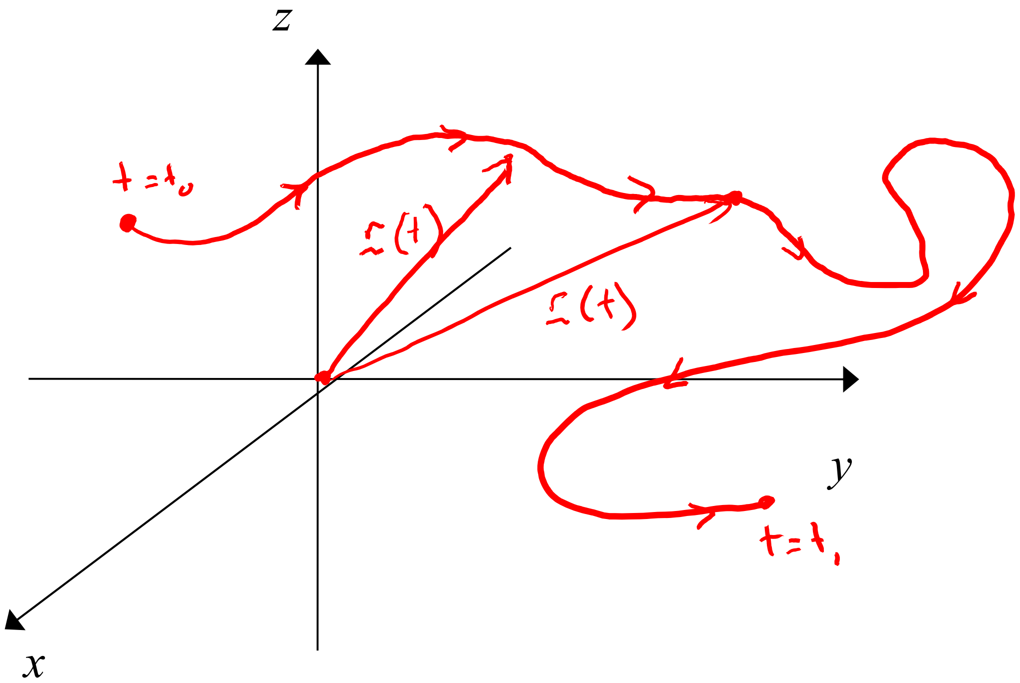
Paths in 3D

A path in three dimensions can be written in *parametric form* as

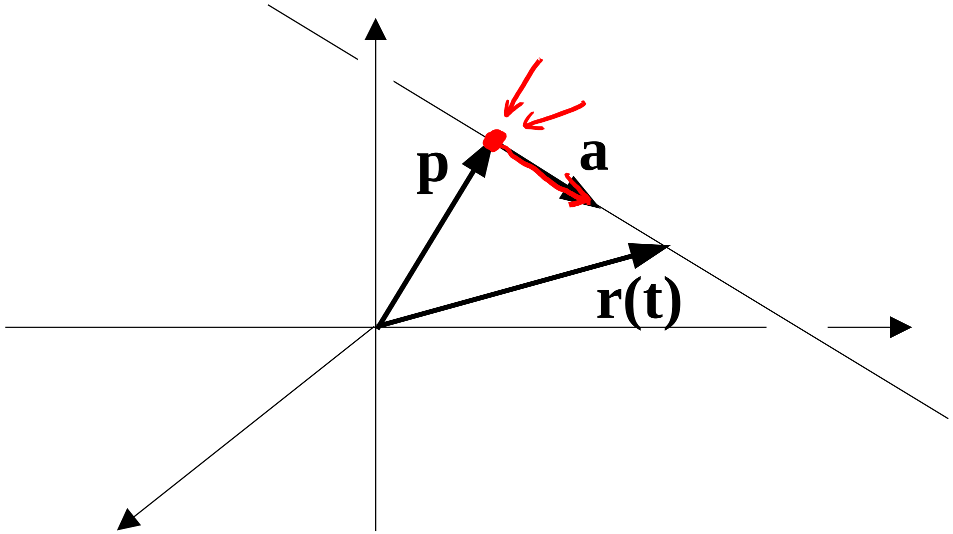
$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} = \langle x(t), y(t), z(t) \rangle.$$

where $t \in [t_0, t_1]$ (start, end)

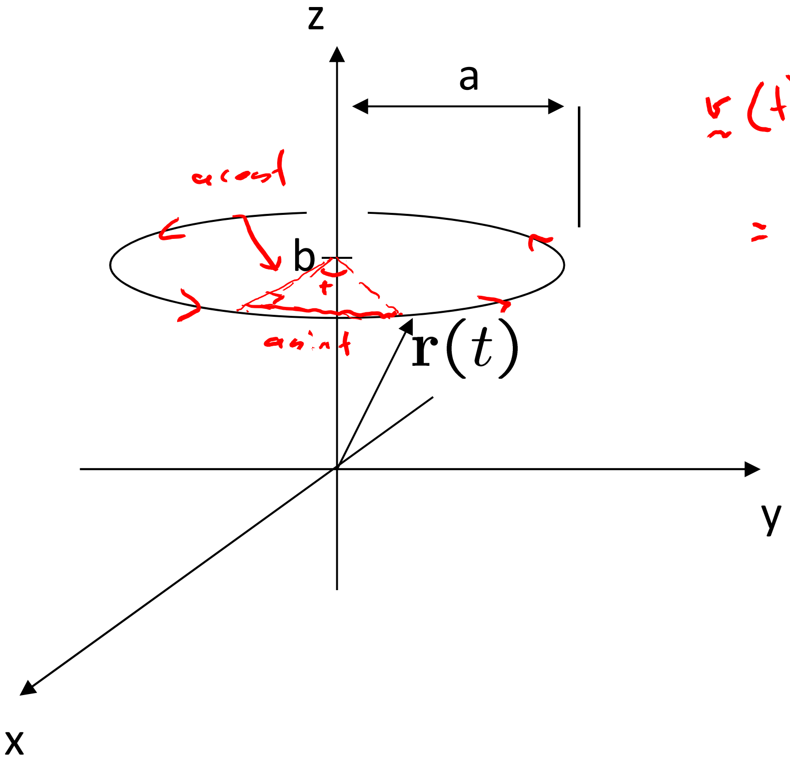


E.g. A straight line through a point **p**, parallel to a vector **a** is

$$\underset{\sim}{r}(t) = \underset{\sim}{p} + \underset{\sim}{a} \cdot t$$



E.g. A circle at constant $z = b$ with radius a:



$$\underset{\sim}{r}(t) = \langle a\cos t, a\sin t, b \rangle$$

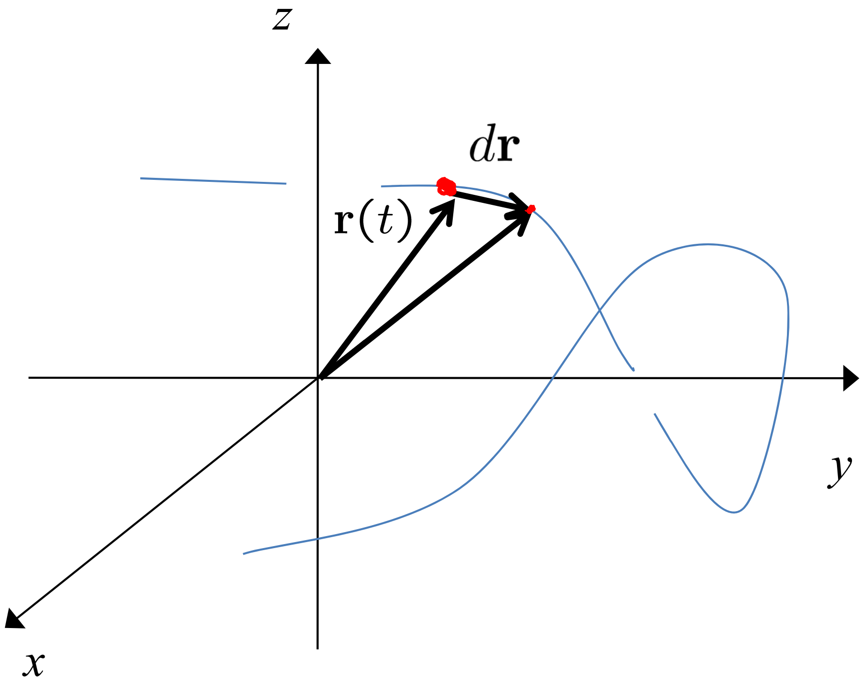
$$= \underset{\sim}{\hat{i}}\, a\cos t + \underset{\sim}{\hat{j}}\, a\sin t + b\underset{\sim}{\hat{k}}$$

