Chapter 4

Line Integrals

Paths and Line Integrals

A path in three dimensional space is usually defined in terms of a *parameter*, which is a variable which changes as we move along the path. If we denote the parameter by the real number t, then any path can be represented by the vector

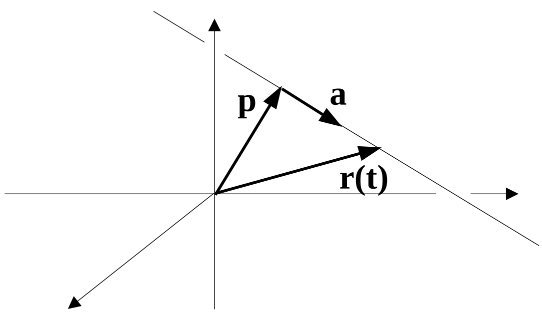
$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \tag{4.1}$$

where

$$t \in [t_0, t_1]$$

The vector equation for a straight line, passing through the point with position vector $\mathbf{p}$ and parallel to the vector $\mathbf{a}$ can be written

$$\mathbf{r} = \mathbf{p} + t\mathbf{a} \quad . \tag{4.2}$$



Motion along a parametrised path

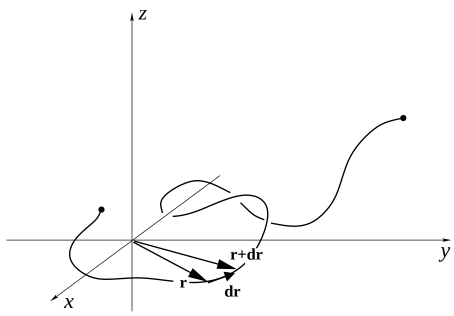
The value of a scalar field ϕ may well depend on the position of the path-vector $\mathbf{r}(t)$. As we move along the path and $\mathbf{r}(t)$ changes, the value of ϕ will also change. Calculus is all about adding up these little changes, and so before we proceed, we need to find a way of characterising the motion along the path if the distance moved is very small.

If we move a short distance along the path $\mathbf{r}(t)$, then we say that we have moved along the vector $d\mathbf{r}$, where

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{4.3}$$

$$= \left(\frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}\right) dt \tag{4.4}$$

The *infinitesimal path length* we denote by ds , and this is equal to



$$ds = |dr| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \tag{4.5}$$

Consequently, in terms of the path parameter t,

$$ds = \frac{ds}{dt}dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{4.6}$$

Note that this last expression is equivalent to

$$ds = \sqrt{\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}}\, dt \tag{4.7}$$

because $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

This quantity under the square-root sign (equal to ds/dt) tells us exactly how far along the path we move if we change the parameter t by a small amount. So if ds/dt is large, then the vector $\mathbf{r}(t)$ will move a long way along the path for a small change in t. Similarly if ds/dt is very small, then $\mathbf{r}(t)$ will stay in more or less the same spot even for large changes in the parameter t.

4.0.1 The tangent to a curve

We can see from the previous figure that the infinitesimal displacement vector $d\mathbf{r}$ is parallel to the derivative $d\mathbf{r}/dt$. This means that

$$\frac{d\mathbf{r}}{dt} \quad \text{is a } \textit{tangent vector} \text{ to } \mathbf{r}(\mathrm{t}) \ .$$

The unit tangent vector to the curves is obtain simply by normalising this vector (i.e. making sure that the length is equal to 1):

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \ , \tag{4.8}$$

provided $\|\mathbf{r}(t)\| \neq 0$. The rate of change of the tangent to the curve with respect to the path length $s(t)$ is known as the *curvature*, defined as

$$\kappa = \|\frac{d\mathbf{T}}{ds}\| \ . \tag{4.9}$$

Differentiating both sides of (4.8) with respect to t gives

$$\frac{d\mathbf{T}}{dt} \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0 \quad \Longrightarrow \quad 2\mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0 \quad \Longrightarrow \quad \mathbf{T} \perp \frac{d\mathbf{T}}{dt}$$

This last statement says that the vector $d\mathbf{T}/dt$ is normal to the tangent vector $\mathbf{T}$. We can thus define the *principal unit normal vector to the curve* as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} . \tag{4.10}$$

