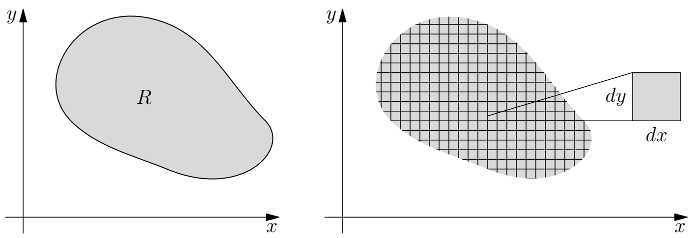
Chapter 5

Area, Volume and Surface Integrals

Double Integrals

In many "real-world" applications we are required to calculate the integrals of functions over two variables. Consider a function of two variables $f(x,y)$ which is defined within a region

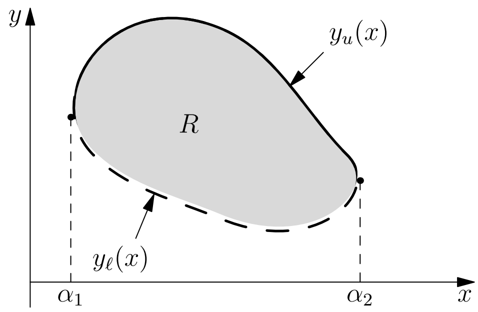


R of the x-y plane. We can divide the region R into a gridwork of tiny pieces all of length Δx and height Δy. The area of each small rectangle is $\Delta A = \Delta x \Delta y$. We can define the double integral of $f(x,y)$ over R to be

$$\iint_R f(x,y)dA = \lim_{\Delta x, \Delta y \to 0} \sum_{(\text{all } \Delta A \text{ in } R)} f(x,y)\Delta A \quad . \tag{5.1}$$

In order to calculate this, we need to decide on the order in which to perform the double integral. First we note that the region R lies entirely with the two curves which we can call $y_u(x)$ and $y_l(x)$, and the the range of x is bounded by the number α_1 and α_2. Then we can say that

$$\iint_R f(x,y)dA = \int_{\alpha_1}^{\alpha_2} \left[\int_{y_l(x)}^{y_u(x)} f(x,y)dy \right] dx \quad . \tag{5.2}$$

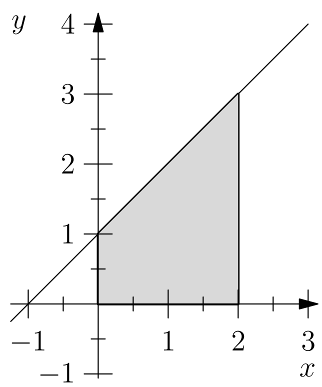


We can do the inner integral first, this leaves us with a function of x which we can then integrate over the range $[\alpha_1, \alpha_2]$.

Example: Find the integral of the function $f(x, y) = 2x^2y - 1$ over the area bounded by the functions

$$\begin{aligned} y_u(x) &= x + 1 \\ y_l(x) &= 0 \end{aligned}$$

and lying between the lines $x = 0$ and $x = 2$.



<u>Solution</u>:

$$\begin{aligned}
I &= \iint_R f(x,y)dA \\
&= \int_0^2 \left[\int_0^{x+1} (2x^2y-1)dy\right] dx \\
&= \int_0^2 \left[x^2y^2 - y\right]_0^{x+1} dx \\
&= \int_0^2 \big(x^2(x+1)^2 - (x+1)\big)\, dx \\
&= \int_0^2 \big(x^4 + 2x^3 + x^2 - x - 1\big)\, dx \\
&= \left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x^2}{2} - x\right]_0^2 = \frac{196}{15}
\end{aligned}$$

Finding the area of a region

In order to find the area of a region R we calculate the double integral of the function $f(x,y) = 1$ over R. i.e.

$$Area_R = \iint_R dA \tag{5.3}$$

This makes sense because in calculating this integral you are just 'adding up' all the small areas dA within the region R, the sum of which has to give you the total area.

We can use the area to define the *average* of a function over a region. For a continuous function of two variables $f(x,y)$ defined on a region R, the average $\bar{f}$ is defined to be

$$\bar{f} = \frac{1}{Area_R} \iint_R f(x,y)dxdy\ . \tag{5.4}$$

This formula can also be used for vectors; by using $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ we can find the average position vector (a.k.a the centroid) $\bar{\mathbf{r}}$:

$$\bar{\mathbf{r}} = \frac{1}{Area_R} \iint_R \mathbf{r}dxdy\ . \tag{5.5}$$

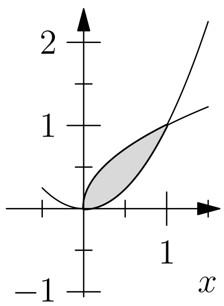
Example: Find the area between the curves

$$y = x^2$$

and

$$y = x^{1/2}$$

in the range $0 \le x \le 1$.

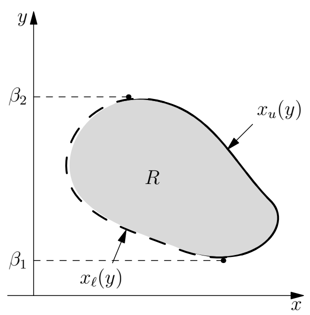


Solution:

$$\begin{aligned} Area &= \iint_R dA \\ &= \int_0^1 \int_{x^2}^{x^{1/2}} 1.dydx \\ &= \int_0^1 \left(x^{1/2} - x^2\right) dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3} \end{aligned}$$

Reversing the order of integration

There is no reason why we have to do the y integration first followed by the integration over the x coordinate.



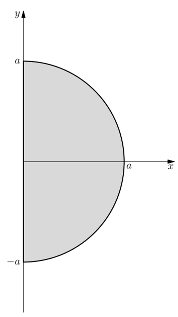
Referring to the diagram above, we note that the region R is bound by the curves $x = x_l(y)$ and $x = x_u(y)$, and extends over the range $\beta_1 \leq y \leq \beta_2$. We can calculate the integral of the function $f(x, y)$ over the region R as

$$\iint_R f(x,y)dA = \int_{\beta_1}^{\beta_2} \left[\int_{x_l(y)}^{x_u(y)} f(x,y)dx \right] dy \quad . \tag{5.6}$$

As with the previous definition (5.2), we do the inner integral first. This leaves us with a function in y which we integrate over the range $[\beta_1, \beta_2]$.