The infinitessimal displacement vector
Is the vector differential

$$d\mathbf{r} = \langle dx, dy, dz \rangle$$

$$= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$\underline{v}$$



The infinitessimal arc-length is

$$ds = |d\mathbf{r}|$$

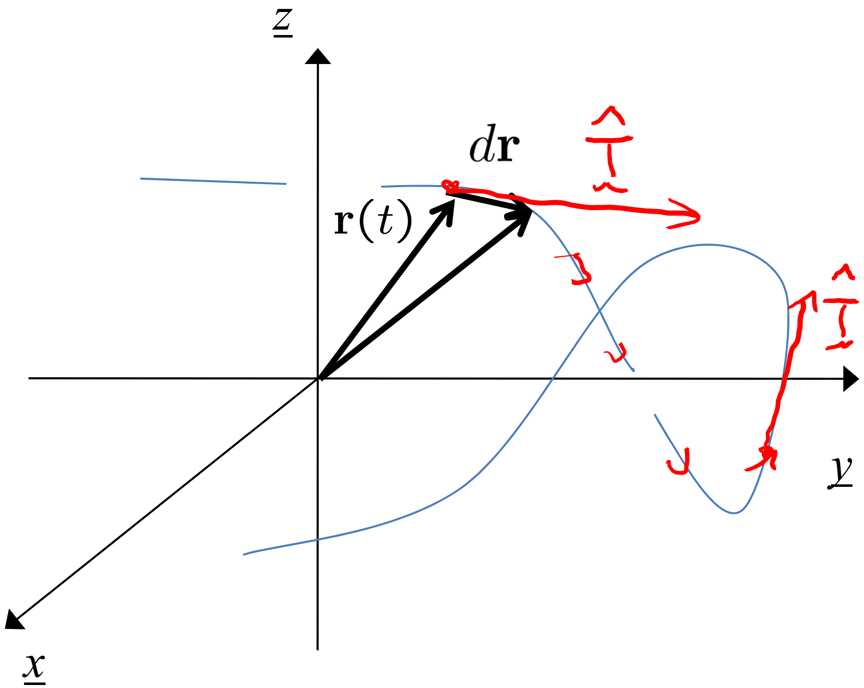
$$= \sqrt{dx^2 + dy^2 + dz^2} = \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \right) dt .$$

Tangents and normals

$d\mathbf{r}/dt$ is always parallel to the curve, and so is a *tangent vector*

The unit tangent vector is then

$$\hat{\underset{\sim}{T}} = \frac{\underset{\sim}{T}}{|\underset{\sim}{T}|} = \frac{\frac{d\underset{\sim}{r}}{dt}}{\left|\frac{d\underset{\sim}{r}}{dt}\right|}.$$



We can define a normal to the curve by noting that

$$\hat{\mathbf{T}} \cdot \hat{\mathbf{T}} = 1$$

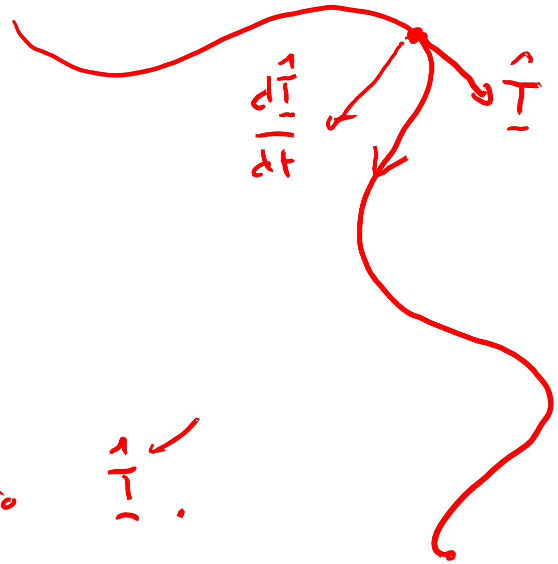
because $|\hat{T}| = 1$ and $\hat{T} \cdot \hat{T} = |\hat{T}|^2$

$$\frac{d}{dt}\left(\hat{T} \cdot \hat{T}\right) = 0$$

$$\left(\frac{d}{dt}\hat{T}\right) \cdot \hat{T} + \left(\frac{d}{dt}\hat{T}\right) \cdot \hat{T} = 0$$

$$2\,\hat{T} \cdot \left(\frac{d\hat{T}}{dt}\right) = 0$$

Therefore $\frac{d\hat{T}}{dt}$ is perpendicular to $\hat{T}$.



Example: Find the unit tangent to the curve

$$
\begin{aligned}
x(t) &= \cos(t) \\
y(t) &= \sin(t) \\
z(t) &= 0
\end{aligned}
$$

$x(0) = 1$

$y(0) = 0$

For $0 \leq t \leq 2\pi$.

$\underset{\sim}{T} = \dfrac{d\underset{\sim}{r}}{dt}$

So $\hat{\underset{\sim}{T}} = \dfrac{d\underset{\sim}{r}}{dt} \Big/ \left\| \dfrac{d\underset{\sim}{r}}{dt} \right\|$.

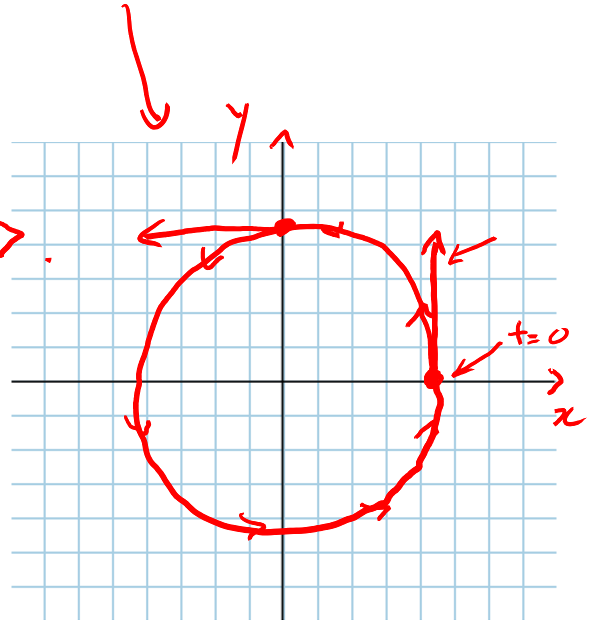
Here $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$

$= \langle \cos t, \sin t, 0 \rangle$

So $\dfrac{d\underset{\sim}{r}}{dt} = \langle -\sin t, \cos t, 0 \rangle$

So $\hat{\underset{\sim}{T}} = \dfrac{\langle -\sin t, \cos t, 0 \rangle}{\sqrt{\sin^2 t + \cos^2 t}}$

$= \langle -\sin t, \cos t, 0 \rangle$.



