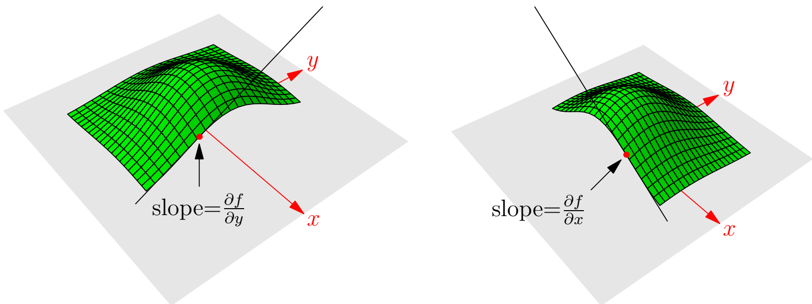
Chapter 2

Partial Differentiation

We will now give a brief overview of the idea of partial differentiation. We consider first functions of two variables, and then generalise the definitions to three and higher dimensions.

First consider a function of two variables $f(x, y)$. This is a *scalar-valued function*, ie. it is a single number which depends on the vector $\langle x, y \rangle$. The function $f(x, y)$ can be visualized as a surface in three-dimensional space, if we write $z = f(x, y)$.



The plane $y =$*constant* intersects the surface $f(x, y)$ at a set of points which form a curve. The slope of this curve is a derivative similar to that for functions of one variable, *but one for which y is kept constant*. The derivative is then called the partial derivative of f *with respect to x* (with the understanding that y is kept constant). It is defined to be

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \quad .$$

Similarly the curve of the plane $x =$*constant* with the surface $z = f(x, y)$ has a slope given by the partial derivative

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \quad .$$

Sometimes we emphasise the *operation* of differentiation by writing the partial derivative in the form $\frac{\partial}{\partial x}(f)$, i.e. the symbol $\frac{\partial}{\partial x}$ represents the process of differentiating f *with respect to* x*, keeping* y *constant* and the result is the new function $\frac{\partial f}{\partial x}$. Although y is kept constant *in this process*, the result is a new function of x and y, which may itself be differentiated with respect to x or y.

Thus we can also define higher derivatives such as $\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial x})$, $\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$, $\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$, and $\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$, with similar definitions for third-order and higher derivatives.

For all the functions in this part of the course it turns out that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad ,$$

so that the order of differentiation is unimportant.

Differentials and the chain rule

Another concept which you will have encountered is that of the *differential* df of $f(x,y)$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \ . \tag{2.1}$$

This quantity can be thought of as 'a small change in the function f'. Strictly speaking it tells us how much the function f changes for each small change dx in the x-coordinate and dy in the y coordinate.

Suppose that we can write our new coordinates (u, v) in terms of our old coordinates (x, y), so that $u = u(x, y)$ and $v = v(x, y)$. Then we can re-write the derivatives with respect to x and y in terms of the derivatives with respect to u and v, via the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \quad . \tag{2.2}$$

Problems

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in the case when $f = xe^{xy} + y\ln(x+y)$.

2. Find all first and second derivatives of the function

$$f(x,y) = x^3 + 5x^2y - 6xy^2 + 4y^3 - 10xy \ .$$

Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ in this case.

3. Verify that the function

$$f(r,\theta) = r^2 \cos 2\theta$$

satisfies

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \quad .$$

This last equation is an important *Partial Differential Equation*, known as Laplace's equation.

4. Given that $r^2 = x^2 + y^2$ and $f(r,\theta) = \frac{1}{r}$, use the chain rule to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

5*. Laplace's equation in two dimensions can be written

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad .$$

Show that in polar coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, and $f = f(r,\theta)$, this equation becomes

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \quad .$$