University of Technology Sydney
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Mathematical Statistics (37262) –
Tutorial 9

1. A negative binomial variable with range $\{r, r+1, r+2, \ldots\}$ can be considered as the sum of r independent identically distributed geometric random variable. It describes how many independent identically distributed Bernoulli variables must be observed until r successes (1s) have been observed.

   For $N \sim NegBin(r,p)$, the probability mass function is

$$f(n) = P(N=n) = \begin{cases} \dfrac{(n-1)!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} & n \in \{r, r+1, r+2, \ldots\} \\ 0 & \text{otherwise} \end{cases}.$$

   i) Show that, if r is known, $N \sim NegBin(r,p)$ belongs to the exponential family.

   ii) Find the natural parameter for this distribution.

   iii) Hence calculate the mean and variance of $N \sim NegBin(r,p)$.

2. The density function of $Y \sim Weibull(\lambda, k)$ for $k > 0,\ \lambda > 0$ .is given by

$$f_Y(y) = \begin{cases} \dfrac{k y^{k-1} e^{-\left(\frac{y}{\lambda}\right)^k}}{\lambda^k} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

   Here, we assume that k is known, but λ is unknown.

   i) Show that $Y \sim Weibull(\lambda, k)$ belongs to the exponential family.

   ii) Hence show that we can take $c(\lambda) = -\dfrac{1}{\lambda^k}$ as the natural parameter for this distribution.

3. Consider independent realisations $\{u_1, u_2, ... u_n\}$ of a continuous uniform random variable $U \sim U[0,b]$ where $b > 0$ is unknown.

That is, U has density function $f(u) = \begin{cases} \frac{1}{b} & u \in [0,b] \\ 0 & \text{otherwise} \end{cases}$.

i) Justifying your answer, find $E(U)$.

ii) Using the Method of Moments applied to the first moment estimates b by $\hat{b}_{MM} = 2\frac{\sum_{i=1}^{n} u_i}{n}$.

Show that $\hat{b}_{MM}$ is an unbiased estimator of b.

iii) Estimating b by maximum likelihood estimation gives an estimator $\hat{b}_{MLE} = \max\{u_1, u_2, ..., u_n\}$.

Calculate the bias of this estimator when we just have a single observation u_1 of $U \sim U[0,b]$.