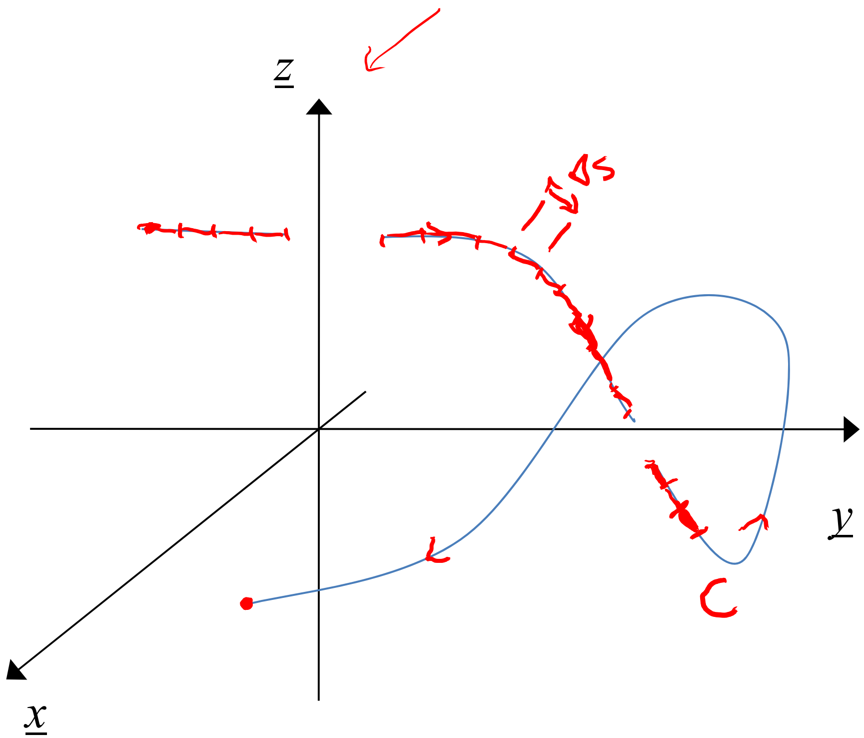
Line integrals of scalar functions

The line integral of a scalar function f(x,y,z) over a path C is

$$\int_{\mathcal{C}} f(x,y,z)ds = \lim_{\Delta S_i \to 0, N \to \infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta S_i$$

We define the symbol *ds* as the infinitessimal arc-length :

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$



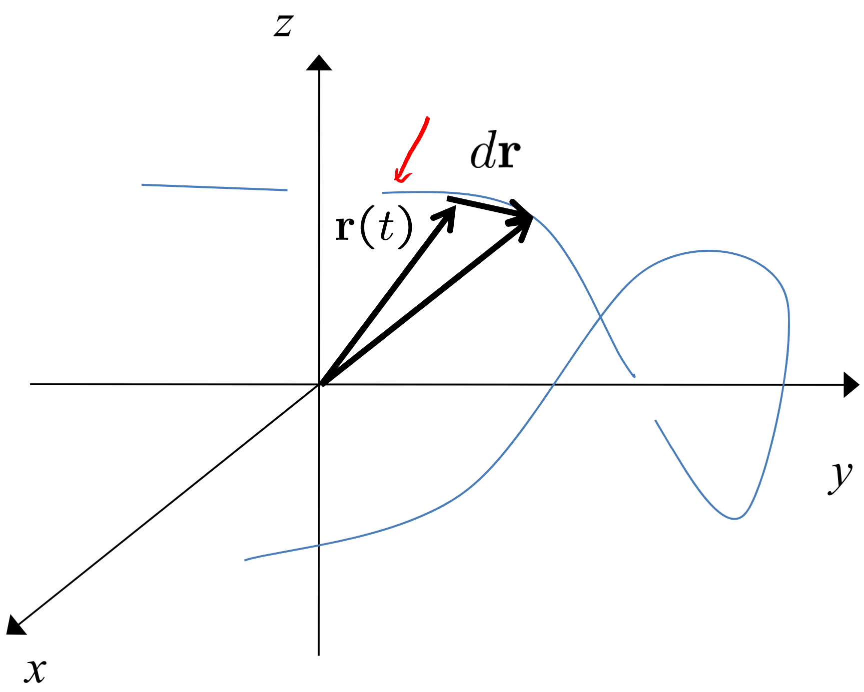
Line integrals of scalar functions

The line integral of a scalar function f(x,y,z) over a path C is

$$\int_C f(x,y,z)ds = \lim_{\Delta S_i \to 0, N \to \infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta S_i$$

We the symbol *ds* is the infinitessimal arc-length :

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$



For a parametrised path,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

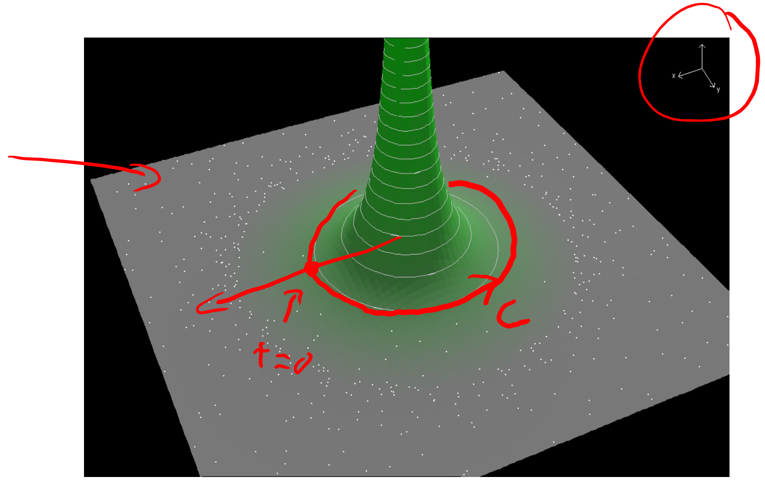
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Then

$$\int_C f(x,y,z)\, ds = \int_{t_0}^{t_1} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example: Evaluate

$$\int_C \frac{1}{\sqrt{x^2+y^2}}\,ds$$



where C is the path given by

$$\begin{aligned} x(t) &= a\cos(t) \\ y(t) &= a\sin(t) \\ z(t) &= 0 \end{aligned}$$

with $0 \le t \le 2\pi$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + 0^2}\,dt$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \sqrt{a^2}\,dt = a\,dt.$$

Now

$$\int_C \frac{1}{\sqrt{x^2+y^2}}\,ds = \int_0^{2\pi} \frac{1}{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}\,a\,dt = \int_0^{2\pi} \frac{1}{a}\,a\,dt$$

$$= \int_0^{2\pi} 1\,dt = \left[t\right]_0^{2\pi} = (2\pi - 0) = 2\pi$$

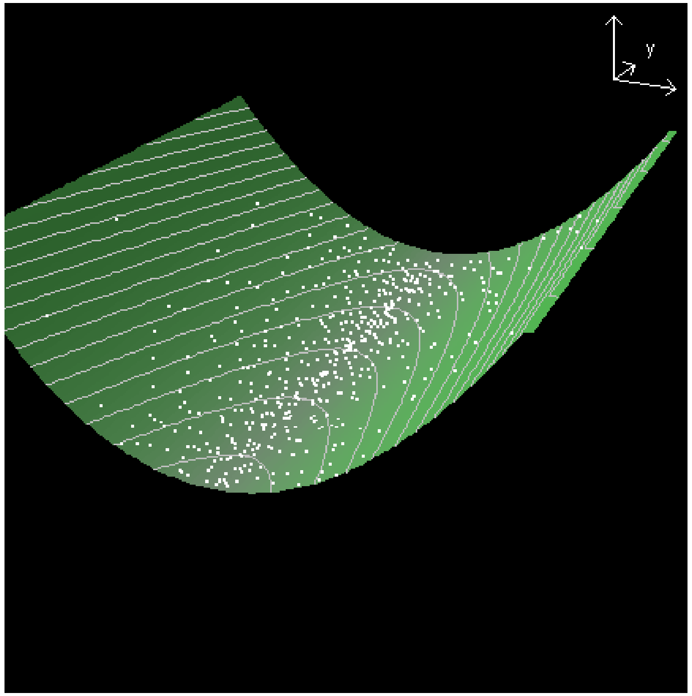
Example: Evaluate

$$\int_C (x^2 + 2y)\,ds$$

Where C is the path given by

$$\begin{aligned} x(t) &= 2t + 2 \\ y(t) &= 2t \\ z(t) &= t \end{aligned}$$

with $0 \leq t \leq 1$



$$\int_C (x^2 + 2y)\,ds = \int_0^1 \left[x^2(t) + 2y(t)\right] \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

