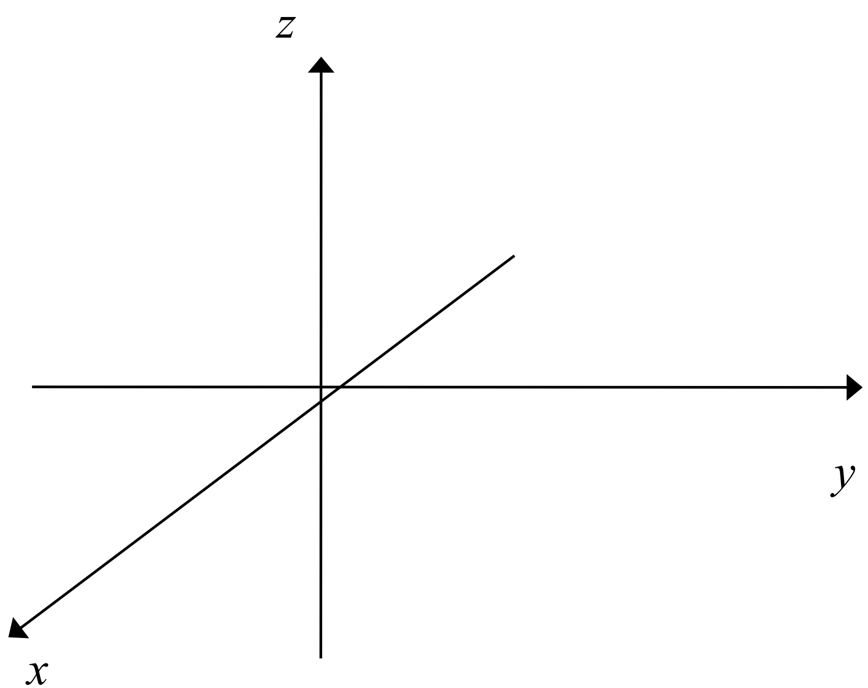
Paths in 3D

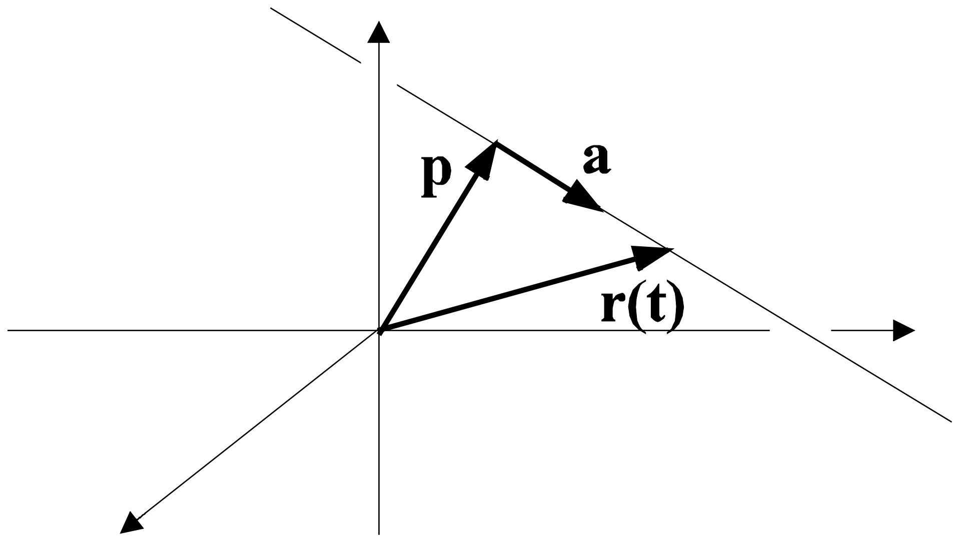
A path in three dimensions can be written in *parametric form* as

