Question 2. (10 marks)

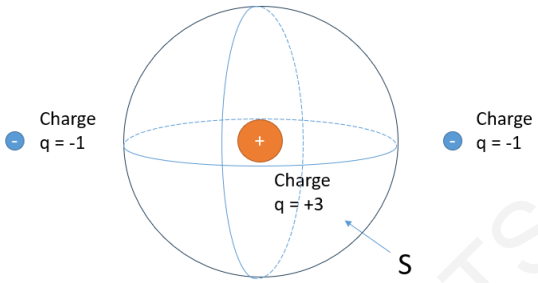
Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) Gauss's law in differential form states that the divergence of the electric field at any point is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the charge density at that point and ϵ_0 is a constant.

Consider the charge distribution shown (units are in Coulombs). The radius of the sphere S is 2.5 m, and the distance between the positive and negative charges is 4 m.



Use Gauss's law to compute the flux integral of the electric field $\mathbf{E}$ through the surface S. What happens to the flux integral if the radius of the surface S is doubled, while the charge distribution remains the same? (5 marks)

(b) The function

$$\phi_3(y) = \sin\left(\frac{3\pi y}{b}\right)$$

is an eigenfunction of the Sturm-Liouville operator

$$\mathcal{L} = -\frac{1}{2}\frac{d^2}{dy^2}$$

on the domain $0 \leq y \leq b$ with the boundary condition $\phi(0) = \phi(b) = 0$.

(i) Find the eigenvalue λ_3 corresponding to ϕ_3.

(ii) Show that the formula

$$\lambda_3 = \frac{\langle \phi_3, \mathcal{L}\phi_3 \rangle}{||\phi_3||^2}$$

gives the correct result for the eigenvalue.

(5 marks)