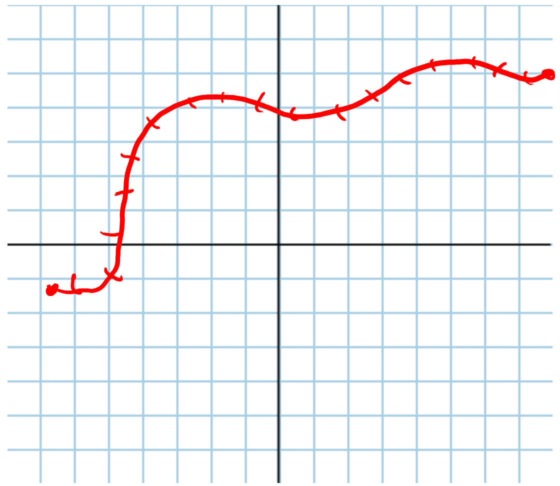
$$= \int_0^1 \left[(2t+2)^2 + 2 \cdot 2t\right] \sqrt{2^2 + 2^2 + 1^2}\,dt = \int_0^1 (4t^2 + 4 + 8t)\sqrt{9}\,dt$$

$$= 3\int_0^1 (4t^2 + 4 + 8t)\,dt = 3\left[\frac{4}{3}t^3 + 4t + 4t^2\right]_0^1 = 3\left(\frac{4}{3} + 4 + 4\right) - 0$$

$$= 4 + 24 = 28.$$

The *length* of a path C is

$$L = \int_C ds \quad = \int_c 1 \, ds$$

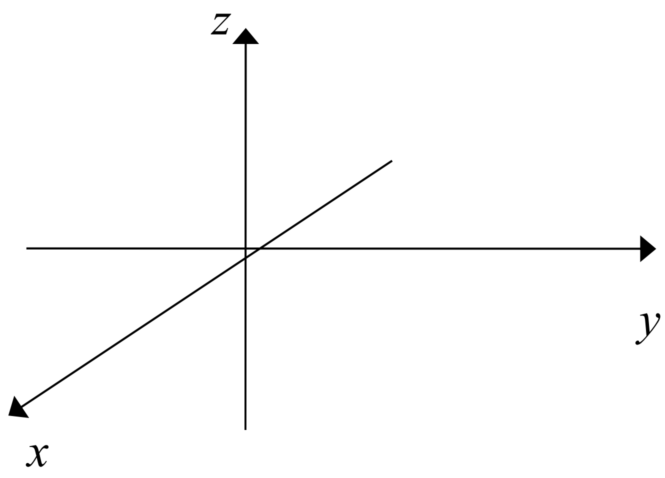


Example: Show that the diameter of a circle with radius R is 2 ` R.

Left as exercise.

Example: find the length of the curve

$$\begin{cases} x(t) = t \\ y(t) = \dfrac{1}{\sqrt{2}}t^2 \\ z(t) = \dfrac{1}{3}t^3 \end{cases}$$



With $0 \leq t \leq 1$

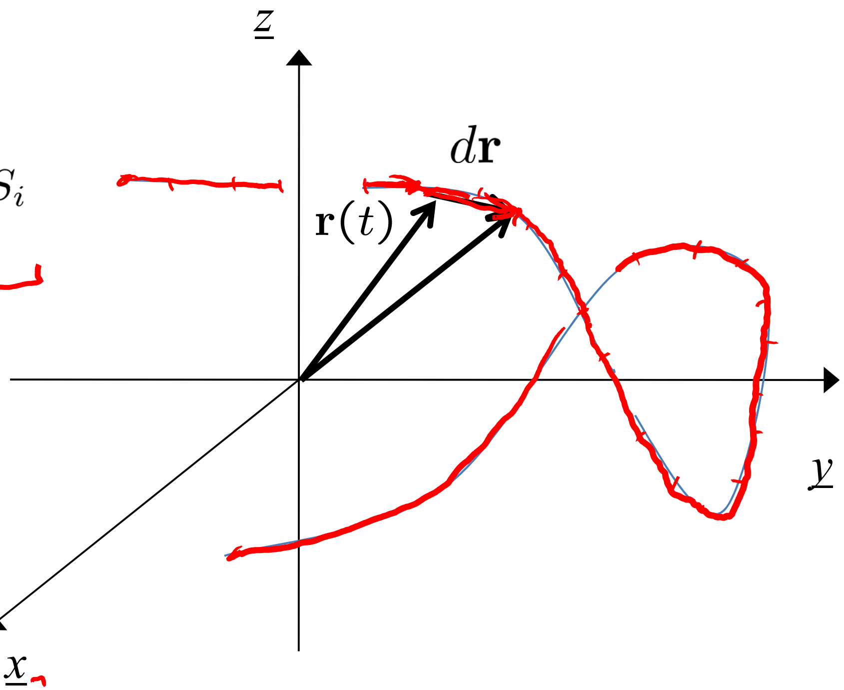
$$L = \int_C 1 \, ds = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

Ans: $\frac{4}{3}$.

Line integrals of vector fields

The line integral of a vector field $\mathbf{F}(x,y,z)$ over a path C is

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \lim_{\Delta S_i \to 0, N\to\infty} \sum_{i=1}^{N} \mathbf{F}(x_i,y_i,z_i)\cdot \hat{\mathbf{T}}\Delta S_i$$



$d\mathbf{r}$ is the infinitessimal displacement vector

$$d\mathbf{r} = \lim_{\Delta S_i \to 0, N\to\infty} \hat{\mathbf{T}}\ \Delta S_i$$

which we can write as

$$d\mathbf{r} = \langle dx, dy, dz\rangle = \underset{\sim}{i}\,dx + \underset{\sim}{j}\,dy + \underset{\sim}{k}\,dz\,.$$

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_C F_x\,dx + \int_C F_y\,dy + \int_C F_z\,dz$$