Example:
Integrate the function $f(x, y) = xy^2$ over the interior of the semicircle

$$x^2 + y^2 \leq a^2 \qquad , \qquad x \geq 0 \quad .$$



Solution: The region is bound by the curves

$$x_u(y) = \sqrt{a^2 - y^2}$$

and

$$x_l(y) = 0 \quad .$$

The integral is then calculated:

$$
\begin{aligned}
I &= \int_{-a}^{+a} \int_{0}^{\sqrt{a^2-y^2}} dx dy \\
&= \int_{-a}^{+a} \left[\frac{1}{2} x^2 y^2\right]_0^{\sqrt{a^2-y^2}} dy \\
&= \int_{-a}^{+a} \frac{1}{2} y^2 (a^2 - y^2) dy \\
&= \left[\frac{1}{6} a^2 y^3 - \frac{1}{10} y^5\right]_{-a}^{+a} \\
&= \frac{2}{15} a^5 \quad .
\end{aligned}
$$

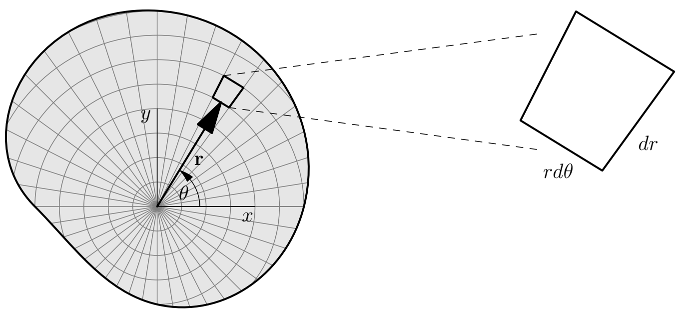
The same answer can be obtained by doing the y integration first, as in the previous section.

Curvilinear coordinates

Many double integrals are made much easier if we use a non-Cartesian coordinate system, such as plane polar coordinates. In this system we replace x and y by the variables r and θ, by making the substitution

$$x = r\cos\theta \tag{5.7}$$
$$y = r\sin\theta \quad . \tag{5.8}$$

If we examine a small element of area in polar coordinates, we see that



$$\Delta A \ \ (= \Delta x \Delta y) \ \ = r \Delta\theta \Delta r \quad . \tag{5.9}$$

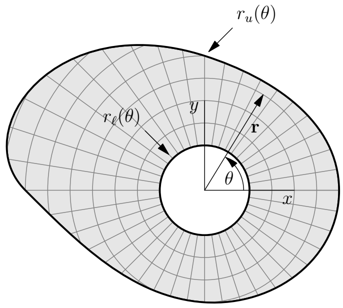
As we take the limit $\Delta A \to 0$, we can replace our 'small elements' by differentials, and so

$$dA = r d\theta dr \quad . \tag{5.10}$$

Hence the double integral of a function f over a region R is

$$\iint_R f(x,y) dA = \iint_R f(r,\theta) r dr d\theta \quad . \tag{5.11}$$

As before, we must do both integrals in the double integral in order. To do this, we write



the region we want to integrate over as being bounded by the curves $r_l(\theta)$ and $r_u(\theta)$, and lying within the range $\theta_0 \leq \theta \leq \theta_1$. Thus,

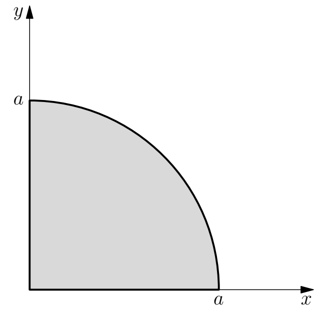
$$\iint_R f(x,y) dA = \int_{\theta_0}^{\theta_1} \left[\int_{r_l(\theta)}^{r_u(\theta)} f(r,\theta) r dr \right] d\theta \tag{5.12}$$

Example:
Integrate the function $f(x,y) = xy$ over the region R, which is the quarter circle

$$x^2 + y^2 \leq a^2 \quad , x \geq 0, \quad y \geq 0 \quad .$$

<u>Solution</u>:
The region R is bound by the curves

$$\begin{aligned} r_u(\theta) &= a \\ r_l(\theta) &= 0 \end{aligned}$$



and lies within the range $0 \leq \theta \leq \pi/2$. We also place the function f into polar coordinates:

$$\begin{aligned} f(r,\theta) &= xy = (r\cos\theta)(r\sin\theta) \\ &= r^2\cos\theta\sin\theta \quad . \end{aligned}$$

The integral is then

$$\begin{aligned} I &= \iint_R f(r,\theta) r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^a r^2 \cos\theta \sin\theta r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[\int_0^a r^3 dr\right] \cos\theta\sin\theta d\theta \\ &= \left[\frac{r^4}{4}\right]_0^a \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta \\ &= \left(\frac{a^4}{4}\right)\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{a^4}{4}\right)\left(\frac{1}{2} - 0\right) \\ &= \frac{a^4}{8} \quad . \end{aligned}$$

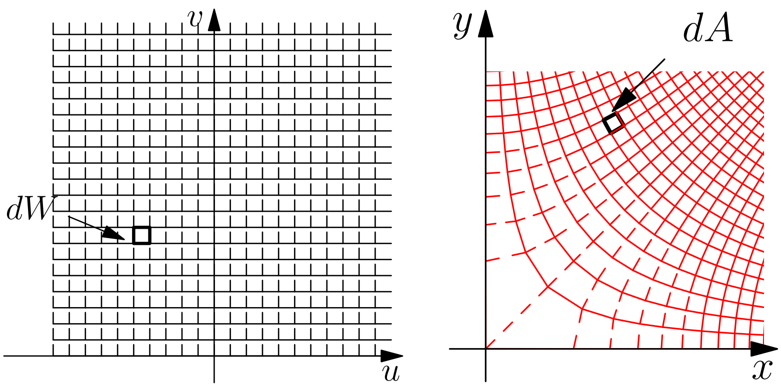
Calculating this double integral using Cartesian coordinates is a little more complicated, and is left as an exercise.

Changing coordinates and the Jacobian

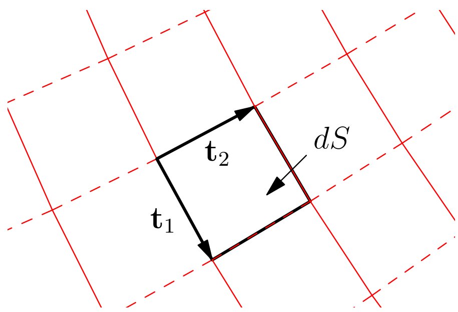
We have seen in the previous section that when we change coordinate systems, say from (x, y) coordinates to (r, θ) coordinates, we also have to change the area element dA which we use in the integration. Going from cartesian to polar coordinates, we had to stretch the area element by a factor of r, so that $dxdy \to rdrd\theta$. This is a specific example for a general rule for coordinate transformations, which we now discuss here.