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

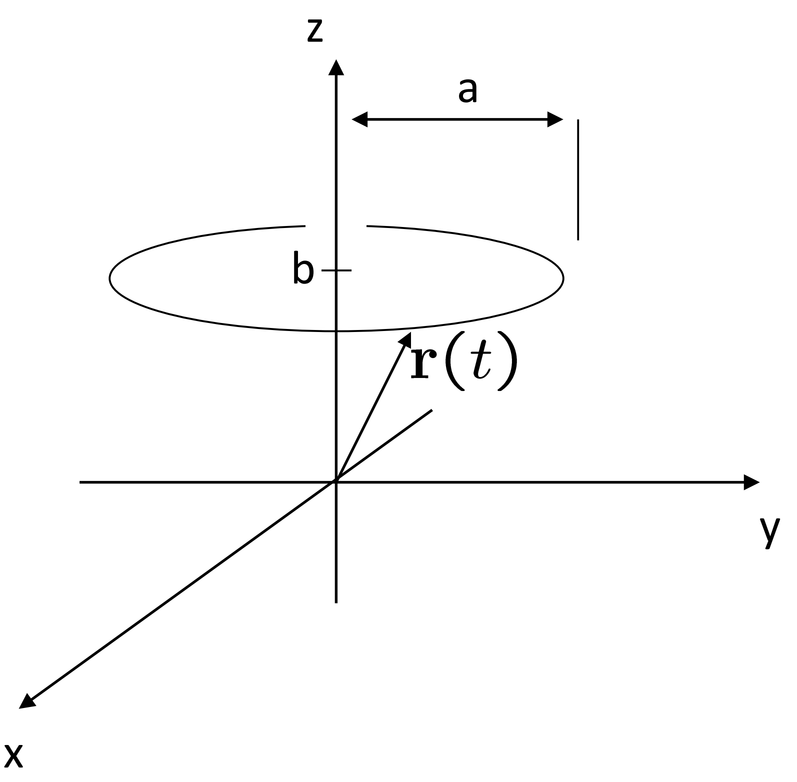
where $t \in [t_0, t_1]$



E.g. A straight line through a point **p**, parallel to a vector **a** is

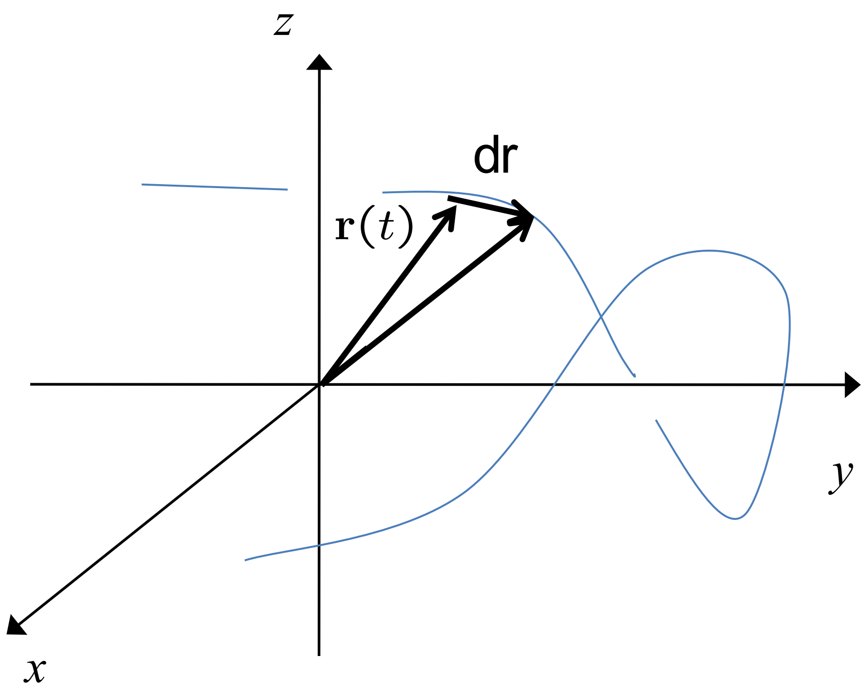


E.g. A circle at constant $z = b$ with radius a:



The infinitessimal displacement vector
Is the vector differential

$$d\mathbf{r} = \langle dx, dy, dz \rangle$$



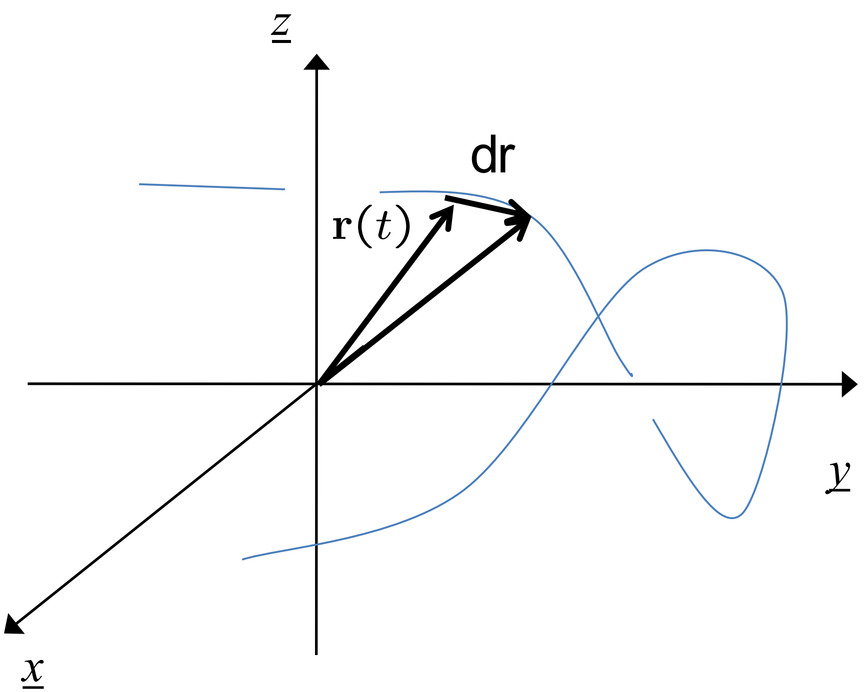
The infinitessimal arc-length is

$$ds = |d\mathbf{r}|$$

Tangents and normals

$d\mathbf{r}/dt$ is always parallel to the curve, and so is a *tangent vector*

The unit tangent vector is then



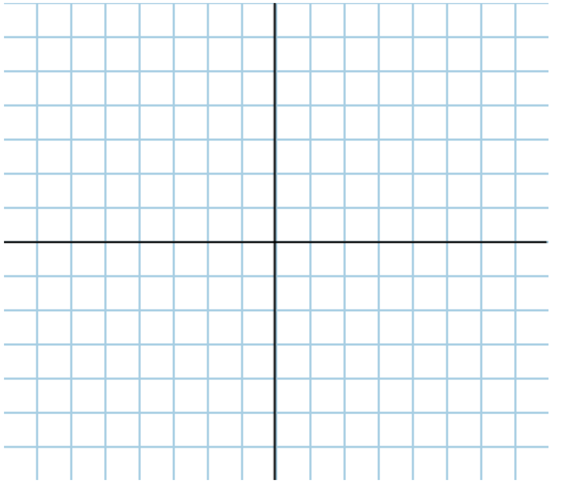
We can define a normal to the curve by noting that

$$\hat{\mathbf{T}} \cdot \hat{\mathbf{T}} = 1$$

Example: Find the unit tangent to the curve

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \\ z(t) &= 0 \end{aligned}$$

For $0 \leq t \leq 2\pi$.