For a parametrised path,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

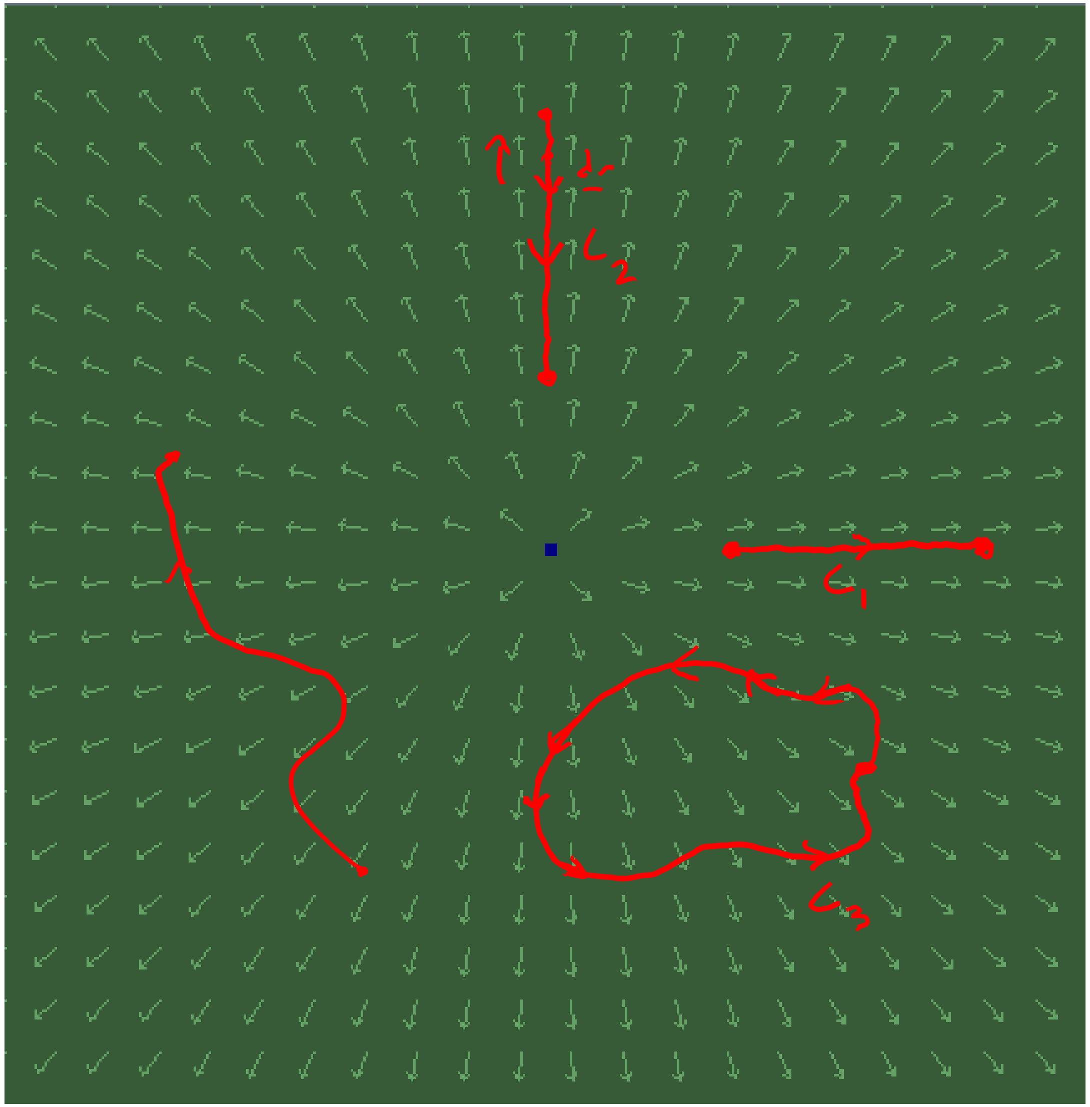
so

$$\int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_C \underline{F} \cdot \left(\frac{d\underline{r}}{dt}\right) dt$$

$$= \int_C \underline{F} \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

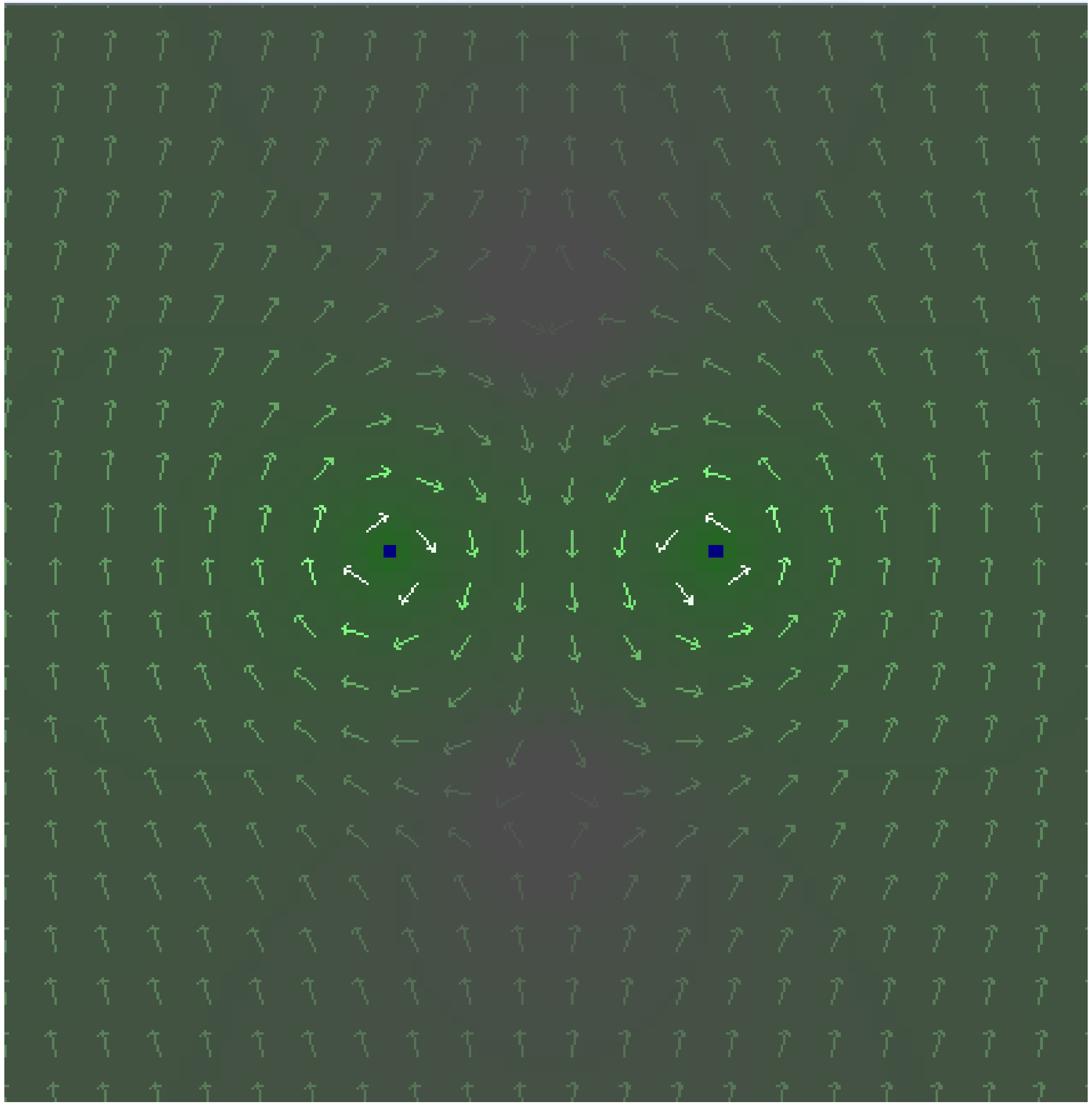
If F represents a *force,* then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ represents the *work done* by the force along the path C.

