probability density function is given by $f(z) = \dfrac{n(z-a)^{n-1}}{(b-a)^n}$. This gives

$$E(Z) = \int_a^b z f(z)dz = \int_a^b z\frac{n(z-a)^{n-1}}{(b-a)^n}dz = \int_a^b (z-a)\frac{n(z-a)^{n-1}}{(b-a)^n}dz + a\int_a^b \frac{n(z-a)^{n-1}}{(b-a)^n}dz$$

$$E(Z) = (b-a)\int_a^b \frac{n(z-a)^n}{(b-a)^{n+1}}dz + a\int_a^b \frac{n(z-a)^{n-1}}{(b-a)^n}dz = \frac{n(b-a)}{n+1} + a = \left(\frac{a+bn}{n+1}\right)$$

Hence $E(\hat{b}) = \left(\dfrac{a+bn}{n+1}\right) = b - \left(\dfrac{b-a}{n+1}\right)$.

Applying the same argument to the minimum we see that

$$E(\hat{a}) = \left(\frac{an+b}{n+1}\right) = a + \left(\frac{b-a}{n+1}\right).$$

iv) It is trivial to see that the biases of $\pm\left(\frac{b-a}{n+1}\right) \to 0$ as $n \to \infty$.