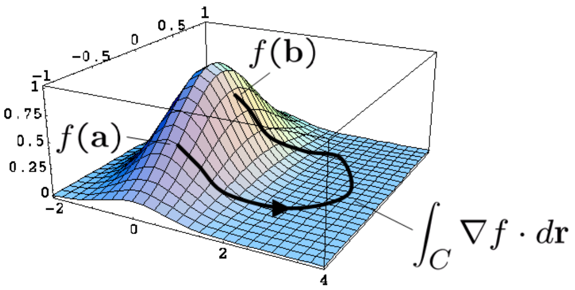
Chapter 6

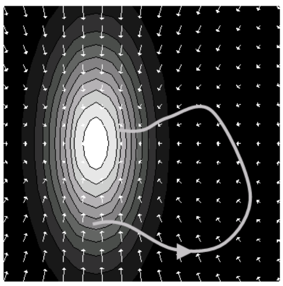
Integral theorems and applications

There are three integral theorems which we consider in this course. In fact the first of them we have already covered, within the context of conservative fields. I called it the 'fundamental' theorem and it is worth re-stating here:

The fundamental theorem of calculus in 3D:
Given a piecewise continuous path in three-dimensions $\mathcal{C}$ which begins at point $\mathbf{a}$ and ends at point $\mathbf{b}$, and given a scalar field $f(x, y, z)$ which is continuous and has continuous partial derivatives on $\mathcal{C}$, then

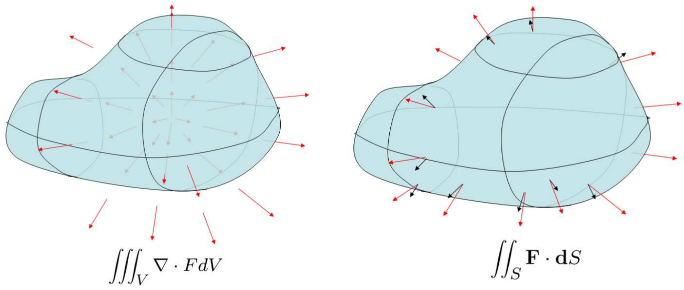
$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{b}) \ . \tag{6.1}$$





It doesn't matter how much the path wriggles around and changes course, the integral depends only on where the path begins and ends. That is, the line integral of the gradient of the function f depends only on the endpoints of the path.

This integral theorem tells us how to reduce a line integral, which is a one-dimensional quantity, to the values of a function at the endpoints. The next integral theorem applies to volume integrals, and is generally known as 'the divergence theorem' or 'Gauss's theorem':



The divergence theorem:
Given a volume V which is bounded by a piecewise continuous surface S, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives in V, then

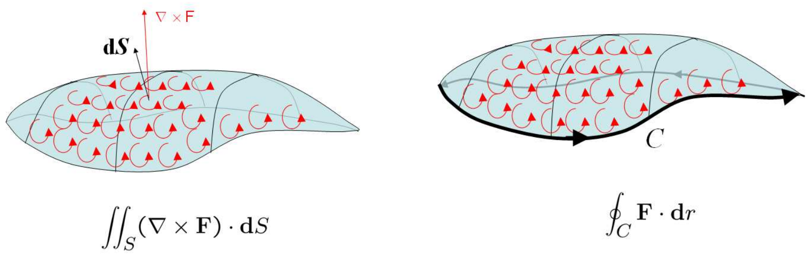
$$\iiint_V \nabla \cdot \mathbf{F} dV = \oiint_S \mathbf{F} \cdot d\mathbf{S} \quad . \tag{6.2}$$

Notice that the surface integral S has been drawn with a circle around it, in order to indicate that this surface is *closed*, i.e. that it entirely encompasses the volume V.

Just as the fundamental theorem relates the integral of a directional derivative along a path to the values at the end points, the divergence theorem relates the integral of a divergence over a volume to the values at the boundary of the volume.