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} < 0$$



$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} > 0$$

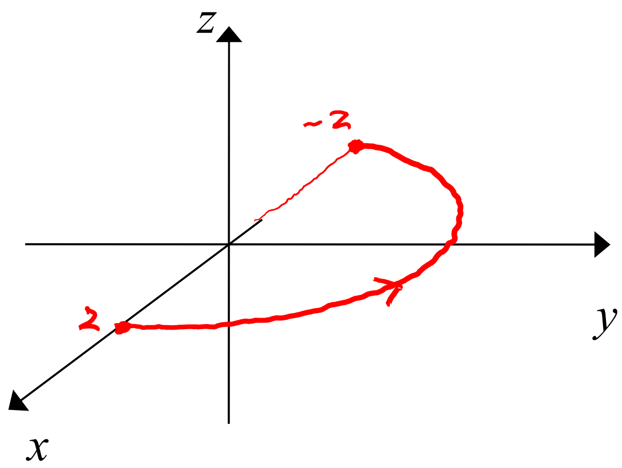
$$\int_{C_3} \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0.$$



Example: Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and C is the curve parametrised by

$$\begin{cases} x(t) = 2\cos(t) \\ y(t) = 2\sin(t) \\ z(t) = 0 \end{cases} \qquad 0 \le t \le \pi$$

begin end



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \right) dt$$

don't forget!

$$= \int_0^{\pi} \langle -y, x, 1 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{\pi} \langle -2\sin t, 2\cos t, 1 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{\pi} \left(4\sin^2 t + 4\cos^2 t + 0 \right) dt = \int_0^{\pi} 4 \, dt = \left[4t \right]_0^{\pi} = 4\pi .$$

Method for doing vector line integrals:

1. Parametrise the curve (i.e. Find $\mathbf{r}(t)$)

2. Write $d\mathbf{r} = (d\mathbf{r}/dt)dt$ $\frac{d\mathbf{r}}{dt} dt$

3. Substitute $\mathbf{F}(x,y,z)$ by $\mathbf{F}(x(t),y(t),z(t))$

4. Integrate!

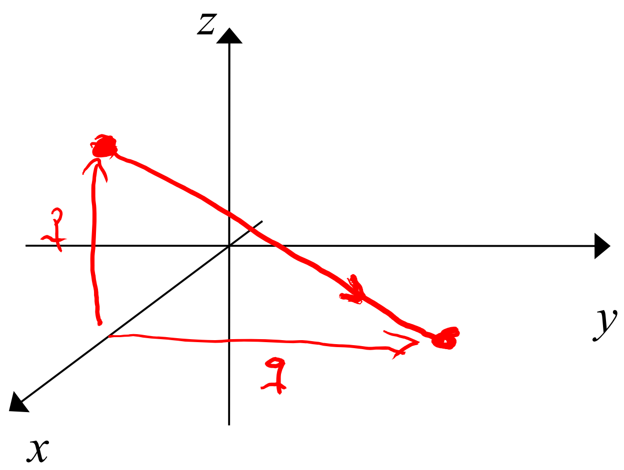
Example: Calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

and C is the straight line going from <2,-1,3> to <4,2,-1>.



1. Find $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$

General formula for a straight line is

$$\underset{\sim}{r}(t) = \underset{\sim}{p} + \underset{\sim}{a}\, t \leftarrow$$

$r = q = p + (q-p)$
$= p + (q-p) \times 1$
$t = 1$

where

$\underset{\sim}{p}$ = a point on the line = $\langle 2,-1,3 \rangle$.

and $\underset{\sim}{a}$ is the direction of the line

$$\longrightarrow \underset{\sim}{a} = \underset{\sim}{q} - \underset{\sim}{p} = \langle 4,2,-1 \rangle - \langle 2,-1,3 \rangle$$

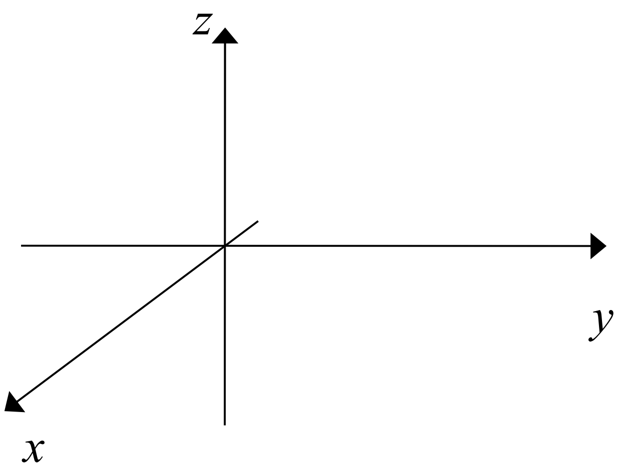
$$= \langle 2,3,-4 \rangle$$

so

$$\underset{\sim}{r}(t) = \langle 2,-1,3 \rangle + \langle 2,3,-4 \rangle t = \langle 2+2t, -1+3t, 3-4t \rangle$$

$\underset{\sim}{r}(t) = \langle 2+2t, -1+3t, 3-4t \rangle$

($x(t)$, $y(t)$, $z(t)$ marked under the components)



2. Differentiate:

$$\frac{d\underset{\sim}{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$= \langle +2, 3, -4 \rangle.$$

3. Find $\underset{\sim}{F}$ in terms of t:

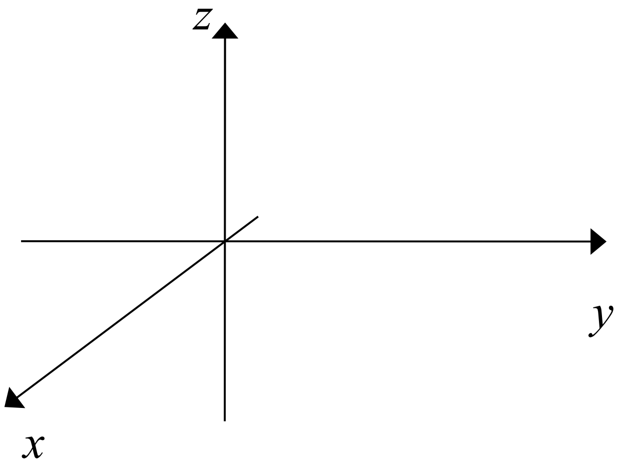
$$\underset{\sim}{F} = \langle yz, xz, xy \rangle = \langle (-1+3t)(3-4t), (2+2t)(3-4t), (2+2t)(-1+3t) \rangle$$

4. Integrating:

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 \langle (-1+3t)(3-4t), (2+2t)(3-4t), (2+2t)(-1+3t) \rangle \cdot \langle 2, 3, -4 \rangle \, dt.$$

$$= \int_0^1 \left[2(-1+3t)(3-4t) + 3(2+2t)(3-4t) - 4(2+2t)(-1+3t) \right] dt.$$

$$= \int_0^1 \Big[2\left(-3 + 4t + 9t - 12t^2\right) + 3\left(6 - 8t + 6t - 8t^2\right) - 4\left(-2 + 6t - 2t + 6t^2\right) \Big] dt$$



$$= \int_0^1 \Big[-6 + 8t + 18t - 24t^2 + 18 - 24t + 18t - 24t^2 + 8 - 24t + 8t - 24t^2 \Big] dt$$

$$= \int_0^1 \left(20 + 4t - 72t^2\right) dt = \left[20t + 2t^2 - \frac{72}{3} t^3 \right]_0^1$$

$$= 20 + 2 - 24 = -2.$$

$\square$

The fundamental theorem of calculus

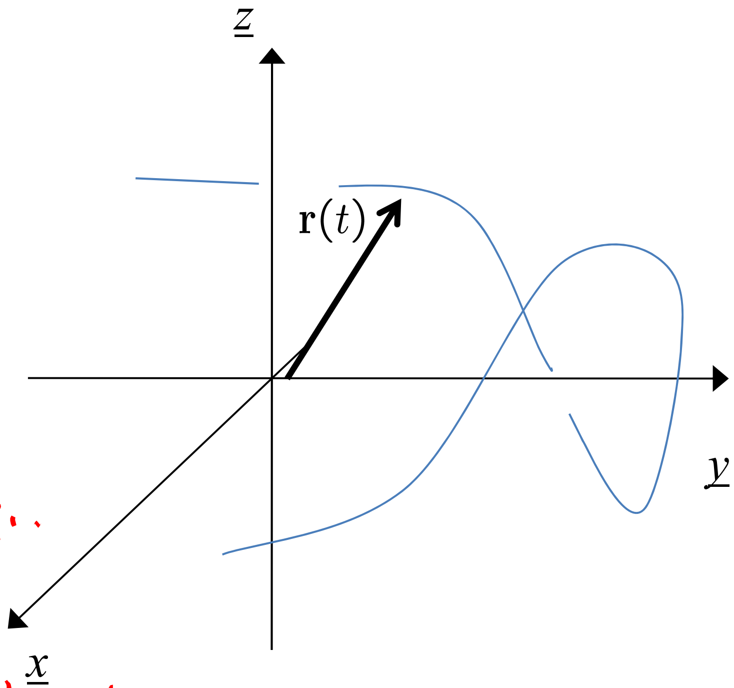
Given a one dimensional function $\phi(x)$, the *fundamental theorem* states that

$$\int_a^b \frac{d\phi}{dx} dx = \phi(b) - \phi(a)$$

$$\int_a^b f(x)\, dx = \left[\phi(x)\right]_a^b \qquad f = \frac{d\phi}{dx}$$

For a function ˘(x,y,z), the *fundamental theorem in 3D* states that

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} (\nabla\phi)\cdot d\mathbf{r} = \phi(\mathbf{r}_1) - \phi(\mathbf{r}_0)$$