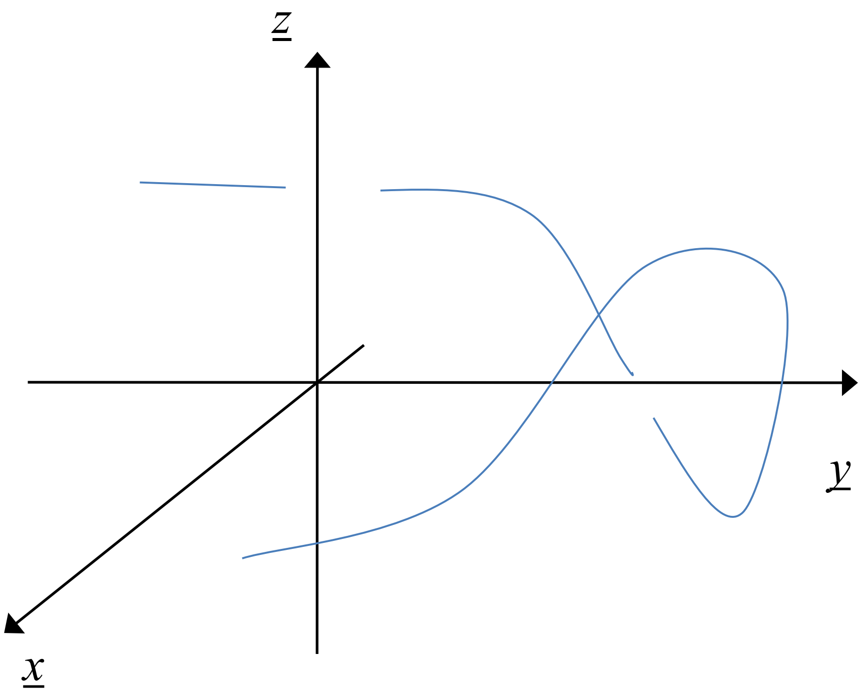
Line integrals of scalar functions

The line integral of a scalar function f(x,y,z) over a path C is

$$\int_C f(x,y,z)ds = \lim_{\Delta S_i \to 0, N \to \infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta S_i$$

We define the symbol *ds* as the infinitessimal arc-length :

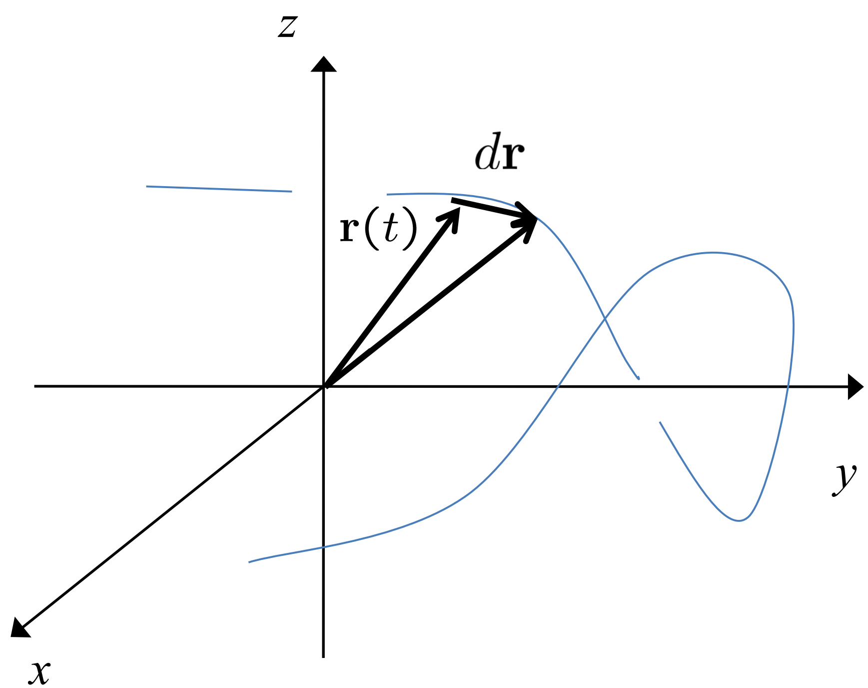
$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$



Line integrals of scalar functions

The line integral of a scalar function f(x,y,z) over a path C is

$$\int_C f(x,y,z)ds = \lim_{\Delta S_i \to 0, N\to\infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta S_i$$



We the symbol *ds* is the infinitessimal arc-length :

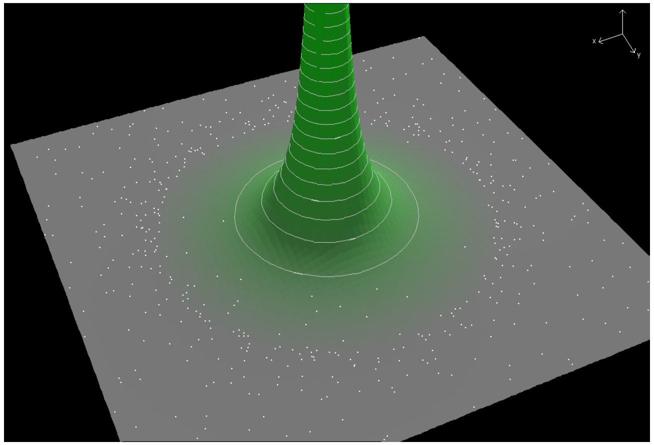
$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

For a parametrised path,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

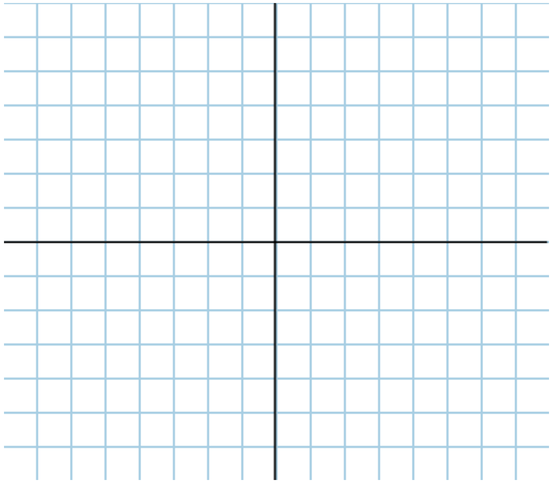
Example: Evaluate

$$\int_C \frac{1}{\sqrt{x^2+y^2}}\,ds$$



where C is the path given by

$$\begin{aligned} x(t) &= a\cos(t) \\ y(t) &= a\sin(t) \\ z(t) &= 0 \end{aligned}$$