If we have an alternative set of parameters (u, v), then we can write the old coordinates in terms of the new ones as

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned} \tag{5.13}$$



The figure above shows a regular grid in the (u, v) plane, and the transformed coordinates in the (x, y) plane. The solid curves in both figures represent constant values of u, whereas the dashed curved represent constant values of v. We would like to know how the rectangle dW in the (u, v) plane maps onto the shape dA in the (x, y) plane.



Examining the sides of the area element dA, we note that it is almost a parallelogram. The sides of the parallelogram are given by the tangent vectors along the lines of constant u and constant v. Using the formulas (4.8) for the unit tangent to a line, the sides of the shape are given by

$$\mathbf{t}_1 = \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}\right) du$$
$$\mathbf{t}_2 = \left(\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}\right) dv$$

The area of this shape is given by the cross product of these two vectors:

$$dA = \|\mathbf{t}_1 \times \mathbf{t}_2\|$$

Using the rules for the cross product (1.7) we obtain

$$dA = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \ . \tag{5.14}$$

Note that we are here taking the determinant of a matrix, as indicated by the straight lines on either side. The matrix is known *Jacobian* of the transformation, and is often written

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \tag{5.15}$$

The determinant of this matrix tells us how much to stretch the area elements when changing coordinates. The integral of a function $f(x,y)$ in the new coordinates (u,v) is then

$$\iint_R f(x,y)dxdy = \iint_W f(x(u,v),y(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \ . \tag{5.16}$$

Note that the *domain of integration* must also be transformed from region R in (x,y) coordinates, to region W in (u,v) coordinates.

Example: Show that the area element $dxdy$ in Cartesian coordinates transforms to $rdrd\theta$ in polar coordinates.

Solution: The area element is

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$

where the Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \tag{5.17}$$

From the definition of polar coordinates, we have

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ ,$$

so

$$\frac{\partial x}{\partial r} = \cos\theta\ ,\ \frac{\partial x}{\partial \theta} = -r\sin\theta\ ,\ \frac{\partial y}{\partial r} = \sin\theta\ ,\ \frac{\partial y}{\partial \theta} = r\cos\theta$$

therefore the determinant of the Jacobian matrix is

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \tag{5.18}$$

We therefore have

$$dA = r dr d\theta\ .$$

The **reciprocal theorem** states that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} \tag{5.19}$$

Note that this is true even though it's not necessarily true that $\frac{\partial x}{\partial u} \neq 1/\frac{\partial u}{\partial x}$. This theorem is very useful if you have a formula for u and v but do not want to solve for x and y before differentiating.

Example

1. Substitute $u = xy$, $v = \frac{y}{x}$ to find the area of the region in the first quadrant bounded by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$, $xy = 2$.

Solution:

$$\begin{aligned} \text{Area} &= \iint_R dxdy \\ &= \iint_R 1.\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \iint_R 1.\frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} \quad \text{using the reciprocal theorem (5.19).} \end{aligned}$$

Now

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2\frac{y}{x} = 2v$$

The lines $y = x$ and $y = 2x$ correspond to the line $v = 1$ and $v = 2$ respectively, while the lines $xy = 1$ and $xy = 2$ correspond to $u = 1$ and $u = 2$. The integral is then

$$\begin{aligned} \text{Area} &= \int_1^2 \int_1^2 \frac{\mathrm{d}u\mathrm{d}v}{2v} \\ &= \frac{1}{2}\int_1^2 \frac{1}{v}[u]_1^2 \\ &= \frac{1}{2}[\ln v]_1^2 \\ &= \frac{\ln 2}{2} . \end{aligned}$$

Problems

1. Let P be the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$ and $(1, 5)$. Evaluate the integral

$$\iint_P (4x + 8y)\mathrm{d}x\mathrm{d}y$$

using the change of variables $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$.

2. Let T be the triangle with vertices $(0, 0)$, $(1, 1)$ and $(2, 0)$. Evaluate the integral

$$\iint_T e^{(y-x)/(y+x)}\mathrm{d}x\mathrm{d}y$$

by the using the substitution $u = y - x$, $v = y + x$.

3*. Prove the reciprocal theorem (5.19).
(Hint: try to evaluate

$$\det\left[\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right] ,$$

and use the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

with f chosen appropriately).

Green's theorem

Green's theorem gives us a way of turning *area integrals* into *line integrals*, and vice-versa.

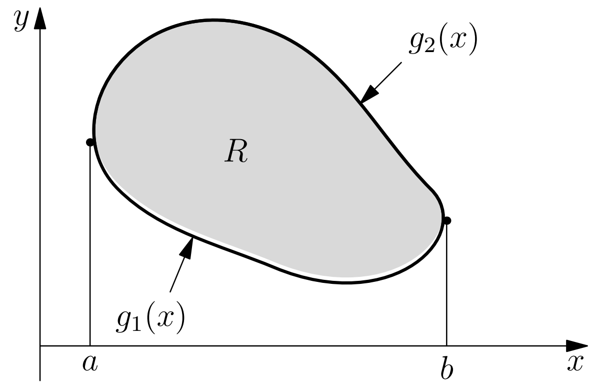
Statement of Green's theorem:
Let $\mathcal{C}$ be a piecewise smooth, simple (non-intersecting) closed curve that forms the boundary of a region R in the (x, y) plane. If $\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$ is continuously differentiable on an open set that contains R, then

$$\oint_{\mathcal{C}} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \qquad (5.20)$$

where the integral on the left hand side is the line integral over $\mathcal{C}$ taken in the **positive** direction (i.e. anticlockwise).

Proof:
Assume that the region R is *convex*. Then we can write

$$R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$



We first prove that $\int_{\mathcal{C}} P dx = -\iint_R \frac{\partial P}{\partial y} dy dx$. Starting from the RHS and using the rules for evaluating double integrals, we find

$$\begin{aligned}
-\iint_R \frac{\partial P}{\partial y} dy dx &= -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx \\
&= -\int_a^b \left[P(x, y) \right]_{y=g_1(x)}^{y=g_2(x)} dx \\
&= -\int_a^b \left[P(x, g_2(x)) - P(x, g_1(x)) \right] dx \\
&= \int_a^b \left[P(x, g_1(x)) - P(x, g_2(x)) \right] dx \qquad (5.21)
\end{aligned}$$