Proof: Consider a path C parametrized by $\underset{\sim}{r}(t)$, with $t_0 \leq t \leq t_1$ and $\underset{\sim}{r}(t_0) = \underset{\sim}{r}_0$, $\underset{\sim}{r}(t_1) = \underset{\sim}{r}_1$.

Then

$$\int_{\underset{\sim}{r}_0}^{\underset{\sim}{r}_1} (\nabla\phi)\cdot d\underset{\sim}{r} = \int_{t_0}^{t_1} \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right\rangle \cdot \frac{d\underset{\sim}{r}}{dt}\, dt$$

$$= \int_{t_0}^{t_1} \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt.$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right] dt$$

But

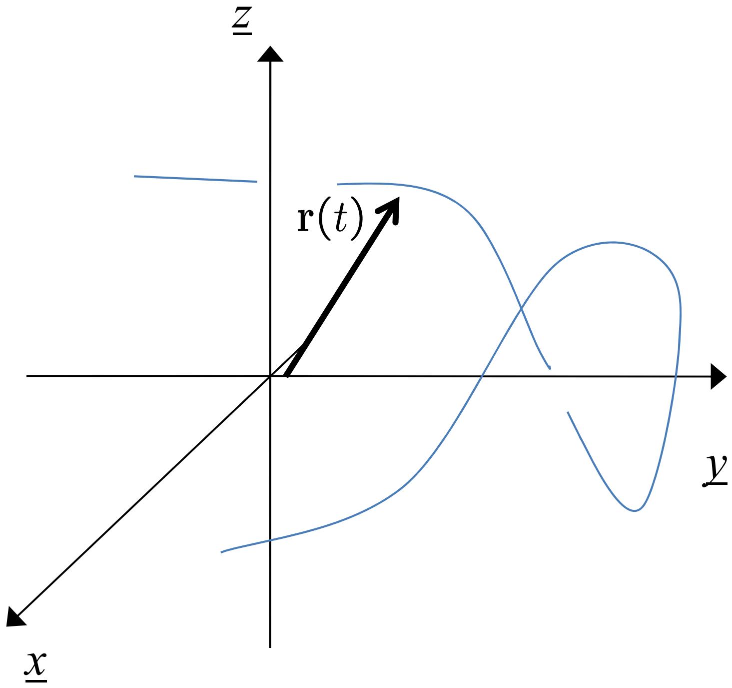
$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

So

$$= \int_{t_0}^{t_1} \frac{d\phi}{dt}\, dt = \phi(t_1) - \phi(t_0)$$

$$= \phi(\underline{r}_1) - \phi(\underline{r}_0).$$

$\square$.



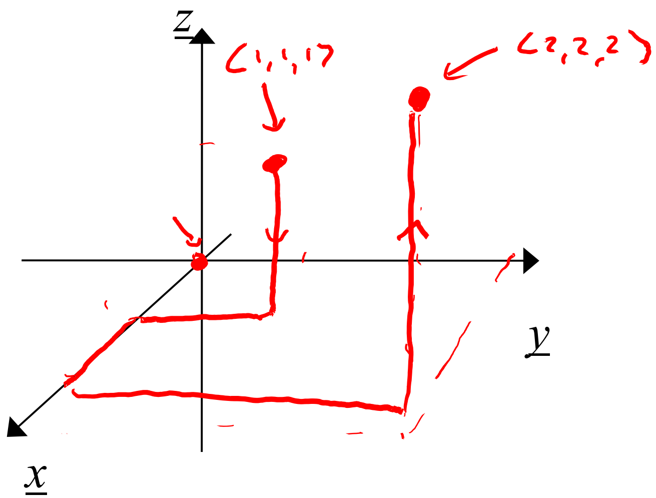
Example: For the scalar function

$$\phi(x, y, z) = \ln(xyz)$$

Calculate

$$\int_C (\nabla\phi) \cdot d\mathbf{r}$$

Along the path from <1,1,1> to <2,2,2>.



$$\int_C (\nabla\phi)\cdot d\mathbf{r} = \phi(\text{end}) - \phi(\text{beginning})$$

$$= \ln(2\times2\times2) - \ln(1\times1\times1)$$

$$= \ln 8 - \ln 1$$

$$= \ln 8$$

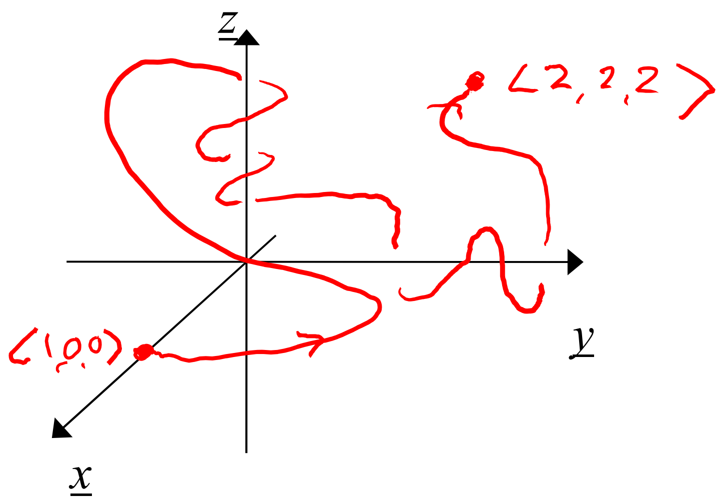
Example: For the scalar function

$$\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Calculate

$$\int_C (\nabla\phi)\cdot d\mathbf{r}$$

where C is the following path:



$$\int_C (\nabla\phi)\cdot d\mathbf{r}$$

$$= \phi(2,2,2) - \phi(1,0,0)$$

$$= \frac{1}{\sqrt{4+4+4}} - \frac{1}{\sqrt{1+0+0}} = \frac{1}{\sqrt{12}} - 1$$

Conservative fields

A vector field F is *conservative* if it can be written

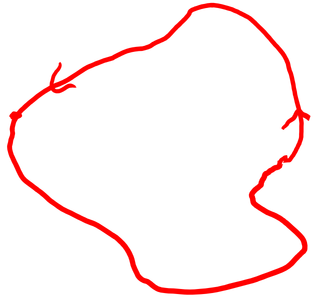
$$\underline{F} = \nabla \phi$$

scalar potential

In this case, the line integral of the field only ever depends on the endpoints:

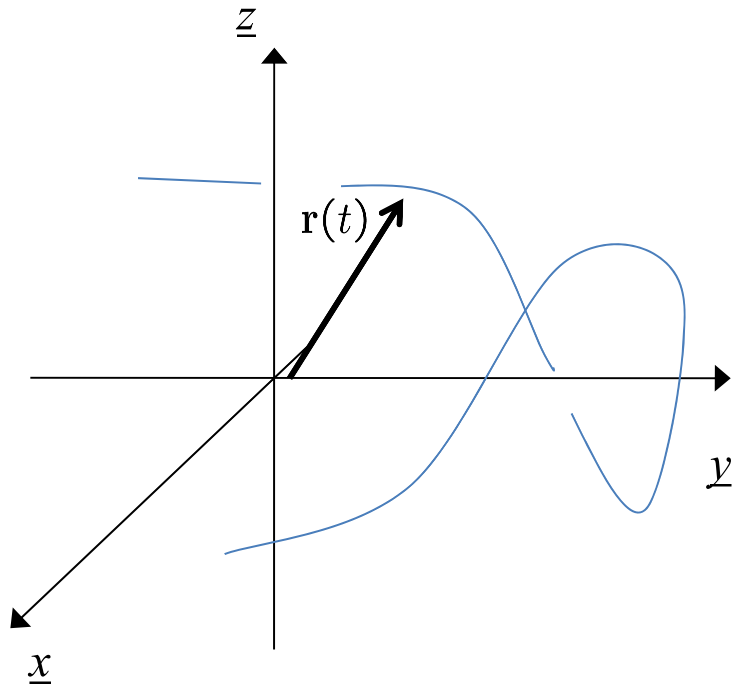
$$\int_C \underline{F} \cdot d\underline{r} = \phi(\text{end}) - \phi(\text{beginning})$$

If the path is closed, then



closed path $\longrightarrow$ $$\oint_C \underline{F} \cdot d\underline{r} = 0.$$

If the line integral is along a *closed path*, then for a conservative field,



Conservative fields are *irrotational*, i.e.

$$\nabla \times \underset{\sim}{F} = 0 .$$

i.e.

→ If $\underset{\sim}{F} = \nabla \phi$ then

$$\nabla \times \underset{\sim}{F} = \nabla \times (\nabla \phi)$$

$$= \underset{\sim}{0} \leftarrow$$

It also works the other way:

→ If $\nabla \times \underset{\sim}{F} = 0$ then there exist some ϕ such that $\underset{\sim}{F} = \nabla \phi$.

Example:

Calculate the curl of the field

$$\mathbf{F} = x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Hence or otherwise calculate the integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the circle of radius 2, centred on the z-axis and lying in the plane z = 57.

$$\nabla \times \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & z \end{vmatrix} = \hat{\underline{i}}(0-0) - \hat{\underline{j}}(0-0) + \hat{\underline{k}}(0-0) = \underline{0}.$$

Therefore $\underline{F}$ is irrotational, so is conservative, and so

$$\oint_C \underline{F} \cdot d\underline{r} = 0.$$

The following statements are <u>equivalent</u>:

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$$\mathbf{F} = \nabla \phi \qquad \text{for some function } \phi$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \qquad \text{for all closed paths } C.$$

Finding the potential function

If a vector field is *irrotational* then we always find a potential function such that

$$\mathbf{F} = \nabla\phi$$

Example:

$$\mathbf{F} = \sin y\hat{\mathbf{i}} + (1 + x\cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

First check that $\nabla \times \underline{F} = \underline{0}$:

$$\nabla \times \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{j} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & 1 + x\cos y & 1 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{\underline{k}}(\cos y - \cos y)$$

$$= \underline{0}. \quad \checkmark$$

$\mathbf{F} = \sin y\hat{\mathbf{i}} + (1 + x\cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\underline{F} = \nabla\phi$

We know that

$$\frac{\partial \phi}{\partial x} = \sin y \quad , \quad \frac{\partial \phi}{\partial y} = 1 + x\cos y \quad , \quad \frac{\partial \phi}{\partial z} = 1$$

F_y

Integrate "partially" w.r.t. x:

$$\Rightarrow \phi(x,y,z) = \int \sin y \, dx + A(y,z)$$

$$= x\sin y + A(y,z)$$

Differentiate this w.r.t. y:

$$\frac{\partial \phi}{\partial y} = x\cos y + \frac{\partial A}{\partial y} \Rightarrow x\cos y + \frac{\partial A}{\partial y} = 1 + x\cos y$$

$$\frac{\partial A}{\partial y} = 1$$

$$\mathbf{F} = \sin y\hat{\mathbf{i}} + (1 + x\cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\frac{\partial A}{\partial y} = 1$$

Integrate w.r.t. y:

$$\rightarrow A(y,z) = \int 1\, dy + B(z)$$

$$= y + B(z)$$

So $\phi(x,y,z) = x\sin y + y + B(z)$.

Differentiate w.r.t. z:

$$\frac{\partial \phi}{\partial z} = 0 + 0 + \frac{\partial B}{\partial z} \Rightarrow \frac{\partial B}{\partial z} = 1$$

So $B(z) = z + C$.

Therefore $\phi(x,y,z) = x\sin y + y + z + C$.

$$\mathbf{F} = \sin y \hat{\mathbf{i}} + (1 + x \cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Example: $$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

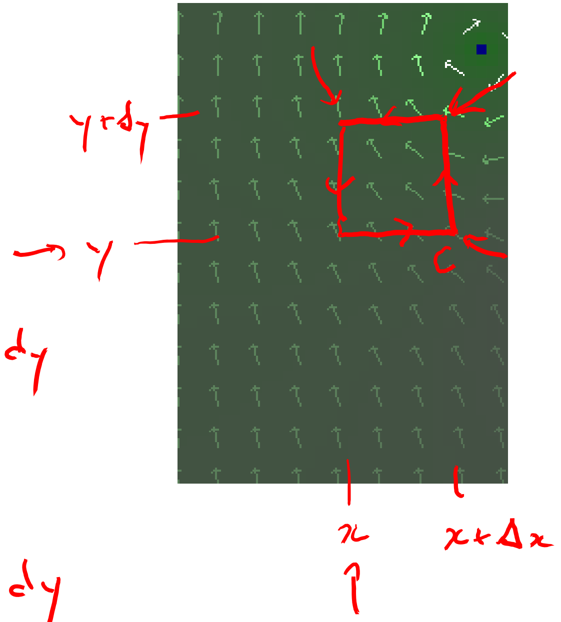
$$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

$$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

The circulation of a vector field

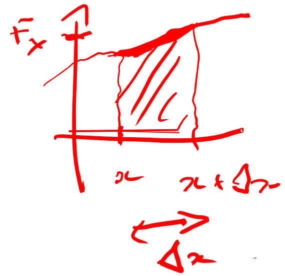
Consider a closed loop integral in a vector field F.

What happens as the area goes to zero?



$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_x^{x+\Delta x} F_x(x,y)\, dx + \int_y^{y+\Delta y} F_y(x+\Delta x, y)\, dy$$

$$- \int_x^{x+\Delta x} F_x(x, y+\Delta y)\, dx - \int_y^{y+\Delta y} F_y(x,y)\, dy$$



$$\approx F_x(x,y)\,\Delta x + F_y(x+\Delta x, y)\,\Delta y - F_x(x, y+\Delta y)\,\Delta x - F_y(x,y)\,\Delta y$$

$$= \Delta x \Delta y \left[\underbrace{\frac{F_y(x+\Delta x, y) - F_y(x,y)}{\Delta x}}_{\frac{\partial F_y}{\partial x}} - \underbrace{\frac{F_x(x, y+\Delta y) - F_x(x,y)}{\Delta y}}_{\frac{\partial F_x}{\partial y}} \right]$$

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} \approx \Delta x \, \Delta y \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right]$$

$$\nabla \times \underset{\sim}{F} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \ldots \ldots + \hat{\underset{\sim}{k}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Alternative definition of the curl:

$$\nabla \times \mathbf{F} = \hat{\mathbf{n}} \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

Where ƒ S is the area of the loop C and **n** is the unit normal vector to this area element.