with $0 \leq t \leq 2\pi.$

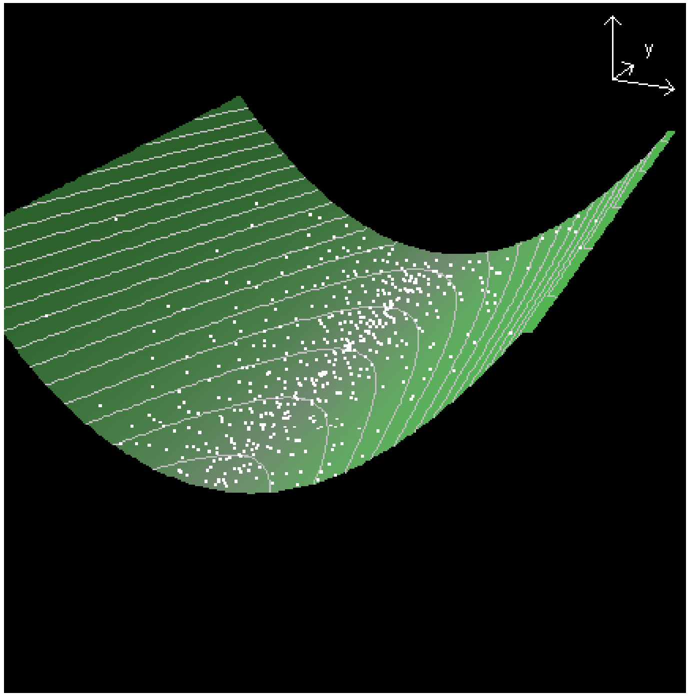
Example: Evaluate

$$\int_C (x^2 + 2y)ds$$

Where C is the path given by

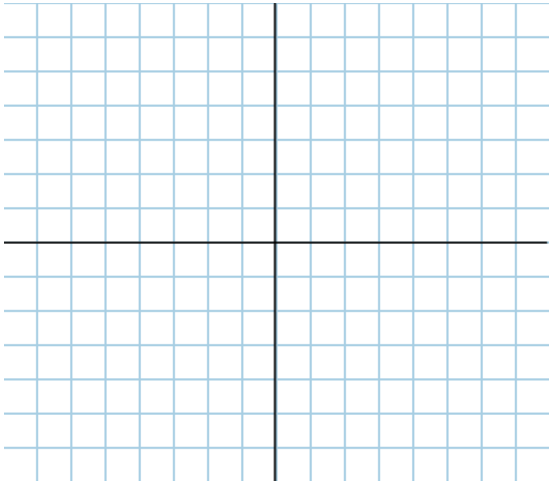
$$\begin{aligned} x(t) &= 2t + 2 \\ y(t) &= t/2 \\ z(t) &= t \end{aligned}$$

with $0 \leq t \leq 1$



The *length* of a path C is

$$L = \int_{\mathcal{C}} ds$$

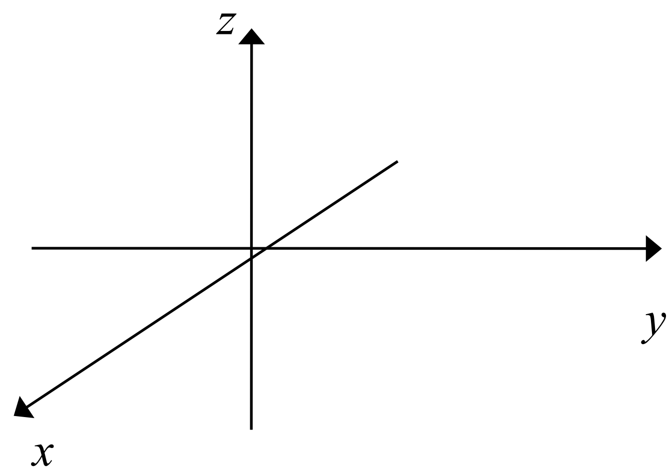


Example: Show that the diameter of a circle with radius R is 2 ¼R.

Example: find the length of the curve

$$
\begin{aligned}
x(t) &= t \\
y(t) &= \frac{1}{\sqrt{2}}t^2 \\
z(t) &= \frac{1}{3}t^3
\end{aligned}
$$

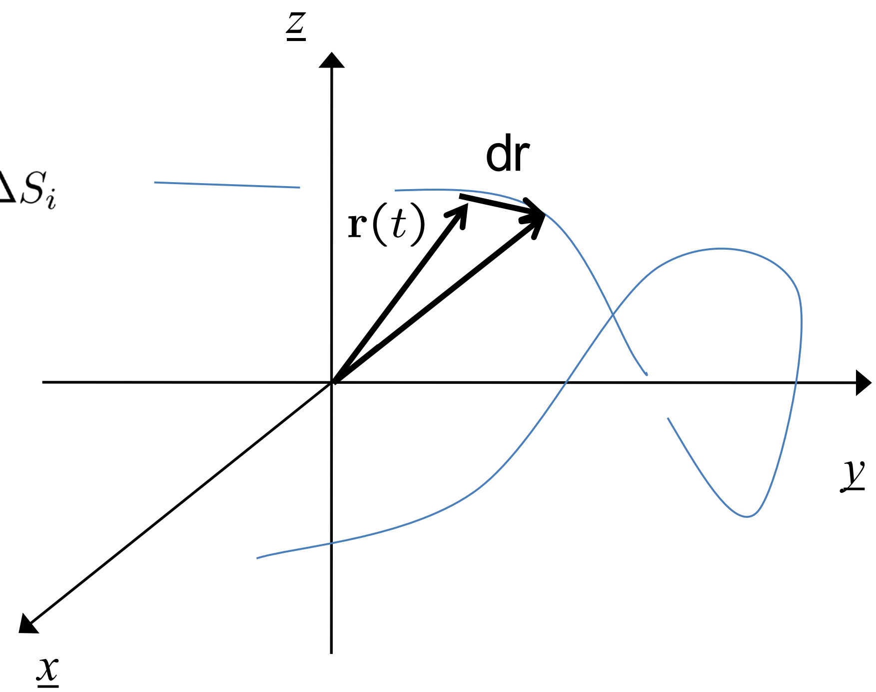
With $0 \leq t \leq 1$



Line integrals of vector fields

The line integral of a vector field $\mathbf{F}(x,y,z)$ over a path C is

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} \lim_{\Delta S_i \to 0, N\to\infty} \sum_{i=1}^{N} \mathbf{F}(x_i,y_i,z_i)\cdot \hat{\mathbf{T}}\Delta S_i$$



