Question 1. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) Use Stokes theorem to evaluate

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} ,$$

where

$$\mathbf{F} = y(z+1)\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z^3\hat{\mathbf{k}} ,$$

and S is the half-ellipsoid $x^2 + y^2 + 2z^2 = 9$, with $z \geq 0$, oriented upward.

(8 marks)

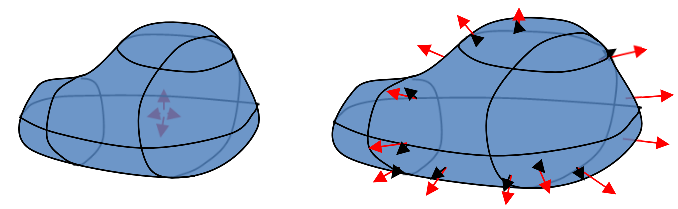
(b) What would be the value of this integral if S is instead the *downward-oriented* hemisphere $x^2 + y^2 + z^2 = 9$, with $z \leq 0$?

(2 marks)

Integral theorems overview

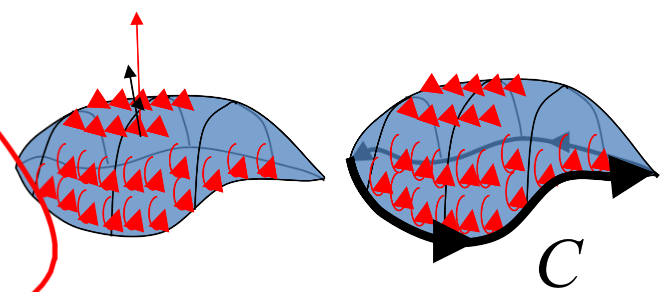
1. The divergence theorem

$$\iiint_V \nabla \cdot F dV = \iint_S \mathbf{F} \cdot \mathbf{d}S$$



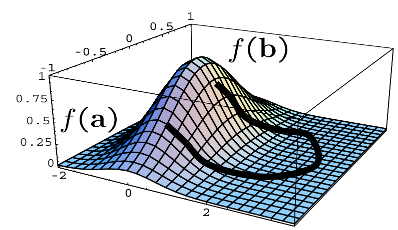
2. Stokes' theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S = \oint_C \mathbf{F} \cdot \mathbf{d}r$$



3. The fundamental theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$



Vector Calculus and PDEs 37336
Problem Set 6: Integral theorems

1. (a) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where
$$\mathbf{F} = -y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
and S is the part of the paraboloid $x^2+y^2+z = 4$ that lies above xy plane, and that is upward-oriented.

(b) State the divergence theorem.

(c)Verify that the divergence theorem is true for $\mathbf{F}$ and for the region bound by the surface S.

2. Use the divergence theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F} = 3xy^2\hat{\mathbf{i}} + xe^z\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$$
and S is the outward-oriented surface of the solid bound by the cylinder $z^2+y^2 = 1$ and the planes $x = -1$ and $x = 2$.

3. Use Stokes theorem to evaluate
$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}\ ,$$
where $\mathbf{F} = x^2e^{yz}\hat{\mathbf{i}} + y^2e^{xz}\hat{\mathbf{j}} + z^2e^{xy}\hat{\mathbf{k}}$, and S is the hemisphere $x^2+y^2+z^2 = 4$, with $z \geq 0$, oriented upward. What would be the value of this integral if S is instead the the upward-oriented paraboloid $x^2 + y^2 + z = 4$, with $z \geq 0$?

4. Use the divergence theorem to evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F}(x, y, z) = x\cos^2 z\hat{\mathbf{i}} + y\sin^2 z\hat{\mathbf{j}} + \sin x\cos x\hat{\mathbf{k}}$$
and S is the sphere centred at the origin with radius R.

5. Use Stokes' theorem to evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F}(x, y, z) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$
and S is the outward-oriented part of the sphere $x^2 + y^2 + (z + 3)^2 = 25$ with $z \geq 0$.

3. Use Stokes theorem to evaluate

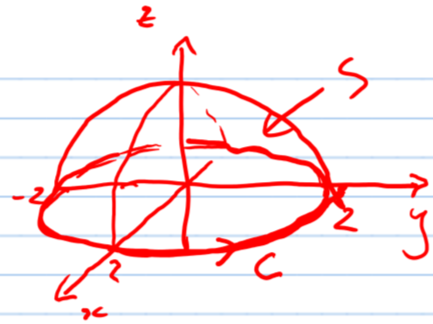
$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}\ ,$$

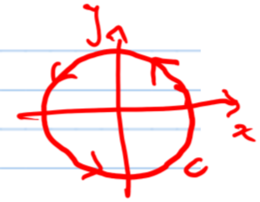
where $\mathbf{F} = x^2 e^{yz}\hat{\mathbf{i}} + y^2 e^{xz}\hat{\mathbf{j}} + z^2 e^{xy}\hat{\mathbf{k}}$, and S is the hemisphere $x^2+y^2+z^2 = 4$, with $z \geq 0$, oriented upward. What would be the value of this integral if S is instead the the upward-oriented paraboloid $x^2 + y^2 + z = 4$, with $z \geq 0$?