Now looking at the line integral, we find from the definition of line integration over the boundary of R,

$$\begin{aligned}\int_{\mathcal{C}} P dx &= \int_{\mathcal{C}_\infty} P(x,y)dx + \int_{\mathcal{C}_\in} P(x,y)dx \\ &= \int_a^b P(x, g_1(x))dx - \int_a^b P(x, g_2(x))dx\end{aligned}$$

where we have swapped the sign in the second integral because the line integral is traversed in the opposite direction. Both integrals are equal, and so we have proved that

$$\int_{\mathcal{C}} P dx = -\iint_R \frac{\partial P}{\partial y} dy dx \ .$$

Similarly, one can prove that

$$\int_{\mathcal{C}} Q dy = \iint_R \frac{\partial Q}{\partial x} dx dy \ ,$$

and thus we have

$$\oint_{\mathcal{C}} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \ .$$

Applications of Green's theorem

Green's theorem can be used to make an area integral quicker to compute, by transforming it into a line integral. There are many ways we can do this; for example, setting $Q = x$ and $P = 0$, we substitute into Green's theorem to find

$$\iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}0\right) \mathrm{d}x\mathrm{d}y = \oint_{\mathcal{C}} 0dx + \oint_{\mathcal{C}} xdy$$

or

$$\text{Area} = \iint_R \mathrm{d}x\mathrm{d}y = \oint_{\mathcal{C}} xdy \ .$$

Alternatively we could also have set $Q = 0$, $P = -y$, to find

$$\text{Area} = \iint_R \mathrm{d}x\mathrm{d}y = -\oint_{\mathcal{C}} ydx \ .$$

However we could run into problems when either of these integrals runs over a region which is parallel to the x or the y coordinate axis. (Why? Because when part of the path $\mathcal{C}$ runs parallel to the x axis, the differential $\mathrm{d}y$ is undefined). For this reason it is safer to take the average of these two quantities. Setting $Q = x/2$ and $P = -y/2$ we find

$$\text{Area} = \iint_R \mathrm{d}x\mathrm{d}y = \frac{1}{2}\oint_{\mathcal{C}} (xdy - ydx) \ . \tag{5.22}$$

Example:

Use Green's theorem to show that the area of a circle of radius a is equal to πa^2.

Solution: From Green's theorem the area is given by

$$\text{Area} = \frac{1}{2}\oint_{\mathcal{C}} (xdy - ydx)$$

where $\mathcal{C}$ describes the exterior of a circle in the anticlockwise direction. This path can be parameterised by the equations

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}$$

where $0 \leq \theta \leq 2\pi$. The area is then

$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_0^{2\pi} a\cos\theta(a\cos\theta)d\theta - a\sin\theta(-a\sin\theta)d\theta \\ &= \frac{a^2}{2}\int_0^{2\pi} d\theta \\ &= \pi a^2 \ . \end{aligned}$$

We can also use Green's theorem to change a complicated closed line integral into an area integral with a simpler form. In this situation it is good to make use of the result (5.5), in the form

$$\bar{x} = \frac{1}{\mathrm{Area}_R} \iint_R x \mathrm{d}x \mathrm{d}y \quad , \quad \bar{y} = \frac{1}{\mathrm{Area}_R} \iint_R \mathrm{d}x \mathrm{d}y \ , \tag{5.23}$$

since the average values $\bar{x}$ and $\bar{y}$ are often known over the region R.

Example 1

Evaluate the integral

$$\oint_{\mathcal{C}} (xy + 3y^2)dx + (5xy + 2x^2)dy, \quad \text{where } \mathcal{C} = \{(x, y)|(x-1)^2 + (y+2)^2 = 1\}.$$

and the path $\mathcal{C}$ is traversed in the clockwise direction.

Solution:

The path C describes a circle of radius 1 centered at the point $(1, -2)$. Substituting $P = xy + 3y^2$, $Q = 5xy + 2x^2$ into Green's theorem, we find

$$\frac{\partial P}{\partial y} = x + 6y \ , \ \frac{\partial Q}{\partial x} = 5y + 4x$$

so the integral is

$$\begin{aligned}
\oint_{\mathcal{C}} P dx + Q dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \mathrm{d}y \\
&= \iint_R (5y + 4x - (x + 6y)) \, \mathrm{d}x \mathrm{d}y \\
&= \iint_R (3x - 4y) \mathrm{d}x \mathrm{d}y \\
&= (3\bar{x} - \bar{y}) \times \mathrm{Area}_R \\
&= (3 \times 1 - (-1)) \times \pi \\
&= 5\pi \ .
\end{aligned}$$

Example 2

Evaluate the integral

$$\oint_{\mathcal{C}} (e^x \cos y) \mathrm{d}x + (e^x \sin y) \mathrm{d}y$$

where the path $\mathcal{C}$ is the rectangle with vertices $(0,0),(1,0),(1,\pi),(0,\pi)$, traversed in the clockwise direction.

Setting $P = e^x \cos y$ and $Q = e^x \sin y$ we find

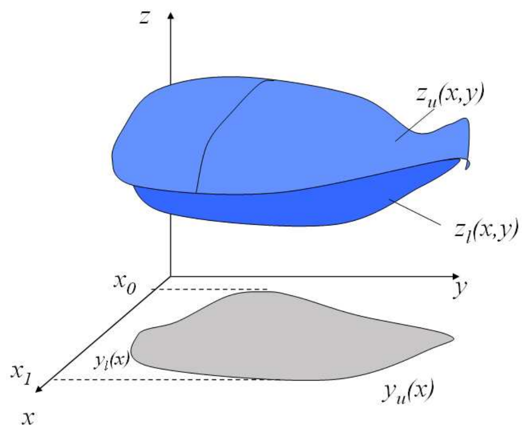
$$\frac{\partial P}{\partial y} = -e^x \sin y \quad , \quad \frac{\partial Q}{\partial x} = e^x \sin y \ .$$

Substituting into Green's theorem we find that

$$\begin{aligned}
\oint_{\mathcal{C}} P\mathrm{d}x + Q\mathrm{d}y &= -\oint_{\mathcal{C}'} (P\mathrm{d}x + Q\mathrm{d}y) \\
&\qquad \text{where C' traverses the region in the } \textit{positive} \text{ direction} \\
&\qquad \text{(i.e. in the anticlockwise direcgtion)} \\
&= -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\
&= -\iint_R (2e^x \sin y)\, \mathrm{d}x\mathrm{d}y \\
&= -\int_0^{\pi} \int_0^1 2e^x \sin y \mathrm{d}x\mathrm{d}y \\
&= -\int_0^{\pi} 2 \sin y \, [e^x]_0^1 \, \mathrm{d}y \\
&= -2(e-1)[-\cos y]_0^{\pi} \\
&= -4(e-1) \ .
\end{aligned}$$

Triple integrals

Consider a volume V in three dimensional space.