*d***r** is the infinitessimal displacement vector

$$d\mathbf{r} = \lim_{\Delta S_i \to 0, N\to\infty} \hat{\mathbf{T}}\ \Delta S_i$$

which we can write as

$$d\mathbf{r} = \langle dx, dy, dz\rangle$$

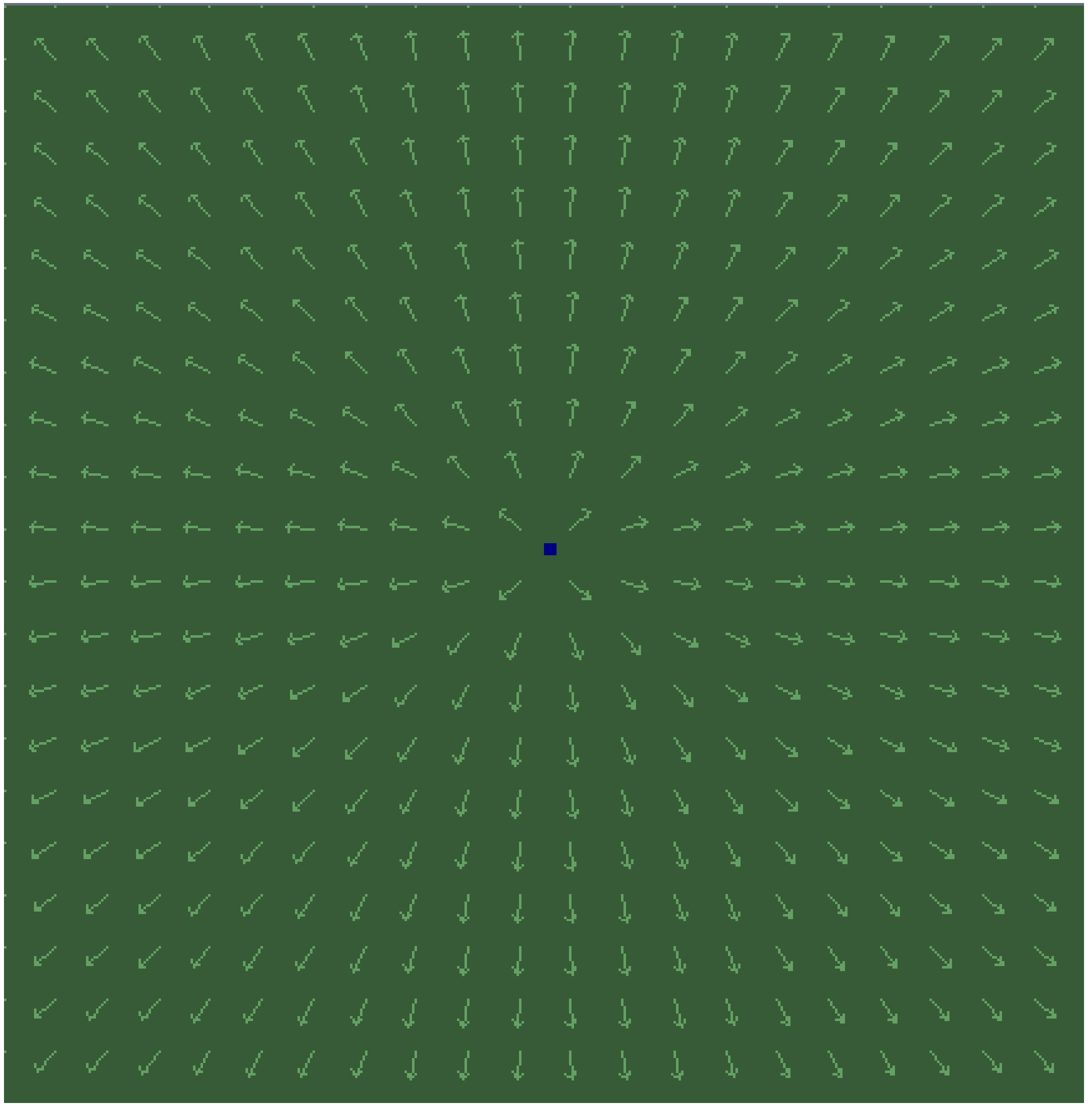
For a parametrised path,

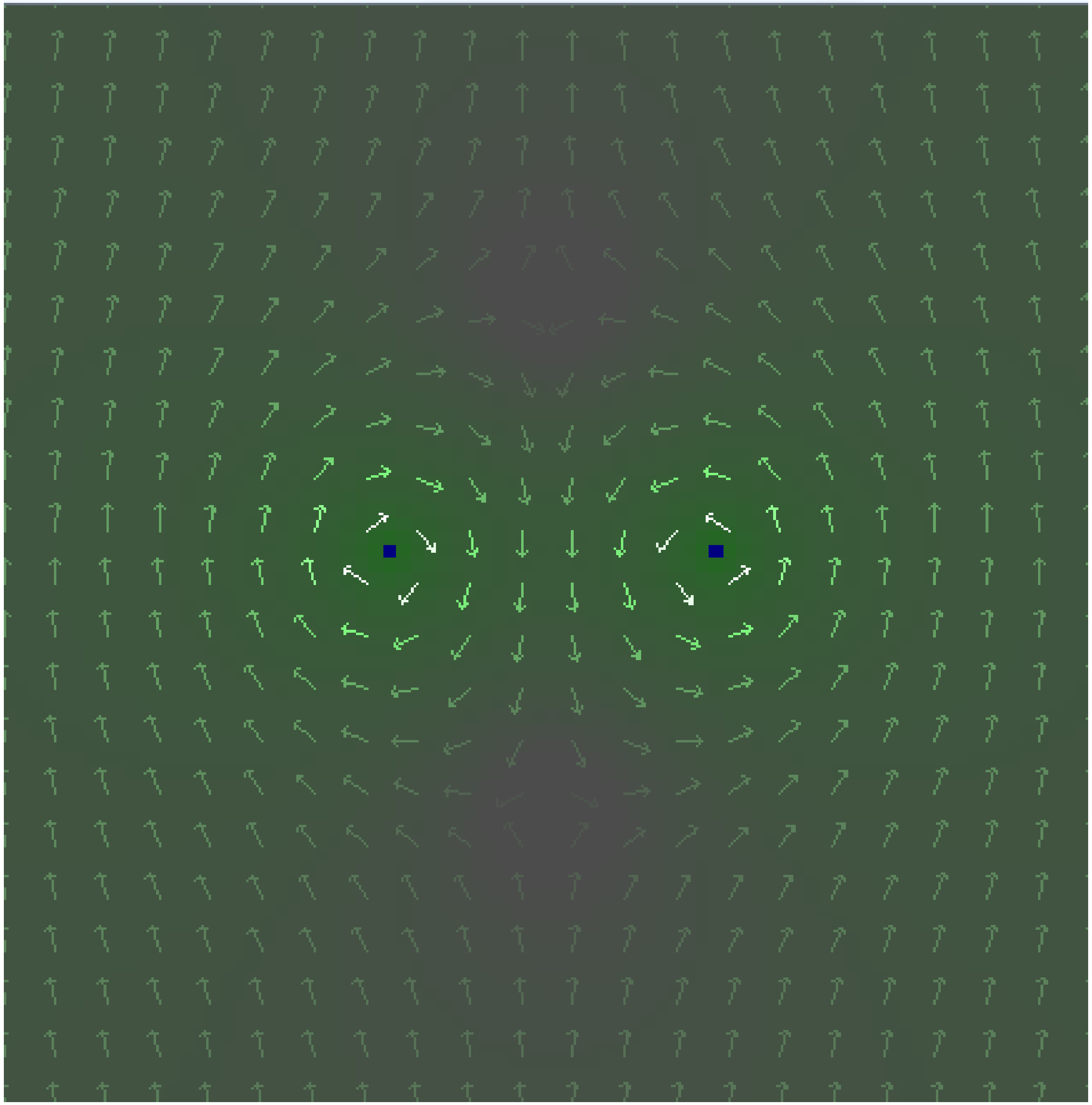
$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

so

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} =$$

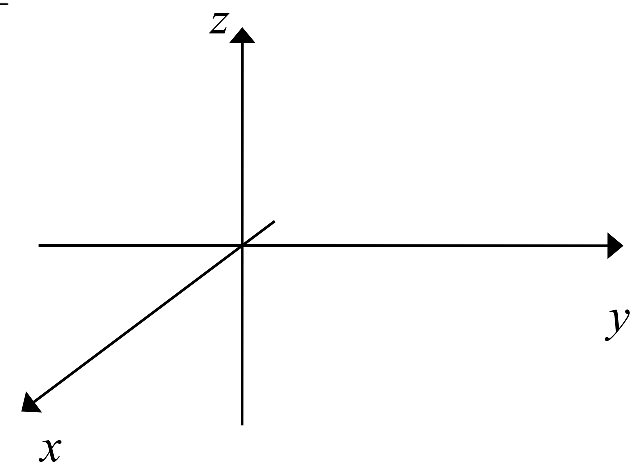
If F represents a *force,* then $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the *work done* by the force along the path C.