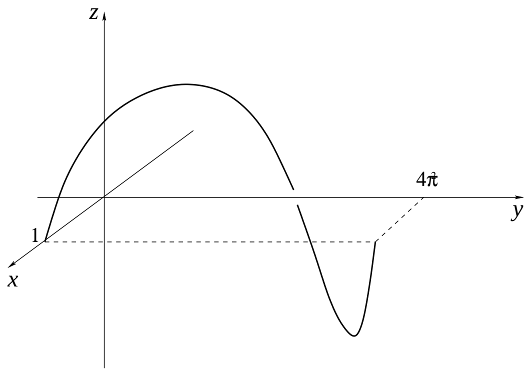
Scalar-type integrals

Scalar-type line integrals occur when we want to integrate the value of a scalar field as it changes along some path. Suppose ϕ is a scalar function which is defined along a path $\mathcal{C}$ and this path can be represented by a parametrised coordinate vector $\mathbf{r}(t)$, as outlined above. Then we define the line integral of ϕ along $\mathcal{C}$ as

$$\int_{\mathcal{C}} \phi(x,y,z) ds = \int_{t_0}^{t_1} \phi(t) \frac{ds}{dt} dt \ . \tag{4.11}$$

where

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \ . \tag{4.12}$$



Example:

Suppose we want to find the integral of the scalar function $\phi(x, y, z) = x^2 + y^2 + z^2$ over the spiral path $\mathcal{C}$ parametrised by the equations

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= t \\ z(t) &= \sin t \end{aligned}$$

where

$$0 \leq t \leq 2\pi \ .$$

First we express the scalar function ϕ in terms of the parameter t. This is:

$$\begin{aligned} \phi(t) &= x(t)^2 + y(t)^2 + z(t)^2 \\ &= (\cos t)^2 + (\sin t)^2 + t^2 = 1 + t^2 \quad . \end{aligned}$$

Next we note that

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \\ &= \sqrt{1+1} = \sqrt{2} \quad . \end{aligned}$$

Substituting into the definition of the scalar line integral (4.11), we obtain

$$\begin{aligned} \int_{\mathcal{C}} \phi ds &= \int_{t_0}^{t_1} \phi(t) \frac{ds}{dt} dt \\ &= \int_0^{2\pi} \left(1 + t^2\right) \sqrt{2} dt = \left[\sqrt{2}(t + \frac{t^3}{3})\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi \left(1 + \frac{(2\pi)^2}{3}\right) \quad . \end{aligned}$$

Path lengths

In order to calculate the *length* of a path $\mathcal{C}$ in three-dimensional space, we integrate over all the infinitesimal path-lengths ds:

$$\begin{aligned} Length &= \int_{\mathcal{C}} ds \\ &= \int_{t_0}^{t_1} \frac{ds}{dt} dt \quad . \end{aligned} \tag{4.13}$$

This is the same as integrating the scalar function $\phi(x, y, z) = 1$ over the path $\mathcal{C}$.

Example

Calculate the length of the path used in the previous example.

Solution:

$$\begin{aligned} Length &= \int_0^{2\pi} \frac{ds}{dt} dt \\ &= \int_0^{2\pi} \sqrt{2} dt = \sqrt{2} \, [t]_0^{2\pi} \\ &= 2\sqrt{2}\pi \quad . \end{aligned}$$

Vector-type line integrals

Suppose $\mathbf{F}$ is a vector field which is defined along a path $\mathcal{C}$. The line integral of $\mathbf{F}$ over this path written

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ . \tag{4.14}$$

where $d\mathbf{r}$ is the infinitesimal displacement vector along the path. Substituting

$$\mathbf{F} = F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}} + F_z \hat{\mathbf{k}} \ , \ \ d\mathbf{r} = dx \hat{\mathbf{i}} + dy \hat{\mathbf{j}} + dz \hat{\mathbf{k}}$$

we find that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} F_x dx + \int_{\mathcal{C}} F_y dy + \int_{\mathcal{C}} F_z dz \ . \tag{4.15}$$

If the path is described by the parameter vector t, i.e.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

where $t_0 \leq t \leq t_1$, then the integral of $\mathbf{F}$ over the path can be calculated as

$$\int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_{t_0}^{t_1} F(t) \cdot \frac{d\mathbf{r}}{dt} dt \quad , \tag{4.16}$$

where

$$\frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \right) \quad . \tag{4.17}$$

If the vector field $\mathbf{F}$ represents a force acting on some object, then the integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

is equal to the *work done* by the force in moving the object from the beginning to the end of the path.