Stokes' theorem is

$$\iint_S (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \oint_C \underbrace{\underset{\sim}{F} \cdot d\underset{\sim}{r}}_{\text{want to find}}$$

where S is the hemisphere
and C is its bounding curve:





Parametrize C:

$$\underset{\sim}{r}(t) = \langle \underset{x}{2\cos t}, \underset{y}{2\sin t}, \underset{z}{0} \rangle \quad \text{for } t \in [0, 2\pi]$$

So $$\frac{d\underset{\sim}{r}}{dt} = \langle -2\sin t, 2\cos t, 0 \rangle$$

Then

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \langle x^2 e^{yz}, y^2 e^{xz}, z^2 e^{xy} \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} \langle (2\cos t)^2 \cdot e^0, (2\sin t)^2 \cdot e^0, 0 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} (-8\cos^2 t \sin t + 8\sin^2 t \cos t + 0)\, dt$$

$$= 8\int_0^{2\pi} \left(\underline{-\cos^2 t \sin t} + \sin^2 t \cos t \right) dt$$

$$= -8\left[\frac{\cos^3 t}{3} \right]_0^{2\pi} + 8\left[\frac{\sin^3 t}{3} \right]_0^{2\pi} .$$

$$= -8\left[\frac{\cos^3 2\pi}{3} - \frac{\cos^3 0}{3} \right]$$

$$+ 8\left[\frac{\sin^3 2\pi}{3} - \frac{\sin 0}{3} \right]$$

$$= 0 .$$

$$\int \cos^2 t \sin t \, dt$$

$$u = \cos t$$

$$\frac{du}{dt} = \sin t$$

$$du = \sin t \, dt.$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} = \frac{\cos^3 t}{3} + C$$

Question 2. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) The electric field in some region is found to depend on the distance from the origin r via the equation

$$\mathbf{E} = \frac{k}{r}\hat{\mathbf{r}}$$

where k is some constant and $\hat{\mathbf{r}}$ is the radial unit vector in spherical polar coordinates. Use Gauss's law to compute the charge density $\rho(\mathbf{r})$.

[Hint: You may need Gauss's law in differential form, which is

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

where $\rho(\mathbf{r})$ is the charge density at a point $\mathbf{r}$, and ε_0 is a constant.]

(5 marks)

(b) Given the operator

$$\mathcal{L} = -\frac{1}{2}\frac{d^2\phi}{dx^2}$$

Find all eigenvalues and eigenfunctions of the Sturm-Lioville problem

$$\mathcal{L}\phi = \lambda\phi$$

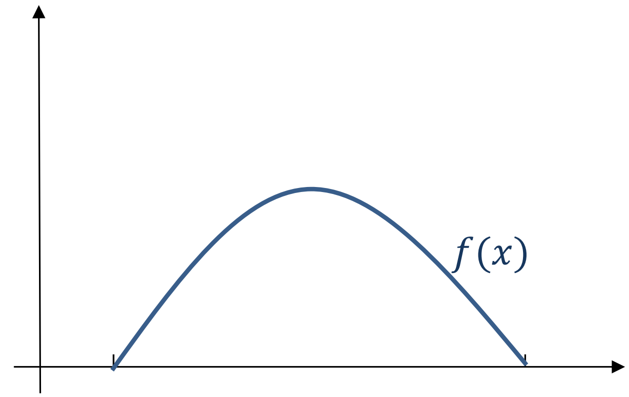
on the domain $0 \leq x \leq a$, with the boundary condition $\phi(0) = \phi(a) = 0$.

(5 marks)

An *operator* is a device for turning one function into another. For the function u, we create a new function

$$g = \mathcal{L}u$$

using the operator $\mathcal{L}$.