Example: Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + \hat{\mathbf{k}}$
and C is the curve parametrised by

$$\begin{aligned} x(t) &= 2\cos(t) \\ y(t) &= 2\sin(t) \\ z(t) &= 0 \end{aligned} \qquad 0 \leq t \leq \pi$$



Method for doing vector line integrals:

1. Parametrise the curve (i.e. Find $\mathbf{r}(t)$)

2. Write $d\mathbf{r} = d\mathbf{r}/dt\ dt$

3. Substitute $\mathbf{F}(x,y,z)$ by $\mathbf{F}(x(t),y(t),z(t))$

4. Integrate!

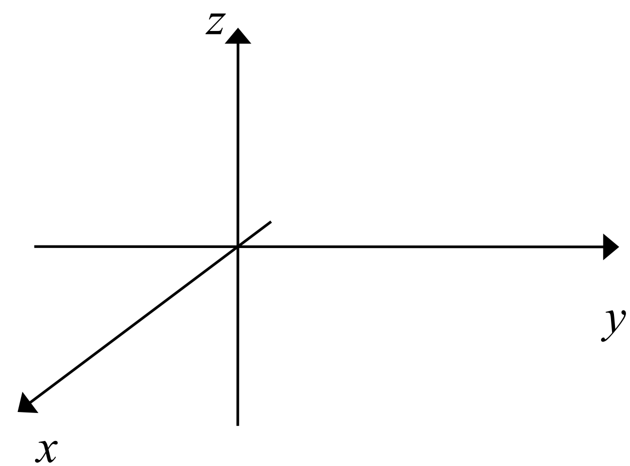
Example: Calculate

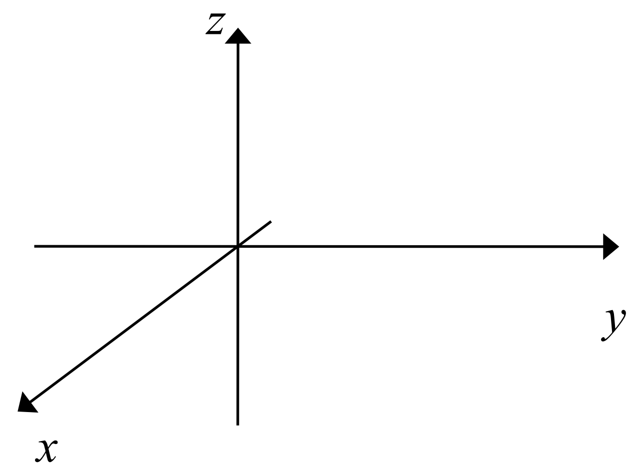
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