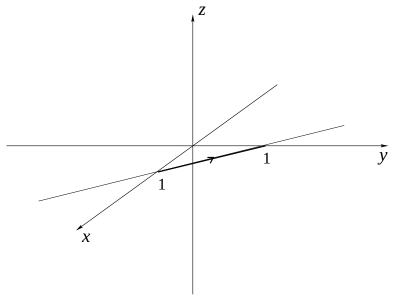
Example 1:

NB: This example is entirely done in two dimensions. For this reason I have simply omitted all the dependencies on z and $\hat{\mathbf{k}}$.

Find the work done by the vector field $\mathbf{F}(x, y) = y\hat{\mathbf{i}} + 2x\hat{\mathbf{j}}$ in moving an object along the path $\mathcal{C}$, which joins the points $(1, 0)$ and $(0, 1)$.

Solution:
First note that the path can be parameterised by

$$\begin{aligned} x &= t \\ y &= 1 - t \end{aligned}$$



where

$$0 \leq t \leq 1 \quad .$$

Now,

$$\mathbf{F}(t) = y\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} = (1-t)\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t)\hat{\mathbf{i}} + \frac{d}{dt}(1-t)\hat{\mathbf{j}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} \quad .$$

So,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_0^1 ((1-t)\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}}) dt \\
&= \int_0^1 [(1-t) - 2t]\, dt = \int_0^1 (1-3t) dt \\
&= \left[t - \frac{3}{2}t^2\right]_1^0 \\
&= \frac{1}{2} \quad .
\end{aligned}$$

Note that because the path begins at $t = 1$ and ends at $t = 0$ the limits on integral have been swapped. The lower limit should *always* correspond to the initial point of the path, whereas the upper limit should always correspond to the endpoint.

Example 2

: Find the line integral of

$$\mathbf{F} = 12x^2\hat{\mathbf{i}} - 5xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}$$

over the path $\mathcal{C}$ defined by

$$y = x^2 \quad , \quad z = x^3$$

from the point $(0, 0, 0)$ to $(2, 4, 8)$.

<u>Solution:</u>
First of all use the parametrisation

$$\begin{aligned} x &= t \\ y &= t^2 \\ z &= t^3 \end{aligned}$$

with

$$0 \leq t \leq 2 \quad .$$

Then

$$\mathbf{F}(t) = 12t^2\hat{\mathbf{i}} - 5t^3 = jh + t^4\hat{\mathbf{k}} \quad ,$$

also

$$\mathbf{r}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$$

so

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}} \quad .$$

The integral is then

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 (12t^2\hat{\mathbf{i}} - 5t^3\hat{\mathbf{j}} + t^4\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})dt \\ &= \left[4t^3 - 2t^5 + \frac{3}{7}t^7\right]_0^2 = \frac{160}{7} \quad . \end{aligned}$$

The fundamental theorem of calculus in 3D

Suppose $f(x)$ is a function of a single variable. The the fundamental theorem of calculus states that

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a) \quad . \tag{4.18}$$

That is, the integral of the derivative df/dx depends only on the value of f at the endpoints a and b. We will now see that there is a generalisation of this theorem to functions of three (or more) variables.

Suppose $\phi(x, y, z)$ is a continuously differentiable scalar field on an open set which contains a smooth curve $\mathcal{C}$. Then

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \\ &= \nabla \phi \cdot \frac{d\mathbf{r}}{dt}\end{aligned}$$

where $\nabla\phi$ is the *gradient*, defined in formula (3.6), and $\mathbf{r} = (x, y, z)$ is a point on the path $\mathcal{C}$.

