

2)

$$a) \nabla \cdot \vec{E} = \frac{\rho_{\text{density}}}{\epsilon_0} = \frac{3}{\epsilon_0} \quad (\text{enclosed charges only})$$

$$\begin{aligned} \vec{I} &= \iint_{\partial V} \vec{E} \cdot d\vec{S} \stackrel{(\text{Gauss})}{=} \iiint_V (\nabla \cdot \vec{E}) dV = \frac{3}{\epsilon_0} \iiint_V dV = \frac{3}{\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^{\frac{5}{2}} \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta \\ &= \frac{3}{\epsilon_0} \frac{4}{3} \pi \left(\frac{5}{2}\right)^3 = \frac{125\pi}{2\epsilon_0} \quad (\text{sphere formula}) \end{aligned}$$

When the radius is doubled ($\rho=5$ instead of $\rho=\frac{5}{2}$, note that ρ is radius and ρ_{density} is charge density) the new sphere contains two negative charges, so $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}$ and therefore

$$\vec{I} = \frac{1}{\epsilon_0} \frac{4}{3} \pi (5)^3 = \frac{500\pi}{3\epsilon_0}$$

$$b) h = -\frac{1}{2} \frac{d^2}{dy^2} \quad \phi(0) = \phi(b) = 0 \quad y \in [0, b]$$

$$i) h\phi = \lambda\phi \quad \text{has solutions } \phi_n(y) = K \sin(\sqrt{\lambda} y) \quad \phi' = -2\lambda\phi$$

Since characteristic method is used to reveal sine + cosine form,

then $\phi(0)$ implies the function is purely sine

Periodicity and $\phi(b)=0$ implies that $\sqrt{\lambda} b = \pi n \quad n \in \mathbb{Z}$

Since πn are the zeroes of \sin . $\therefore \lambda_n = \frac{\pi^2 n^2}{2b^2} \quad n \in \mathbb{N} - \{0\}$
(λ_0 is trivial, so omit it).

$$\therefore \lambda_3 = \frac{9\pi^2}{2b^2} \quad \text{for } \phi_3(y) = \sin\left(\frac{3\pi y}{b}\right)$$

$$ii) \langle \phi_3, h\phi_3 \rangle = \int_0^b \sin^2\left(\frac{3\pi}{b} y\right) \left(+\frac{3\pi^2}{2b^2}\right) dy = \frac{3\pi^2}{2b^2} \int_0^b \sin^2\left(\frac{3\pi}{b} y\right) dy$$

$$\|\phi_3\|^2 = \langle \phi_3, \phi_3 \rangle = \int_0^b \sin^2\left(\frac{3\pi y}{b}\right) dy$$

$$\therefore \frac{\langle \phi_3, h\phi_3 \rangle}{\|\phi_3\|^2} = \frac{\frac{3\pi^2}{2b^2} \int_0^b \sin^2\left(\frac{3\pi y}{b}\right) dy}{\int_0^b \sin^2\left(\frac{3\pi y}{b}\right) dy} = \frac{3\pi^2}{2b^2} = \lambda_3 \quad \blacksquare$$

Integrals i.e. $\frac{3\pi^2}{2b^2} \int_0^b \sin^2\left(\frac{3\pi y}{b}\right) dy$ cancel out!

Integrals are also non zero since it's norm of eigenfunction.