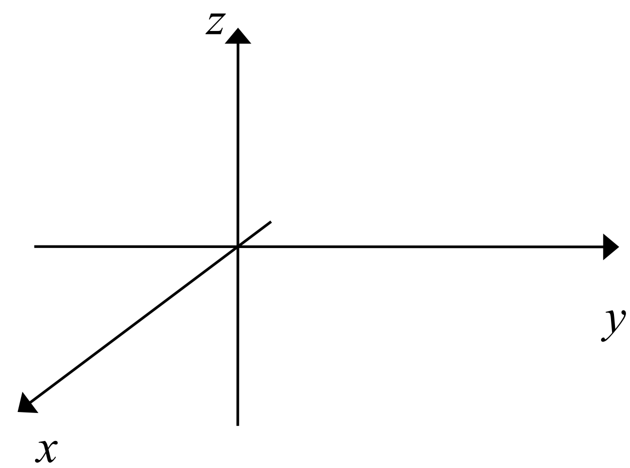
where

$$\mathbf{F} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

and C is the straight line going from <2,-1,3> to <4,2,-1>.




z
y
x


z
y
x

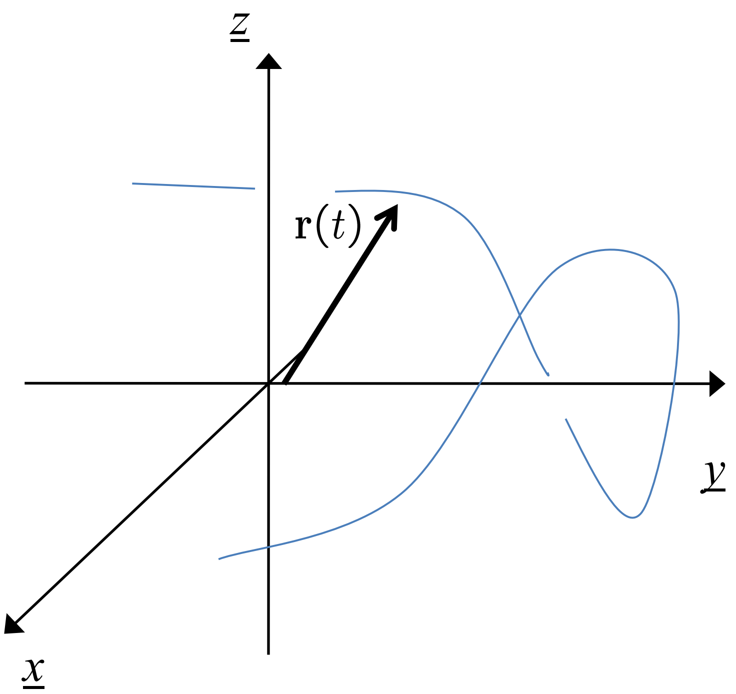
The fundamental theorem of calculus

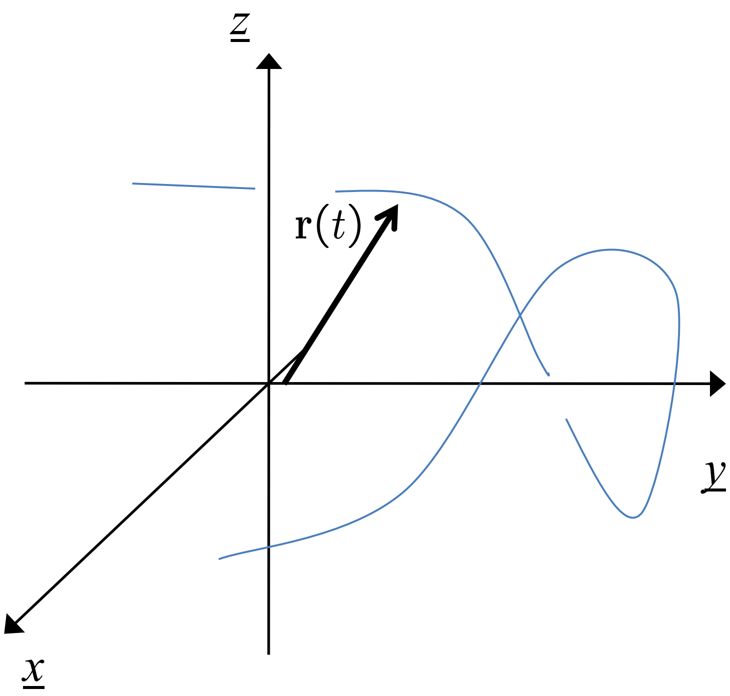
Given a one dimensional function $\phi(x)$, the *fundamental theorem* states that

$$\int_a^b \frac{d\phi}{dx} dx = \phi(b) - \phi(a)$$

For a function Á(x,y,z), the *fundamental theorem in 3D* states that

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} (\nabla \phi) \cdot d\mathbf{r} = \phi(\mathbf{r}_1) - \phi(\mathbf{r}_0)$$




z
r(t)
y
x

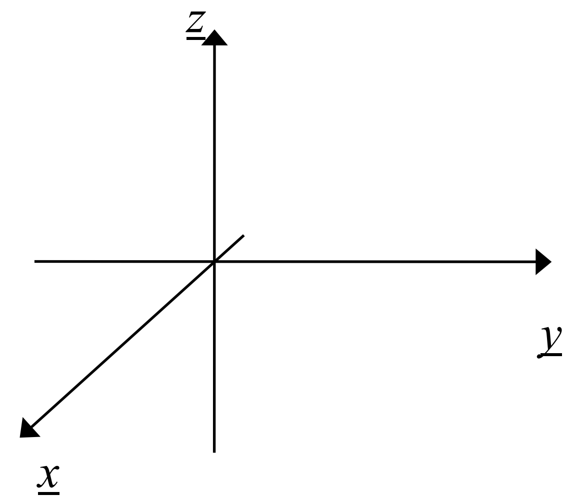
Example: For the scalar function

$$\phi(x, y, z) = \ln(xyz)$$

Calculate

$$\int_{\mathcal{C}} (\nabla \phi) \cdot d\mathbf{r}$$

Along the path from <1,1,1> to <2,2,2>.