We can then calculate the line integral of the vector field $\nabla\phi$ along the path $\mathcal{C}$:

$$\begin{aligned}\int_{\mathcal{C}} (\nabla\phi) \cdot d\mathbf{r} &= \int_{t_0}^{t_1} \nabla\phi \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_{t_0}^{t_1} \frac{d\phi}{dt} dt \\ &= \phi(t_1) - \phi(t_0) \quad .\end{aligned}$$

That is, if the path stretches from point A to point B, then

$$\int_A^B \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A) \quad . \tag{4.19}$$

This equation is the generalisation of equation (4.18) to three dimensions, and states that the line integral of a gradient of a function depends only on the value of that function at the endpoints. Critically, the right-hand side of this equation does not depend on the actual path taken, only on where the path begins and ends. This means that *the integral* $\int_A^B \nabla\phi \cdot d\mathbf{r}$ *is independent of the path taken from A to B.*

Example:

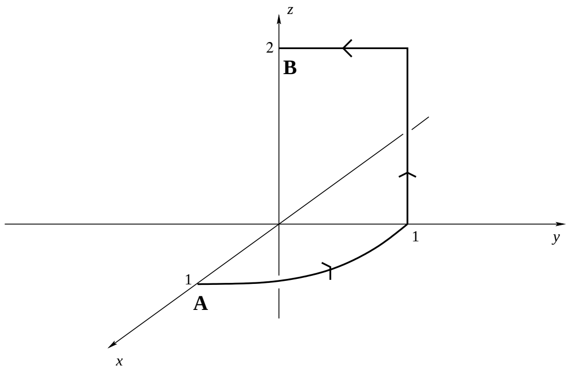
Consider the function

$$\phi = \frac{1}{r} \quad .$$

Calculate the line integral

$$I = \int_{\mathcal{C}} \nabla\phi \cdot d\mathbf{r}$$

over the path C shown in the figure below.



Solution:
From equation (4.19), we know that

$$\begin{aligned}\int_A^B \nabla\phi \cdot d\mathbf{r} &= \phi(B) - \phi(A) \\ &= \phi(0,0,2) - \phi(1,0,0) \\ &= \frac{1}{2} - \frac{1}{1} \\ &= -\frac{1}{2} \ .\end{aligned}$$

Conservative fields

If the path we choose begins and ends at the same point, then the line integral is usually written

$$\oint \mathbf{F} \cdot d\mathbf{r}$$

with the circle around the integral sign to denote the fact that the path is *closed*.

If the vector function $\mathbf{F}$ is the gradient of some scalar function ϕ, i.e.

$$\mathbf{F} = \nabla\phi$$

Then it is a corollary of (4.19) that

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint \nabla\phi \cdot d\mathbf{r} = 0 \quad . \tag{4.20}$$

Field of this type are called *conservative* , for the reason that if $\mathbf{F} = \nabla\phi$ represents a force, then the work done by the force in moving an object around a closed path is exactly zero.

The scalar field ϕ is known as the *potential* function of $\mathbf{F}$. Electrostatic and gravitational fields are of this type. Forces such as friction or air resistance however are not be conservative, since the work done by these forces in moving an object around a closed path is always positive, energy is always 'lost' from the system, in the form of, say, heat or pressure waves.

The test to tell if a field is conservative or not is:

$$\mathbf{F} = \nabla\phi \quad \text{if and only if} \quad \nabla \times \mathbf{F} = \mathbf{0} \ . \tag{4.21}$$

Clearly the "from left to right" argument is true - if $\mathbf{F}$ can be written $\mathbf{F} = \nabla\phi$ then from (3.13) it follows that $\nabla \times \mathbf{F} = \mathbf{0}$. To prove that if $\nabla \times \mathbf{F} = \mathbf{0}$ means that we can *always* find a potential function ϕ is more complicated and is left as an exercise.

To summarise, the following statements are equivalent:

- A vector field $\mathbf{F}$ can be written as the gradient of a scalar field $\mathbf{F} = \nabla\phi$
- $\mathbf{F}$ is irrotational, i.e.
$$\nabla \times \mathbf{F} = 0 \quad .$$
- The line integral of $\mathbf{F}$ depends only on the values of ϕ at the endpoints, i.e.
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$$
- $\mathbf{F}$ is conservative, i.e.
$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad .$$

Note that in the last equation we didn't need to specify the path which we are integrating over. This is because the line integral is zero for *any* path within the domain where $\mathbf{F}$ is defined.