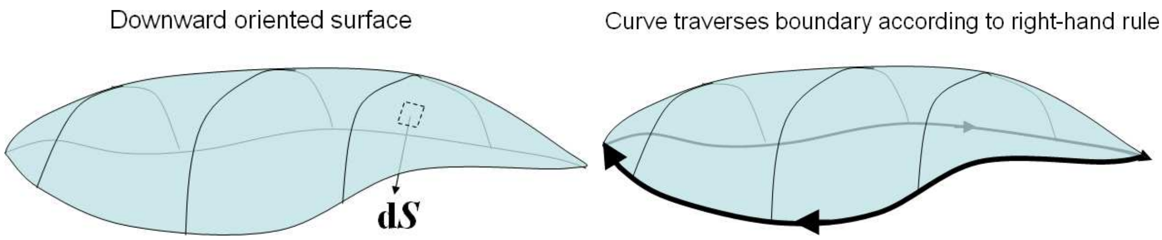
Stokes' theorem:
Given a surface S which is bounded by a piecewise continuous curve $\mathcal{C}$, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives on S, then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{6.3}$$



Note first of all that there seems to be some sort of ambiguity in the above equation; The line integral on the right-hand side definitely depends on the direction in which you traverse the path $\mathcal{C}$. If you around in the opposite direction, then the line integral changes in sign. Which is correct?

The way we get around this is by noting that the surface S also has an ambiguity in direction: we can choose the normal to the surface S so that it points upwards (as in the above figure) or downwards (as in the one below). Clearly either choice will define the same surface, however the normal vector itself will be opposite in sign in both cases. In order to enforce consistency, we adopt once again the *right-hand rule*. This states that the path C should be chosen so that it traverses the surface S in the *positive* sense; that is, if your fingers of your right hand are curled in the direction of the path, then your thumb should point in the direction of the normal.



Examples

We now include some examples of the application of the divergence theorem and Stokes' theorem.

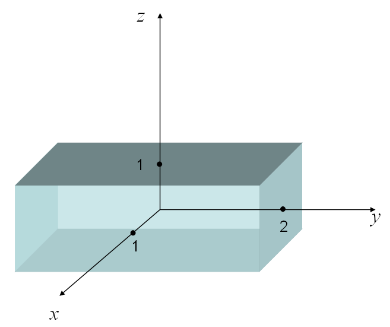
Example 1:
Use the divergence theorem to calculate the surface integral of the vector function $\mathbf{F} = x\hat{\mathbf{i}} + 2y^2\hat{\mathbf{j}} + 3z\hat{\mathbf{k}}$ integrated over the rectangular box S with corners $(\pm 1, -2, -1)$, $(\pm 1, -2, 1)$, $(\pm 1, 2 - 1)$ and $(\pm 1, 2, 1)$.

<u>Solution</u>:
By the divergence theorem, we know that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F}) dV$$

where V is the interior of S. Now

$$\nabla \cdot \mathbf{F} = 1 + 4y + 3$$



and so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_V (1+4y+3)dV \\
&= \int_{-1}^{1} dx \int_{-2}^{2} dy \int_{-1}^{1} dz(4+4y) \\
&= \int_{-1}^{1} dx \int_{-2}^{2} dy \left[(4+4y)z\right]_{-1}^{1} \\
&= \int_{-1}^{1} dx \int_{-2}^{2} dy \left[8(1+y)\right] \\
&= 8\int_{-1}^{1} dx \left[y+\frac{y^2}{2}\right]_{-2}^{2} \\
&= 32\int_{-1}^{1} dx \\
&= 64 \quad .
\end{aligned}$$

Example 2:
Verify Stokes' theorem

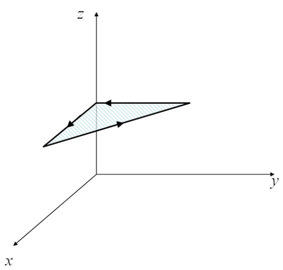
$$\iint (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

where

$$\mathbf{F} = xy\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}} \quad ,$$

the surface S is the upwards-oriented triangle with corners at $(0,0,1),(1,0,1)$ and $(0,1,1)$, and $\mathcal{C}$ is the anti-clockwise path around this surface.