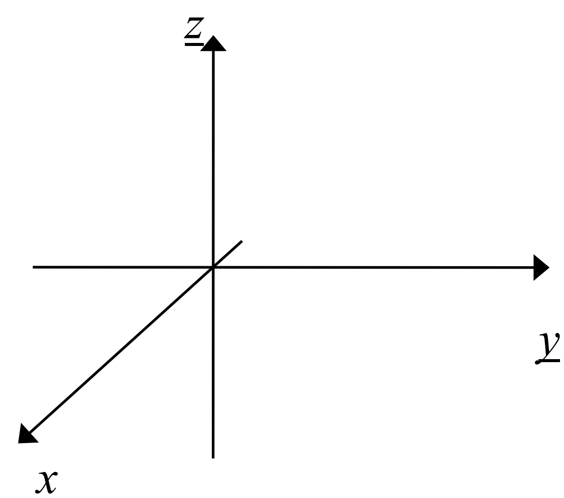
Example: For the scalar function

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Calculate

$$\int_C (\nabla\phi) \cdot d\mathbf{r}$$

where C is the following path:

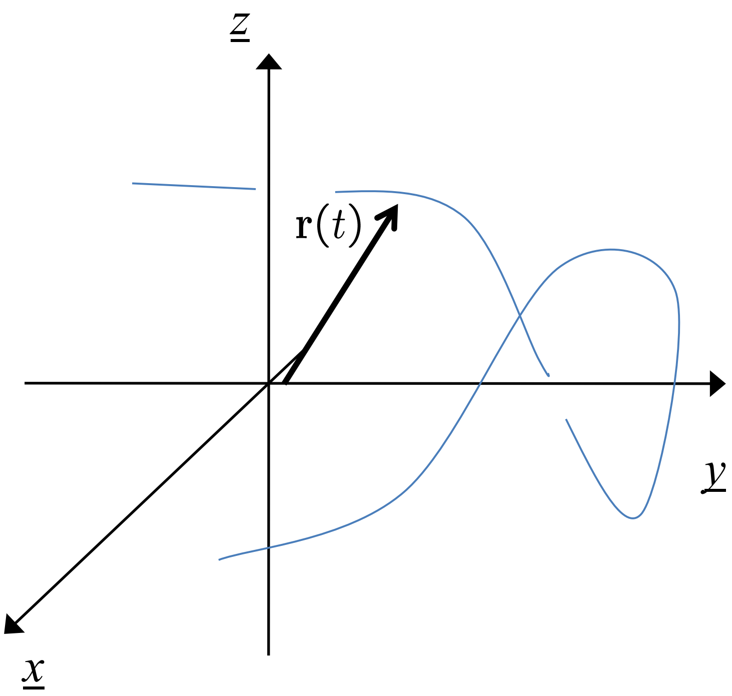


Conservative fields

A vector field F is *conservative* if it can be written

In this case, the line integral of the field only ever depends on the endpoints:

If the line integral is along a *closed path*, then for a conservative field,



Conservative fields are *irrotational*, i.e.

Example:
Calculate the curl of the field

$$\mathbf{F} = x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Hence or otherwise calculate the integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the circle of radius 2, centred on the z-axis and lying in the plane z = 57.

Finding the potential function

If a vector field is *irrotational* then we always find a potential function such that

$$\mathbf{F} = \nabla\phi$$

Example:

$$\mathbf{F} = \sin y\,\hat{\mathbf{i}} + (1 + x\cos y)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\mathbf{F} = \sin y\hat{\mathbf{i}} + (1 + x\cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

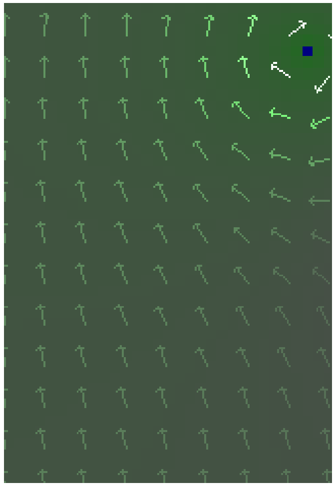
$$\mathbf{F} = \sin y \hat{\mathbf{i}} + (1 + x \cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\mathbf{F} = \sin y\hat{\mathbf{i}} + (1 + x\cos y)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The circulation of a vector field

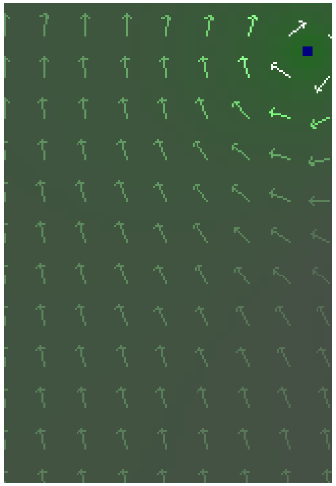
Consider a closed loop integral in a vector field F.

What happens as the area goes to zero?



Alternative definition of the curl:

$$\nabla \times \mathbf{F} = \hat{\mathbf{n}} \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Where ¢ S is the area of the loop C and **n** is the unit normal vector to this area element.