Finding the potential

Given an irrotational vector function $\mathbf{F}$ we can always construct a potential using successive integrations. First we write the field $\mathbf{F}$ in terms of its components

$$\mathbf{F} = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

and note that if a potential function ϕ exists, then

$$\frac{\partial \phi}{\partial x} = P(x,y,z) \ , \quad \frac{\partial \phi}{\partial y} = Q(x,y,z) \ , \quad \frac{\partial \phi}{\partial z} = R(x,y,z) \ .$$

We then take one of these equations (usually the most complicated one!) and integrate it partially with respect to the appropriate variable, holding the other variables constant. For example,

$$\phi(x,y,z) = \int^x P(x',y,z)dx' + C(y,z)$$

Because we have integrated with respect to the x coordinate, the "constant of integration" is a function of y and z. We then differentiate with respect to the other variables in order to recover Q and R.

Example: Find a potential function for

$$\mathbf{F} = (y^2+z^3)\hat{\mathbf{i}} + 2y(x+z^3)\hat{\mathbf{j}} + (x+y^2)\hat{\mathbf{k}} \ .$$

Solution:
We first write

$$\nabla\phi = \mathbf{F} \ ,$$

so that

$$\frac{\partial \phi}{\partial x} = y^2+z^3 \ , \quad \frac{\partial \phi}{\partial y} = 2y(x+z^3) \ , \quad \frac{\partial \phi}{\partial z} = 3z^2(x+y^2) \ .$$

Starting with the third equation, we integrate with respect to z, keeping x and y constant:

$$\begin{aligned} \phi &= \int 3z^2(x+y^2)dz + C(x,y) \\ &= z^3(x+y^2) + C(x,y) \end{aligned}$$

Differentiating with respect to x,

$$\frac{\partial \phi}{\partial x} = z^3 + 0 + \frac{\partial C}{\partial x}$$

But $\frac{\partial \phi}{\partial x} = y^2+z^3$, so

$$\frac{\partial C}{\partial x} = y^2 \qquad \Longrightarrow \ C(x,y) = y^2x + D(y)$$

where D is a function of y. So we have discovered

$$\phi(x,y,z) = z^3(x+y^2) + y^2x + D(y) \ .$$

Differentiating w.r.t. y, we find

$$\frac{\partial \phi}{\partial y} = 2z^3 y + 2xy + \frac{dD}{y}$$

but we already know that $\frac{\partial \phi}{\partial y} = 2y(x + z^3)$, so

$$2z^3 y + 2xy + \frac{dD}{dy} = 2xy + 2yz^3 \qquad \Longrightarrow \frac{dD}{dy} = 0 \ ,$$

therefore D is a constant. Thus the potential is

$$\phi(x, y, z) = xz^3 + y^2 z^3 + xy^2 + \mathrm{C}onstant \ .$$

Problems

1. A path $\mathcal{C}$ in 3D space is parameterised by the equations

$$\begin{aligned} x &= 2\cos t \\ y &= 1 \\ z &= 2\sin t \ , \end{aligned}$$

where t starts at 0 and ends at 2π.

Sketch $\mathcal{C}$. Evaluate the integral

$$\int_{\mathcal{C}} (x^2 + yz) ds \ .$$

[Hint: To do the integral in the last part you may find it helpful to note that $\cos^2 t = \frac{1}{2}(\cos(2t) + 1)$.]

2. Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

when $\mathbf{F} = xy\hat{\mathbf{i}} + (x - y)\hat{\mathbf{j}}$ and $\mathcal{C}$ is the triangle joining the points $(0, 1, 0)$, $(1, -1, 0)$ and $(-1, -1, 0)$, traversed in a clockwise direction.

3. Show that the vector field

$$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + (xy + 2z)\hat{\mathbf{k}}$$

is irrotational. Hence find a function ϕ such that $\mathbf{F} = \nabla\phi$ and evaluate the integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is the line segment from $\langle -1, 0, 1\rangle$ to $\langle 2, 3, 2\rangle$.

4. Show that only one of the vector fields

$$\mathbf{F}_1 = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}\ , \qquad \mathbf{F}_2 = (x^2+y^2)\hat{\mathbf{i}} + zy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}$$

can be expressed as the gradient of a scalar field, and find that scalar field.