As with the two-dimensional integrals, we divide the region up into (an infinite number of) infinitesimal chunks,

$$dV = dx\ dy\ dz$$

We then 'sum' over the totality of these small volume elements to give us our integral. The volume integral of a scalar function $f(x, y, z)$ is written

$$\iiint_V f\ dV = \iiint_V f(x, y, z)\ dxdydz\ .$$

The procedure by which we evaluate these integrals is almost identical to that used for calculating double integrals; it differes only in that there is an additional variable to integrate over. In most simple cases (see above figure) the volume is entirely enclosed between two surfaces: the volume consists of all points z which lie above the lower surface $z = z_l(x, y)$ and below the upper surface $z = z_u(x, y)$, these surfaces lie between the curves $y = y_l(x)$ and $y = y_u(x)$, and these curves lie in turn between the points x_0 and x_1. In terms of equations, the bounds of the volume are:

$$\begin{aligned} x_0 \leq & \quad x & \leq x_1 \\ y_l(x) \leq & \quad y & \leq y_u(x) \\ z_l(x, y) \leq & \quad z & \leq z_u(x, y) \end{aligned}$$

The volume integral can then be evaluated:

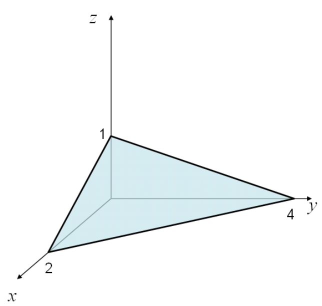
$$\iiint_V f dV = \int_{x_0}^{x_1} \int_{y_l(x)}^{y_u(x)} \int_{z_l(x,y)}^{z_u(x,y)} f(x, y, z) dz\ dy\ dx \quad . \tag{5.24}$$

Example: Evaluate the volume integral

$$\iiint_V xyz \; dxdydz$$

where V is the pyramid bounded by the plane surfaces

$$x = 0 \; , \quad y = 0 \; , \quad z = 0 \; , \quad 2x + y + 4z = 4 \quad .$$



Solution:
Note that within the region V, z is confined such that

$$0 \leq z \leq 1 - \frac{x}{2} - \frac{y}{4} \quad .$$

Also, y is bound by the inequality

$$0 \leq y \leq 4 - 2x$$

and x lies between

$$0 \leq x \leq 2 \quad .$$

The integral can then be written

$$\mathcal{I} = \iiint_V (xyz) \; dxdydz = \int_0^2 dx \int_0^{4-2x} dy \int_0^{1-x/2-y/4} dz \; [xyz]$$

All that is left to do now is to evaluate this integral. Starting with z,

$$
\begin{aligned}
\mathcal{I} &= \int_0^2 dx \int_0^{4-2x} dy \ \left[xy\frac{z^2}{2}\right]_0^{1-x/2-y/4} \\
&= \int_0^2 dx \int_0^{4-2x} dy \ \frac{1}{2}xy\left(1-\frac{x}{2}-\frac{y}{4}\right)^2 \\
&= \int_0^2 dx \ \frac{1}{2}x\left[\frac{y^2}{2}\left(1-\frac{x}{2}\right)^2-\frac{y^3}{6}\left(1-\frac{x}{2}\right)+\frac{y^4}{64}\right]_0^{4-2x} \\
&= \int_0^2 dx \ \frac{2}{3}x\left(1-\frac{x}{2}\right)^4 \\
&= \frac{2}{3}\left[-x\frac{2}{5}\left(1-\frac{x}{2}\right)^5\right]_0^2+\frac{2}{3}\int_0^2 dx \ \frac{2}{5}\left(1-\frac{x}{2}\right)^5 \\
&= 0+\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\left[-\frac{2}{6}\left(1-\frac{x}{2}\right)^6\right]_0^2 \\
&= \frac{4}{45}
\end{aligned}
$$

Finding the volume of a region

In order to find a volume of a region we integrate the scalar function $f(x, y, z) = 1$ over that region.
i.e.,

$$Volume = \iiint_V dV \tag{5.25}$$

Example: Find the volume of the squashed box-shape bound by the six plane surfaces

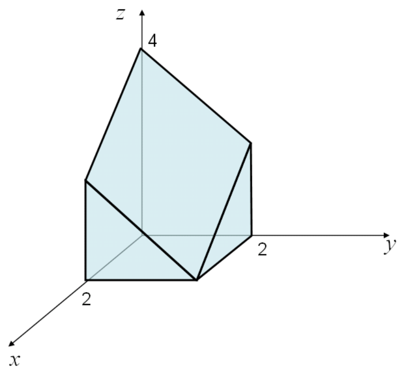
$$x = 0\ ,\quad x = 2\ ,\quad y = 0\ ,\quad y = 2\ , z = 0\ ,\quad x + y + z = 4\quad .$$

Solution:
First note that z lies in the range

$$0 \leq z \leq 4 - x - y\quad .$$

Also x and y lie in the ranges

$$0 \leq x \leq 2\ ,\quad 0 \leq y \leq 2\quad .$$



So the volume is calculated:

$$
\begin{aligned}
Volume &= \int_V dV \\
&= \int_0^2 dx \int_0^2 dy \int_0^{4-x-y} dz \ (1) \\
&= \int_0^2 dx \int_0^2 dy \ (4-x-y) \\
&= \int_0^2 dx \ (6-2x) \\
&= 8 \quad .
\end{aligned}
$$

Change of variables and the Jacobian

If we transform our coordinates from (x, y, z) to another set of variables (u, v, w) then the three-dimensional volume element should also be changed. This transformation is given once again by the Jacobian:

$$dxdydz = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw \tag{5.26}$$

where the Jacobian matrix is defined to be

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \tag{5.27}$$

Volume integrals can then be written in the form:

$$\iiint_V f(x, y, z) dxdydz = \iiint_V f(x(u, v, w), y(u, v, w), z(u, v, w) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw\ . \tag{5.28}$$

The Jacobian also satisfies the reciprocity relation:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} \tag{5.29}$$

in exactly the same way as in two dimensions.

Problems

1. Show that in cylindrical coordinates, the volume element is given by

$$dxdydz = rdrd\theta dz$$

2. Show that the volume of a sphere of radius a is given by $\frac{4}{3}\pi a^3$.

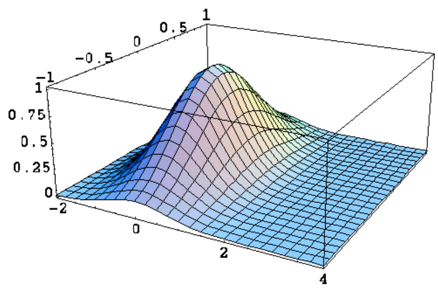
Surfaces in 3D

A surface in three dimensions can be represented in three ways: explicitly, implicity, and parametrically.