Examples:

The operator $\mathcal{L} = \frac{d}{dx}$ acts on a function f to form $\mathcal{L}f = \frac{df}{dx}$

$$\mathcal{L} = 2$$

$$\mathcal{L} = -\frac{d^2}{dx^2}$$

$$\mathcal{L} = -\frac{1}{2}\frac{d^2}{dx^2}.$$

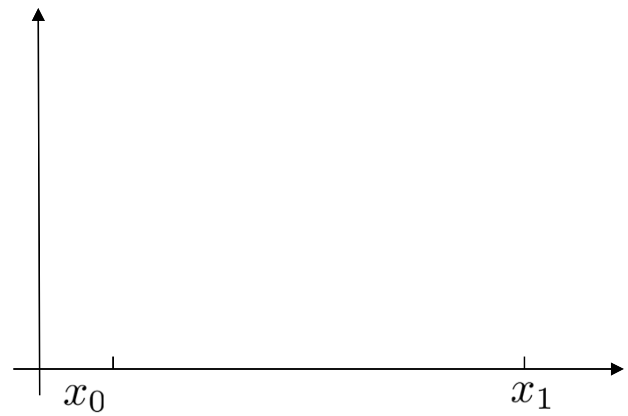
$$\mathcal{L} = e^{-x}$$

Once we have an operator, we can see how it behaves *in conjunction with an inner product.*

$$\langle f, \mathcal{L}g \rangle = \int_D f^*(x)\mathcal{L}g(x)\ w(x)\mathrm{d}x$$

The Sturm-Liouville eigenvalue problem

We consider the following problem on a domain $D = \{x: x_0 \leq x \leq x_1\}$:



$$\mathcal{L}\phi = \lambda\phi \quad \text{on } D$$

where $\mathcal{L}$ is the S-L differential operator.

$$\mathcal{L} = -\frac{1}{w(x)}\left[\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right]$$

Plus boundary conditions of either $\phi(x_0) = \phi(x_1) = 0$

or $\phi'(x_0) = \phi'(x_1) = 0$.

This type of problem is called an *eigenvalue problem*. The solutions form a set:

By convention: $\phi = 0$, $\lambda = 0$ is always a solution, and we omit this from the set.

The solution ϕ_n is the eigenfunction corresponding to the eigenvalue n.

(b) Given the operator

$$\mathcal{L} = -\frac{1}{2}\frac{d^2\phi}{dx^2}$$

Find all eigenvalues and eigenfunctions of the Sturm-Lioville problem

$$\mathcal{L}\phi = \lambda\phi$$

on the domain $0 \le x \le a$, with the boundary condition $\phi(0) = \phi(a) = 0$.

(5 marks)

Want to solve

$$\mathcal{L}\phi = \lambda\phi$$

or

$$-\frac{1}{2}\frac{d^2\phi}{dx^2} = \lambda\phi$$

Re-arrange:

$$\frac{1}{2}\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

The characteristic eqn is

$$\frac{1}{2}m^2 + \lambda = 0$$

note!

$$m^2 = -2\lambda$$

$$m = \pm i\sqrt{2\lambda}$$

The general solⁿ is

$$\phi(x) = A\cos(\sqrt{2\lambda}x) + B\sin(\sqrt{2\lambda}x).$$

We know $\phi(0) = 0$, so

$$0 = A\cos(0) + B\sin(0)$$

$$\Rightarrow 0 = A \times 1 + B \times 0 \Rightarrow A = 0$$

Then $\phi(x) = B\sin(\sqrt{2\lambda}x)$.

Also, $\phi(a) = 0$, so $0 = B\sin(\sqrt{2\lambda}a)$.

Therefore $\sqrt{2\lambda}a = n\pi$, $n \in \mathbb{Z}^+$

$$\Rightarrow 2\lambda a^2 = n^2\pi^2$$

eigenvalues are $\Rightarrow \lambda_n = \frac{n^2\pi^2}{2a^2}$

eigenfunctions are $\phi_n(x) = B\sin(\sqrt{2\lambda_n}x)$

$$= B\sin\left(\frac{n\pi x}{a}\right).$$

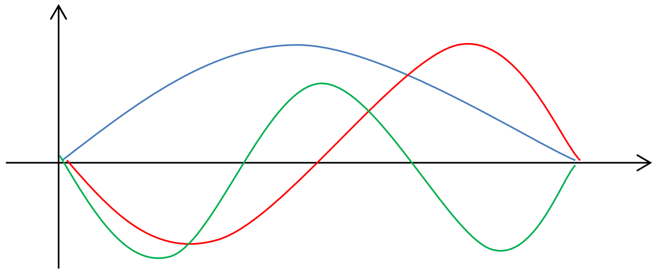
Properties of Sturm-Liouville problems

S-L problems all have the following important properties:

$$\mathcal{L}\phi = \lambda\phi \quad \text{on } D$$

1. There is an infinite set of eigenfunctions and eigenvalues

$$\{\phi_m, \lambda_m\}$$



2. All Eigenvalues are real and positive

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots$$

3. Eigenfunctions are *orthogonal*

$$\langle \phi_m, \phi_n \rangle = \delta_{n,m} ||\phi_m||^2$$

4. The Eigenfunctions form a complete set

Question 4. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) The general solution to Laplace's equation in 2D polar coordinates is