<u>Solution</u>:



Start with the surface integral. Note first that

$$\begin{aligned}\nabla \times \mathbf{F} &= \hat{\mathbf{i}}(0-0) - \hat{\mathbf{j}}(0-z^2) + \hat{\mathbf{k}}(0-x) \\ &= -z^2\hat{\mathbf{j}} - x\hat{\mathbf{k}} \\ &= -\hat{\mathbf{j}} - x\hat{\mathbf{k}} \quad \text{on } S.\end{aligned}$$

Also on S we have

$$d\mathbf{S} = \hat{\mathbf{k}}dx\ dy \quad .$$

So the surface integral is

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_S (-\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}dx\ dy \\ &= \int_S dx\ dy(-x) \quad .\end{aligned}$$

$$\begin{aligned}&= \int_0^1 dx \int_0^{1-x} dy(-x) \\ &= \int_0^1 dx(-x)\,[y]_0^{1-x} \\ &= \int_0^1 dx(-x+x^2) \\ &= -\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{1}{6}\end{aligned}$$

Now looking at the line integral, we split it up into three sections:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}$$

On the line segment $\mathcal{C}_1$, we have $z = 1, x = 0$ and y starts at 1 and ends at 0. Hence we choose the parameters:

$$\begin{aligned} y &= t \\ x &= 0 \\ z &= 1 \end{aligned} \tag{6.4}$$

Hence

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + \hat{\mathbf{k}}\frac{dz}{dt} \\ &= \hat{\mathbf{j}} \end{aligned} \tag{6.5}$$

So,

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (y^3\hat{\mathbf{j}}) \cdot \hat{\mathbf{j}} \\ &= t^3 \quad . \end{aligned} \tag{6.6}$$

Hence

$$\begin{aligned} \int_{\mathcal{C}_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_1^0 t^3 dt \\ &= \left[\frac{t^4}{4}\right]_1^0 = -\frac{1}{4} \end{aligned}$$

On the line segment $\mathcal{C}_2$, we have $z = 1, y = 0$ and x varies between 0 and 1. Hence we choose $x = t$ as our parameter, giving

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}}$$

and hence

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t\hat{\mathbf{k}} \cdot \hat{\mathbf{i}} \cdot = 0 \ .$$

and so

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = 0 \quad .$$

Therefore this line segment contributes nothing to the final integral.

On the line segment $\mathcal{C}_3$, we have $z = 1$, $x + y = 1$ and x starts at $x = 1$ and finishes at $x = 0$.

Hence we chose the parameters

$$\begin{aligned} z &= 1 \\ x &= t \\ y &= 1 - t \end{aligned} \tag{6.7}$$

Thus we have

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

and so

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (xy\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \\ &= t(1-t) - (1-t)^3 \end{aligned} \tag{6.8}$$

So the integral over the line segment is

$$\begin{aligned} \int_{\mathcal{C}_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_1^0 (t - t^2) - (1-t)^3 dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3} + \frac{(1-t)^4}{4} \right]_1^0 \\ &= \frac{1}{4} - (\frac{1}{2} - \frac{1}{3}) = \frac{1}{12} \end{aligned}$$

The final integral is then

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{S} &= -\frac{1}{4} + 0 + \frac{1}{12} \\ &= -\frac{1}{6} \end{aligned} \tag{6.9}$$

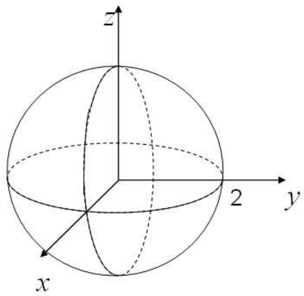
And so Stokes theorem is verified.

Example 3:
Use the divergence theorem to calculate the surface integral

$$\iint_S (x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot d\mathbf{S} \quad ,$$

where S is the spherical surface defined by the equation

$$x^2 + y^2 + z^2 = 4 \quad .$$



Solution:
Now let $\mathbf{F} = x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Then, by Gauss's theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} \, dV \quad ,$$

where V is the interior of the sphere with bounding surface S. The divergence is readily evaluated:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + 2 + 1 \\ &= 4 \quad . \end{aligned}$$

So the surface integral is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_V 4 \, dV \\ &= 4 \iiint_V dV \\ &= 4 \times (\textit{Volume of the sphere}) = 4 \times \frac{4\pi}{3} \times 2^3 \\ &= \frac{128}{3}\pi \quad . \end{aligned}$$