Explicit representation

A point on the surface $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is said to be represented explicitly if one of the variables x, y or z can be written in terms of the other two. For example

$$z = g(x, y)$$



$$z = g(x, y)$$

Implicit representation

A point on the surface $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ can be represented implicitly if there exists a single equation which connects all three coordinates x, y and z. The most general way of writing this connection is

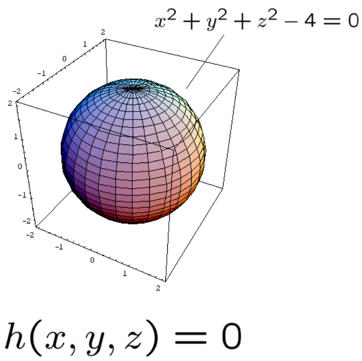
$$h(x, y, z) = 0 \ .$$

Note that if we set $h(x, y, z) = z - g(x, y)$ then we recover the *explicit* formulation.

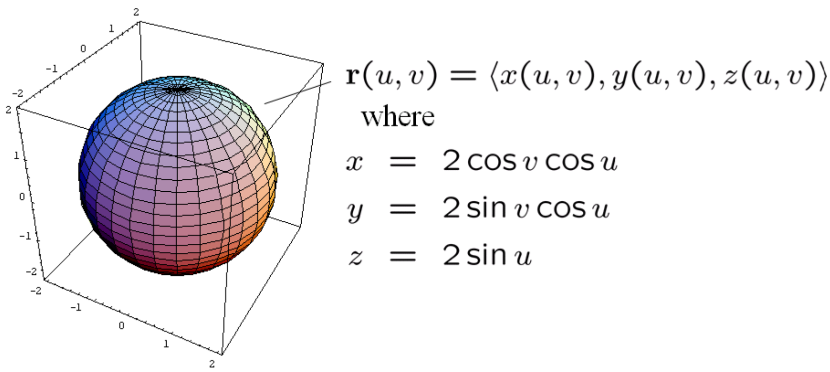
Parametric representation

A surface can be represented *parametrically* by the position vector $\mathbf{r}$:

$$\mathbf{r}(u, v) = x(u, v)\hat{\mathbf{i}} + y(u, v)\hat{\mathbf{j}} + z(u, v)\hat{\mathbf{k}}$$



where (u, v) are a pair of real numbers, known as the *coordinates* on the surface. The surface is then represented by a set of *coordinate lines*, along which the values of u and v do not change.



Surface Integrals

Like line integrals, there are two types of surface integrals, which we will refer to as *scalar* and *vector* type integrals. We approach both in more or less the same way, which is to divide the surface up into a large number of small rectangles of equal area ΔS, and then sum over these as the area approaches zero. By projecting either onto one of the coordinate planes or onto a region in the parametric coordinates (u, v), we can convert an integral over a surface into a double integral, something which we already know how to calculate.

Scalar-type surface integrals

Suppose that $f(x, y, z)$ is some scalar function which is defined on a surface S in 3D space. We then divide the surface up into a large number of small squares, each of area ΔS. We then take the limit, in which we get an infinite number of these small squares and at the same time they become infinitely small, i.e. $\Delta S \to dS$. The surface integral of f over S can then be defined as

$$\iint_S f(x, y, z) dS = \lim_{\Delta S \to 0} \sum_{(\text{all } \Delta S \text{ in } S)} f(x, y, z) \Delta S \tag{5.30}$$

Clearly the evaluation of these integrals is made more difficult because each infinitesimal element dS is aligned at a different angle. That is, each element has a different *direction* associated with it. This direction is the normal to the surface $\hat{\mathbf{n}}$, which will take a different form depending on whether the surface is represented explicitly, implicitly, or parametrically.

Explicit representation

If the surface is represented explicitly, then we can calculate this type of integral by *projection* onto one of the coordinate planes.

If the surface can be written as $z = g(x, y)$ then we project onto the $x - y$ plane. We can see from the figure below that the projected area $dxdy$ can be written[1]

$$dxdy = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}\, dS \ ,$$

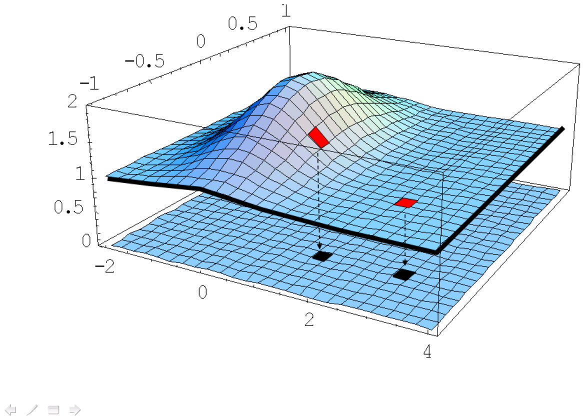
where $\hat{\mathbf{k}}$ is the coordinate vector pointing in the z direction. We can re-arrange this equation to obtain

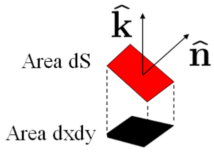
$$dS = \frac{dxdy}{\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}} \tag{5.31}$$

[1] Sometimes we find that it is not convenient to project onto the x-y plane. For example we might want to integrate over the surface $\Phi(x, y, z) = x + y = 1$ which is always perpendicular to the x-y plane. In this case we have to project onto one of the other coordinate planes, either the x-z plane or the z-y plane. It is a simple matter then to obtain the relations

$$dS = \frac{dx\ dy}{\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}} = \frac{dy\ dz}{\hat{\mathbf{n}} \cdot \hat{\mathbf{i}}} = \frac{dz\ dx}{\hat{\mathbf{n}} \cdot \hat{\mathbf{j}}}$$

where, as usual, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the coordinate vectors in the x, y and z directions respectively.





$$\begin{aligned} dxdy &= \cos\theta dS \\ &= |\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|dS \end{aligned}$$

The normal vector $\hat{\mathbf{n}}$ can be calculated in the following way: if we write the equation for the surface as

$$h(x, y, z) = z - g(x, y)$$

then we have seen in chapter 3 that the gradient ∇h points in the direction in which the function h increases most rapidly. Because h does not change along the surface, this means that ∇h *always points perpendicular to the surface.* We can use this to define the **unit normal** to the surface:

$$\begin{aligned} \hat{\mathbf{n}} &= \frac{\nabla h}{\|h\|} \\ &= \frac{\nabla[z - g(x, y)]}{\|\nabla[z - g(x, y)]\|} \\ &= \frac{-\frac{\partial g}{\partial x}\hat{\mathbf{i}} - \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \end{aligned} \tag{5.32}$$

The surface element is then

$$dS = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dxdy\ , \tag{5.33}$$

and the surface integral becomes

$$\int_S f(x, y, z)\ dS = \int_R f(x, y, g(x, y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dxdy \tag{5.34}$$

where R is the 'projected' region of S onto the $x - y$ plane.