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{m=-\infty}^{\infty} \left[a_m r^{|m|} + b_m r^{-|m|}\right] e^{im\theta} \, .$$

Use this to solve

$$\nabla^2 u = 0$$

on the 2D circular domain $r \leq \frac{1}{2}$, with boundary condition

$$u(\frac{1}{2}, \theta) = \sin(2\theta) + \frac{1}{2}\cos(2\theta)$$

at $r = \frac{1}{2}$. Make sure that you state all your reasoning.

(7 marks)

(b) Check you answer from part (a) by showing that it obeys the partial differential equation. (3 marks)

Vector Calculus and PDEs 37336
Problem Set 10: Separation of variables with polar coordinates

1. Find a separation ansatz and hence the general solution to the equation

$$au_{xx} + bu_x + u_t = 0$$

where a and b are constants.

2. Find all eigenvalues and eigenvectors of the periodic Sturm-Liouville problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

defined on the domain $-L \leq x \leq L$, with boundary conditions

$$\phi(-L) = \phi(L) \quad , \phi'(-L) = \phi'(L) \ .$$

3. Show that the function

$$u(r, \theta) = r^m e^{im\theta}$$

is a solution to the Laplace equation in polar coordinates.

4. Starting with the general solution for the Laplace equation in 2D polar coordinates, solve

$$\nabla^2 u = 0$$

on the domain $a \leq r \leq b$, with Dirichlet boundary conditions $u = 0$ at $r = a$ and $u = \cos(2\theta)$ at $r = b$.

5. The Bessel function

$$f(z) = Y_m(z)$$

satisfies the equation

$$z^2 f'' + zf' + (z^2 - m^2)f = 0 \ .$$

Write the solution to the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(\frac{1}{a} - \frac{m^2}{t^2}\right) y = 0$$

in terms of the Y_m Bessel functions.

6. Find all eigenfunctions for the problem

$$\nabla\phi = \lambda\phi$$

defined on the domain in polar coordinates $r \leq a$, and with Neumann boundary conditions

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0$$

defined on the edge of the domain.

7. A large circular plate of radius a is struck by a hammer, causing vibrations to be set up in the material of the plate. The displacement $\psi(r, \theta)$ of the plate from equilibrium is governed by the wave equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

and Neumann boundary conditions $\frac{\partial\psi}{\partial r} = 0$ hold on the edge of the plate. The effect of the hammer is embodied in the initial condition

$$\psi(0, r, \theta) = 0 \quad , \frac{\partial\psi}{\partial t}(0, r, \theta) = e^{-r^2/b^2} \ .$$

Use your eigenfunctions from the previous question, solve this problem to find ψ for all values of t. You can leave your answer as an infinite series of integrals.

4. Starting with the general solution for the Laplace equation in 2D polar coordinates, solve

$$\nabla^2 u = 0$$

on the domain $a \leq r \leq b$, with Dirichlet boundary conditions $u = 0$ at $r = a$ and $u = \cos(2\theta)$ at $r = b$.

Question 4. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) By separating variables in two dimensions, find the general solution for $u(r, \theta)$ if u obeys the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

on the domain $r \leq 2$, subject to the restriction that u remain finite on this domain.

(b) Write the general solution if u does not depend on θ.

(c) Find the three smallest values of k given that $u(2, \theta) = 0$. The first three zeros of the Bessel function of the first kind are

$$\begin{aligned} j_{0,1} &= 2.4048 \\ j_{0,2} &= 5.5201 \\ j_{0,3} &= 8.6537\ . \end{aligned}$$

(d) How many possible values of k are there for the boundary conditions given in part (c)? Give a brief justification for your answer.

Vector Calculus and PDEs 37336
Problem Set 10: Separation of variables with polar coordinates

1. Find a separation ansatz and hence the general solution to the equation

$$au_{xx} + bu_x + u_t = 0$$

where a and b are constants.

2. Find all eigenvalues and eigenvectors of the periodic Sturm-Liouville problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

defined on the domain $-L \leq x \leq L$, with boundary conditions

$$\phi(-L) = \phi(L) \quad , \phi'(-L) = \phi'(L) \ .$$

3. Show that the function

$$u(r, \theta) = r^m e^{im\theta}$$

is a solution to the Laplace equation in polar coordinates.

