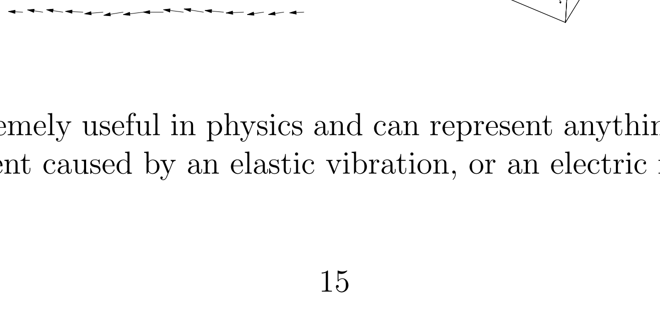
Chapter 3

Div, Grad and Curl

Vector Fields

One interesting quality of vectors is that they have no *position* - a vector by itself can always be pictured as starting at the origin (This is in fact why you are allowed to add them by placing the tip of one against the tail of another). In physics however we often find that a vector quantity (such as force) will vary from place to place. A function which attributes a unique vector to each point in space is called a *vector field*.

I've drawn a couple of vector fields below. Clearly it is not possible to represent every vector in the field, since there is a different vector at every point and so they are packed infinitely densely.



Vector fields are extremely useful in physics and can represent anything from the velocity of fluids, the displacement caused by an elastic vibration, or an electric field.

Differentiation of vector fields

Happily, differentiation of vectors follows the same set of rules which we use differentiation of scalars. If a vector $\mathbf{F}$ depends on some parameter t, such that

$$\mathbf{F}(t) = F_x(t)\hat{\mathbf{i}} + F_y(t)\hat{\mathbf{j}} + F_z(t)\hat{\mathbf{k}}$$

then the derivative with respect to t is

$$\frac{d\mathbf{F}}{dt} = \frac{dF_x}{dt}\hat{\mathbf{i}} + \frac{dF_y}{dt}\hat{\mathbf{j}} + \frac{dF_z}{dt}\hat{\mathbf{k}} \ . \tag{3.1}$$

In this way we can see that vector differentiation inherits all the rules for scalar differentiation. These are:

Addition:
$$\frac{d}{dt}(\mathbf{F}+\mathbf{G}) = \frac{d\mathbf{F}}{dt} + \frac{d\mathbf{G}}{dt} \tag{3.2}$$

The chain rule:
$$\frac{d}{dt}(\mathbf{F}(u(t))) = \frac{d\mathbf{F}}{dt}\frac{du}{dt} \tag{3.3}$$

Multiplication by a constant:
$$\frac{d}{dt}(c\mathbf{F}) = c\frac{d\mathbf{F}}{dt} \tag{3.4}$$

$$\frac{d}{dt}(u\mathbf{F}) = \frac{du}{dt}\mathbf{F} + u\frac{d\mathbf{F}}{dt}$$

Product rules:
$$\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G}) = \frac{d\mathbf{F}}{dt}\cdot\mathbf{G} + \mathbf{F}\cdot\frac{d\mathbf{G}}{dt} \tag{3.5}$$

$$\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \frac{d\mathbf{F}}{dt}\times\mathbf{G} + \mathbf{F}\times\frac{d\mathbf{G}}{dt}$$

Note that in the product rules we have specified the extra rules for using the dot $(\cdot)$ and cross $(\times)$ products.

The gradient

Differentiation describes the rate of change of a function with respect to a given parameter. If this parameter is one of the coordinates x, y, or z, then the partial derivative $\partial f(x,y,z)/\partial x$ denotes the *slope* of the function, as measured parallel to the x axis. That is, $\partial f(x,y,z)/\partial x$ measures the rate of change of the function f in the x direction.

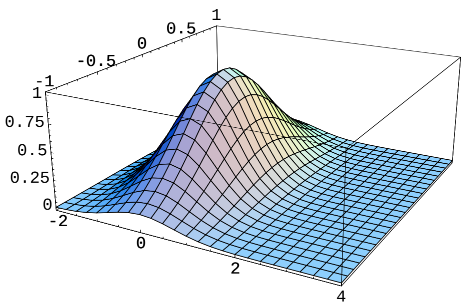
We will now introduce a generalization of differentiation to three dimensions. Because the function $f(x,y,z)$ depends on each of three coordinates, we require a vector field to describe the slope of f in each of the three coordinate directions. The simplest way to contain all this information is to put all the partial derivatives of the function into a vector. This vector field is known as the *gradient* of the function.

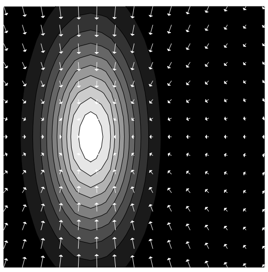
Given a scalar function $f(x,y,z)$, we define the gradient of f to be the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right). \tag{3.6}$$

The gradient has the following properties:

- ∇f is a vector field
- ∇f measures the rate of increase of the scalar function f in each of the three coordinate directions
- ∇f points in the direction in which f increases the most.