Implicit representation

If the surface is represented implicitly, i.e. in the form

$$h(x, y, z) = 0$$

then the normal to the surface is given by the equation

$$\hat{\mathbf{n}} \ = \frac{\nabla h}{\|h\|} \ . \tag{5.35}$$

One must then project onto one of the coordinate planes in order to evaluate the integral. Projecting onto the $x - y$ plane, the surface element becomes

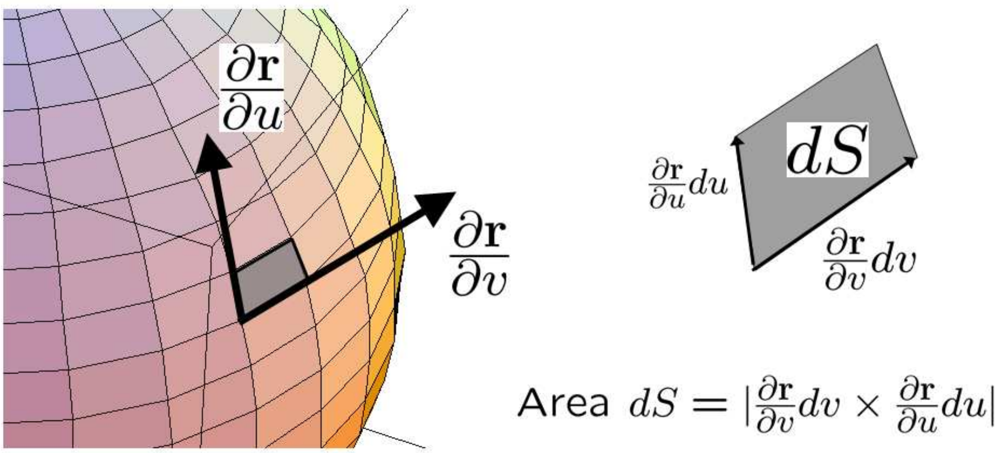
$$dS = \frac{dxdy}{\|\hat{\mathbf{n}} \ \cdot \hat{\mathbf{k}}\|}$$

and the integral is then

$$\iint_S f dS = \iint_R \frac{f(x, y, z)}{\|\hat{\mathbf{n}} \ \cdot \hat{\mathbf{k}}\|} dxdy \tag{5.36}$$

where R is the projection of the surface S onto the $x - y$ plane and $f(x, y, z)$ is taken to be evaluated only at the points (x, y, z) defined by the equation $h(x, y, z) = 0$. This function is generally difficult to invert, and this makes implicitly defined surfaces hard to integrate over. The best strategy is usually to re-define the surface either explicitly or parametrically.

Parametric representation



For a surface defined parametrically we have

$$\mathbf{r}(u,v) = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}}$$

We know that along a coordinate line the values of either u or v do not change, so the derivatives of $\mathbf{r}$ with respect to u and v must define tangents to the coordinate lines. That is,

$$\mathbf{t}_u = \left.\frac{\partial \mathbf{r}}{\partial u}\right|_{v=v_0} \quad \text{describes the tangent to the curve } v = v_0$$

and

$$\mathbf{t}_v = \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{u=u_0} \quad \text{describes the tangent to the curve } u = u_0 \ .$$

Consequently, the direction of the normal to the surface at the point $\mathbf{r}(u_0, v_0)$ is given by the cross-product of these two vectors, i.e.

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \quad \text{is a } \textit{normal} \text{ to the surface}$$

Furthermore, we know that

$$\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| dudv \quad \text{gives the area of the parallelogram } dS.$$

Therefore we can write the surface element dS in terms of the infinitessimals du and dv:

$$dS = \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| \ dudv \ . \tag{5.37}$$

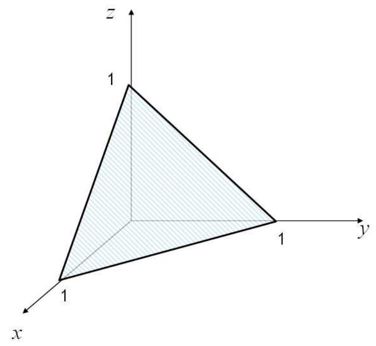
The surface integral of $f(x, y, z)$ over S is then

$$\int_S f \ dS = \int_R f(x(u,v), y(u,v), z(u,v))\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| \ dudv \ . \tag{5.38}$$

where R is now the projection of the surface S onto the $u - v$ plane. Note that this element $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$ is *different* to the Jacobian matrix defined earlier (to see this, write it out explicitly!).

Example: Evaluate the integral of the function $f(x, y, z) = xyz$ over the surface S, which is defined by the equations

$$\begin{array}{c} x + y + z = 1 \\ x \geq 0 \ , \quad y \geq 0 \ , \quad z \geq 0 \ . \end{array}$$



<u>Solution:</u>
We can re-express the surface as

$$\begin{aligned} z &= g(x, y) \\ &= 1 - x - y \ . \end{aligned} \tag{5.39}$$

Then,

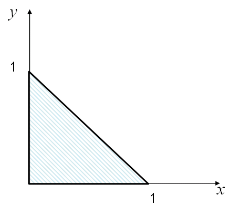
$$\begin{aligned} f(x, y, z) &= xyz \\ &= xy(1 - x - y) \\ &= xy - x^2y - y^2x \ . \end{aligned}$$

Projecting onto the x-y plane,

$$\begin{aligned} dS &= \frac{dxdy}{\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}} \\ &= \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \ dxdy \\ &= \sqrt{3} \ dxdy \end{aligned}$$

The integral is now

$$\mathcal{I} = \int_S f \ dS = \int_R (xy - x^2y - y^2x)\sqrt{3} \ dxdy$$



where the region R is shown below:

This is a double integral, which is the kind of thing we tackled in the previous section. It is evaluated:

$$
\begin{aligned}
\mathcal{I} &= \int_0^1 dx \int_0^{1-x} dy\sqrt{3}(xy - x^2y - y^2x) \\
&= \int_0^1 \left[\frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{y^3x}{3}\right]_0^{1-x} dx \\
&= \int_0^1 \left[-\frac{5}{6}(1-x)^3\right]\sqrt{3}\ dx \\
&= \frac{5\sqrt{3}}{24}
\end{aligned}
$$

Finding the area of a surface

The area of a given surface in 3D can be calculated by integrating the scalar function $f(x, y, z) = 1$ over that surface, i.e.

$$Area = \iint_S dS \ . \tag{5.40}$$

Example:
Calculate the area of the surface S given in the previous example.