4. Starting with the general solution for the Laplace equation in 2D polar coordinates, solve

$$\nabla^2 u = 0$$

on the domain $a \leq r \leq b$, with Dirichlet boundary conditions $u = 0$ at $r = a$ and $u = \cos(2\theta)$ at $r = b$.

5. The Bessel function

$$f(z) = Y_m(z)$$

satisfies the equation

$$z^2 f'' + zf' + (z^2 - m^2)f = 0 \ .$$

Write the solution to the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(\frac{1}{a} - \frac{m^2}{t^2}\right) y = 0$$

in terms of the Y_m Bessel functions.

6. Find all eigenfunctions for the problem

$$\nabla\phi = \lambda\phi$$

defined on the domain in polar coordinates $r \leq a$, and with Neumann boundary conditions

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0$$

defined on the edge of the domain.

7. A large circular plate of radius a is struck by a hammer, causing vibrations to be set up in the material of the plate. The displacement $\psi(r, \theta)$ of the plate from equilibrium is governed by the wave equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

and Neumann boundary conditions $\frac{\partial\psi}{\partial r} = 0$ hold on the edge of the plate. The effect of the hammer is embodied in the initial condition

$$\psi(0, r, \theta) = 0 \quad , \frac{\partial\psi}{\partial t}(0, r, \theta) = e^{-r^2/b^2} \ .$$

Use your eigenfunctions from the previous question, solve this problem to find ψ for all values of t. You can leave your answer as an infinite series of integrals.

$$\Big[f(r)\Big]_{r=a}^{r=b} = f(b) - f(a)$$

$$f(r)\Big|_{r=a} = f(a).$$

Question 4. (10 marks)

Important: marks will only be awarded for fully worked solutions, showing all steps.

(a) By separating variables in two dimensions, find the general solution for $u(r, \theta)$ if u obeys the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

on the domain $r \leq 2$, subject to the restriction that u remain finite on this domain.

Let

$$u(r,\theta) = R(r)\,T(\theta)$$

Sub into the PDE:

$$\underbrace{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)}_{\nabla^2} u + k^2 u = 0$$

So,

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' + k^2 RT = 0.$$

Divide by RT:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{T''}{T} + k^2 = 0$$

multiply by r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{T''}{T} + k^2r^2 = 0$$

or

$$\underline{r^2\frac{R''}{R} + r\frac{R'}{R} + k^2r^2} = -\frac{T''}{T} = \lambda$$

so the variable separate.

The problem for T is

$$\underbrace{-\frac{T''}{T} = \lambda}$$

with $T(2\pi+\theta) = T(\theta)$.

This is a S-L problem with solutions

$$T(\theta) = e^{in\theta} \quad n \in \mathbb{Z}.$$

with $\lambda = n^2$.

The problem for r is

$$\frac{r^2R''}{R} + \frac{rR'}{R} + k^2r^2 = m^2$$

or

$$r^2R'' + rR' + (k^2r^2 - m^2)R = 0.$$

This is Bessel's equation. The solutions are Bessel functions:

$$R(r) = AJ_m(kr) + BY_m(kr).$$

The general solution is then

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} \underbrace{e^{im\theta}}_{T(t)} \underbrace{\left(A_mJ_m(kr) + B_mY_m(kr)\right)}_{R(r)}.$$

But u must remain finite, so $B_m = 0$. Therefore

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} e^{im\theta} A_m J_m(kr).$$

(b) Write the general solution if u does not depend on θ.

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} A_m J_m(kr) e^{im\theta}.$$

Only the $m=0$ term in the sum is independent of θ, so

$$u(r,\theta) = A_0 J_0(kr)$$

(c) Find the three smallest values of k given that $u(2,\theta) = 0$. The first three zeros of the Bessel function of the first kind are

$$j_{0,1} = 2.4048$$
$$j_{0,2} = 5.5201$$
$$j_{0,3} = 8.6537\ .$$

$J_0(2.4048) = 0$

$J_0(5.5201) = 0$

(d) How many possible values of k are there for the boundary conditions given in part (c)? Give a brief justification for your answer.

c) The general solution is

$$u(r,\theta) = A_0 J_0(kr).$$

We would like to apply the boundary condition

$$u(2,\theta) = 0$$

$$A_0 J_0(k.2) = 0$$

$$\Rightarrow \quad J_0(2k) = 0$$

So $2k_0 = 2.4048 \Rightarrow k_0 = \dfrac{2.4048}{2}$

$= 1.2024.$

Also

$$k_1 = \frac{5.5201}{2}$$

$$= \underline{2.7601}$$

$$k_2 = \frac{4.6537}{2}$$

$$= \underline{4.3269}$$

d)