To illustrate the last point, we consider the (two-dimensional) scalar function above, which has been represented both as a surface (left) and as a contour graph (right). Over the top of the contour function have been drawn the vectors representing the scalar field ∇f. Each vector points in the direction in which the function f increases the most, and the length of each vector is proportional to the 'steepness' of the slope.

Problems

1. Given a scalar field

$$f(\mathbf{r}) = r^n \quad ,$$

where $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = |\mathbf{r}|$, show that $\nabla f = nr^{n-2}\mathbf{r}$.

2. Find the gradient of the field

$$f(x, y, z) = \frac{q}{\sqrt{x^2 + y^2 + z^2}} \ ,$$

where q is a constant.

The operator ∇

The gradient ∇f has the form of a vector multiplying a scalar function:

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f \quad .$$

The object in parentheses is not in fact a vector, although behaves very much like one. Instead it is an example of what is known as a *vector operator*, and we can re-write it as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad , \tag{3.7}$$

or, equivalently,

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{3.8}$$

An operator is something which operates on a function to produce another function. In this case ∇ is perhaps best looked at as an *instruction to differentiate* a function in each of the three coordinate directions.

As with an ordinary vector, we can use ∇ to *multiply* in three different ways:

1. Multiplication by a scalar function f: the result is the vector ∇f (the gradient).
2. 'Dot' product with a vector function $\mathbf{v}$: the result is a scalar $\nabla \cdot \mathbf{v}$, known as the *divergence*
3. 'Cross' product with a vector function $\mathbf{v}$: the result is a vector $\nabla \times \mathbf{v}$, known as the *curl*

The Divergence of a vector field

For a vector field $\mathbf{v} = (v_x, v_y, v_z)$, the *divergence* of the field at any point is defined by the number

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (v_x, v_y, v_z) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad . \end{aligned} \tag{3.9}$$

Note that this is clearly a *scalar* quantity. You will often see the divergence written

$$\mathrm{div}\ \mathbf{v} = \nabla \cdot \mathbf{v} \quad .$$

The divergence is essentially a measure of the expansion of a vector field at any given point. That is, it gives the amount that the vector field *spreads out* (or diverges) from a particular point.

Example:

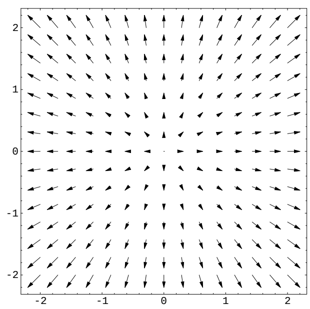
Consider the vector function

$$\mathbf{v}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The divergence of this function is

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}) \cdot (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) \\
&= (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}) \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3 \quad .
\end{aligned}$$

Thus the function has a positive divergence, or 'spreading out' at every point, as is illustrated



by the figure above.

The curl of a vector field

From the definition of the vector, or 'cross' product, we have:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \tag{3.10}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{\mathbf{j}}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \quad . \tag{3.11}$$

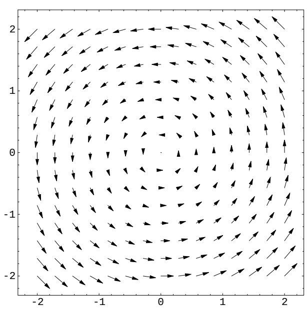
The curl is a measure of how much (and in which direction) a vector field 'curls about' itself. Vector functions which twist dramatically typically have large curls in the vicinity of the twisting, whereas smooth, laminar functions tend to have small values of curl. To illustrate:

Example (i)

Consider the vector function

$$\mathbf{v}(x, y, z) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The curl of this function is



$$\begin{aligned}
\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 1 \end{vmatrix} \\
&= \hat{\mathbf{i}}\left(\frac{\partial(1)}{\partial y} - \frac{\partial(x)}{\partial z}\right) - \hat{\mathbf{j}}\left(\frac{\partial(1)}{\partial x} - \frac{\partial(-y)}{\partial z}\right) \\
&\quad + \hat{\mathbf{k}}\left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right) \\
&= \hat{\mathbf{i}}(0-0) + \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}(1+1) = 2\hat{\mathbf{k}} \quad .
\end{aligned}$$

Example (ii)

By way of contrast, consider the vector function we looked at previously

$$\mathbf{v}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The curl of this function is

$$\begin{aligned}
\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \hat{\mathbf{i}}\left(\frac{\partial(z)}{\partial y} - \frac{\partial(y)}{\partial z}\right) - \hat{\mathbf{j}}\left(\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z}\right) \\
&\quad + \hat{\mathbf{k}}\left(\frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y}\right) \\
&= \hat{\mathbf{i}}(0-0) + \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}(0-0) = \mathbf{0} \quad .
\end{aligned}$$

This function has zero curl. Vector fields of this type are known as *irrotational*.

