Question 2. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) The electric field in some region is found to depend on the distance from the origin r via the equation

$$\mathbf{E} = \frac{k}{r}\hat{\mathbf{r}}$$

where k is some constant and $\hat{\mathbf{r}}$ is the radial unit vector in spherical polar coordinates. Use Gauss's law to compute the charge density $\rho(\mathbf{r})$.

[Hint: You may need Gauss's law in differential form, which is

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

where $\rho(\mathbf{r})$ is the charge density at a point $\mathbf{r}$, and ε_0 is a constant.]

(5 marks)

(b) Given the operator

$$\mathcal{L} = -\frac{1}{2}\frac{d^2\phi}{dx^2}$$

Find all eigenvalues and eigenfunctions of the Sturm-Lioville problem

$$\mathcal{L}\phi = \lambda\phi$$

on the domain $0 \leq x \leq a$, with the boundary condition $\phi(0) = \phi(a) = 0$.

(5 marks)