Solution:
Projecting onto the x-y plane, we know that the area element is

$$dS = \sqrt{3}\ dxdy \quad ,$$

so the total area of the surface is

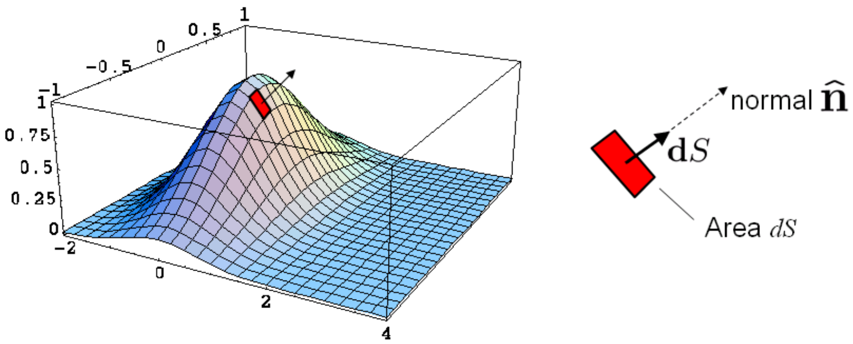
$$\begin{aligned}
Area &= \iint_S dS \\
&= \iint_R 1.\ dxdy \\
&= \int_0^1 dx \int_0^{1-x} dy\ (1) \\
&= \int_0^1 [y]_0^{1-x} \\
&= \int_0^1 dx\ (1-x) \\
&= \left[-\frac{1}{2}(1-x)^2\right]_0^1 \\
&= \frac{1}{2} \ .
\end{aligned}$$

Vector-type surface integrals (flux integrals)

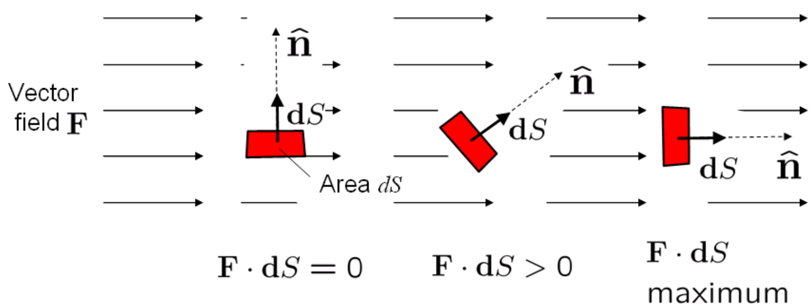
Recall that we have divided our surface up into tiny rectangles of area ΔS, and that each of these elements has a direction associated with it, a direction which is specified by the normal vector $\hat{\mathbf{n}}$. We can use this to define a new sort of area element,

$$\Delta \mathbf{S} = \hat{\mathbf{n}}\ \Delta S \tag{5.41}$$

which is a *vector* quantity, and represents both the area of a surface element and its orientation.



The reason this new quantity is useful is that if we 'dot' product it with a vector field, then it measures *how much* of that field goes through the small surface element ΔS. That is, if the vector field is totally parallel to the surface then it doesn't really 'go through' the surface element at all, it just glides across it, resulting in $\mathbf{F} \cdot \Delta\mathbf{S} = 0$. However if the surface element is oriented so that it is perpendicular to the vector field, then $\mathbf{F} \cdot \Delta\mathbf{S}$ is as large as it can possibly get.



The quantity $\mathbf{F} \cdot \Delta\mathbf{S}$ is known as the *flux* of the vector field $\mathbf{F}$ across the surface element ΔS. If we *add up* the totality of all these fluxes then we can form an integral, often known as the flux integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \lim_{\Delta S \to 0} \sum_{(\text{all } \Delta S \text{ in } S)} \mathbf{F} \cdot \Delta\mathbf{S} \quad , \tag{5.42}$$

where we have replaced the area vector $\Delta\mathbf{S}$ by its infinitesimal counterpart $d\mathbf{S}$ in the limit as $\Delta S \to 0$.

In practise this integral can be evaluated by re-writing it as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \;\; dS \ . \tag{5.43}$$

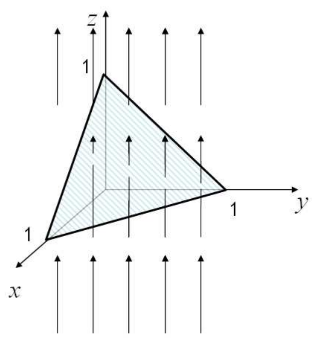
Note that the infinitesimal vector element $d\mathbf{S}$ has been replaced by the scalar quantity dS on the right-hand side. In fact the quantity $\mathbf{F} \cdot \hat{\mathbf{n}}$ is also a scalar, and so this integral is now a *scalar-type* surface integral, which we already know how to calculate.

Example: Calculate the vector-type surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \ ,$$

where $\mathbf{F} = z^2\hat{\mathbf{k}}$ and the surface S is defined by the equations

$$x + y + z = 1 \ , \quad x, y, z \geq 0 \quad .$$



Solution: The surface can be written explicitly as

$$z = g(x, y)$$

where $g(x, y) = 1 - x - y$. The unit normal to the surface is then

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\frac{\partial g}{\partial x}\hat{\mathbf{i}} - \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \\
&= \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad .
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{F} \cdot \hat{\mathbf{n}} &= (z^2\hat{\mathbf{k}}) \cdot (\frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}) \\
&= \frac{z^2}{\sqrt{3}} = \frac{1}{\sqrt{3}}(1 - x - y)^2 \ .
\end{aligned}$$

Therefore we now have to evaluate a scalar integral,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \frac{1}{\sqrt{3}}(1 - x - y)^2 \ dS \ .$$

Projecting onto the x-y plane,

$$\begin{aligned}
dS &= \frac{dx \ dy}{\hat{\mathbf{n}} \ \cdot \hat{\mathbf{k}}} \\
&= \sqrt{3} dx \ dy
\end{aligned}$$

The rest is just evaluating a double integral:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_R \sqrt{3}(1 - x - y)^2 \sqrt{3} \ dxdy \\
&= \int_0^1 dx \int_0^{1-x} dy \ (1 - x - y)^2 \\
&= \int_0^1 dx \ \left[(1 - x)^2 y + \frac{1}{3}y^3 - y^2(1 - x)\right]_0^{1-x} \\
&= \int_0^1 dx \ \frac{1}{3}(1 - x)^3 \\
&= \frac{1}{12} \quad .
\end{aligned}$$