Problems

1. Find the gradients of the following functions:
(a) $f(x, y, z) = 2x^2 + y^2 + 3z^3$
(b) $f(x, y, z) = x^2 y^3 z^4$
(c) $f(x, y, z) = e^z \sin x \ln(y)$

2. Find (i) the divergence, and (ii) the curl of the following vector functions:
(a) $\mathbf{v} = x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + 3z^2\hat{\mathbf{k}}$
(b) $\mathbf{v} = xy\hat{\mathbf{i}} + 2yz\hat{\mathbf{j}} + 3zx\hat{\mathbf{k}}$

3. Calculate the divergence of the function

$$\mathbf{v} = \frac{1}{r}\mathbf{r} \quad ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $r = |\mathbf{r}|$.

Second derivatives

We can form more complicated operations by combining the different permutations of gradient, divergence and curl. If ϕ is a scalar field, and $\mathbf{v} = (v_x, v_y, v_z)$ is a vector field, we obtain:

1. The divergence of a gradient:

$$\begin{aligned} \nabla \cdot (\nabla \phi) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \end{aligned}$$

This operation is known as *the Laplacian* of ϕ, and is usually written

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^\phi}{\partial z^2} \quad . \tag{3.12}$$

This differential operation is one of the most widely used in the entire field of physics. Important examples are:

$$\begin{aligned} \nabla^2 \phi = 0 &\qquad \text{Laplace's equation} \\ \nabla^2 \theta = \frac{1}{\kappa} \frac{\partial \theta}{\partial t} &\qquad \text{Heat conduction, or `diffusion' equation} \\ \nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} &\qquad \text{Wave equation} \end{aligned}$$

Note that the Laplacian is a *scalar* quantity.

2. The curl of a gradient:

$$\begin{aligned} \nabla \times (\nabla \phi) &= \hat{\mathbf{i}} \left(\frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y} \right) - \hat{\mathbf{j}} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} \right) \\ &= \hat{\mathbf{i}} \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) - \hat{\mathbf{j}} \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} = \mathbf{0} \end{aligned} \tag{3.13}$$

So this is in general true: *the curl of a gradient is always zero.* Vector fields for which the curl is zero are called *irrotational.*

3. The gradient of a divergence $\nabla(\nabla \cdot \mathbf{v})$:

This second derivative has no special name, but appears in the equations of elasticity.

4. The divergence of a curl:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0 \quad . \tag{3.14}$$

The proof of this identity is along the same lines as that for calculating the curl of a gradient, given above. Vector fields which have zero divergence everywhere are called *solenoidal*.

5. The curl of a curl:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \tag{3.15}$$

The proof of this identity is left as an exercise. Note that the Laplacian operator ∇^2 operates on each of the three components of the vector $\mathbf{v}$ separately. The curl of a curl is a *vector* quantity.

Example (i)

Calculate the Laplacian of the function $f = x^2 + 2xy^2 + 4z^3$.

Solution:

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2 + 4x + 24z \end{aligned}$$

Example (ii)

Given $r = \sqrt{x^2 + y^2 + z^2}$, show that the scalar function

$$\phi = \frac{1}{r}$$

obeys Laplace's equation, given above.

Solution:

$$
\begin{aligned}
\frac{\partial^2 (r^{-1})}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{x}{(x^2+y^2+z^2)^{1/2}}\right) \\
&= -x\left(\frac{-1}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{3/2}} - \frac{1}{(x^2+y^2+z^2)^{1/2}} \\
&= \frac{x^2}{r^3} - \frac{1}{r}
\end{aligned}
$$

Similarly,

$$\frac{\partial^2 (r^{-1})}{\partial y^2} = \frac{y^2}{r^3} - \frac{1}{r} \quad ,$$

and

$$\frac{\partial^2 (r^{-1})}{\partial z^2} = \frac{z^2}{r^3} - \frac{1}{r} \quad .$$

Hence,

$$
\begin{aligned}
\nabla^2 \phi &= \frac{\partial^2 (r^{-1})}{\partial x^2} + \frac{\partial^2 (r^{-1})}{\partial y^2} + \frac{\partial^2 (r^{-1})}{\partial z^2} \\
&= \frac{x^2+y^2+z^2}{r^3} - \frac{1}{r} = \frac{r^2}{r^2} - \frac{1}{r} = 0 \quad .
\end{aligned}
$$

Problems

1. Calculate the Laplacian ($\nabla^2\phi$) of the following functions:
(a) $\phi = x^2 + (y+1)^2 + xz^2$
(b) $\phi = \cos x \cos y \cos z$

2*. Prove the identity (3.14), i.e.

$$\nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{0} \ .$$

3*. From the definition of ∇, prove the product identity

$$\nabla\left(\phi\psi\right) = \psi\nabla\phi + \phi\nabla\psi \quad .$$

given that ϕ and ψ are both scalar fields.