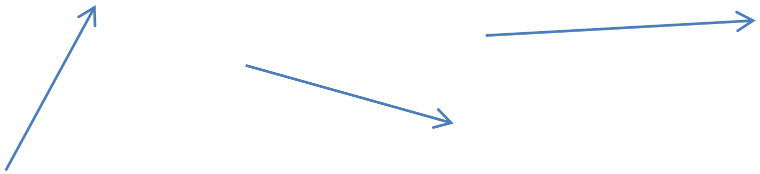
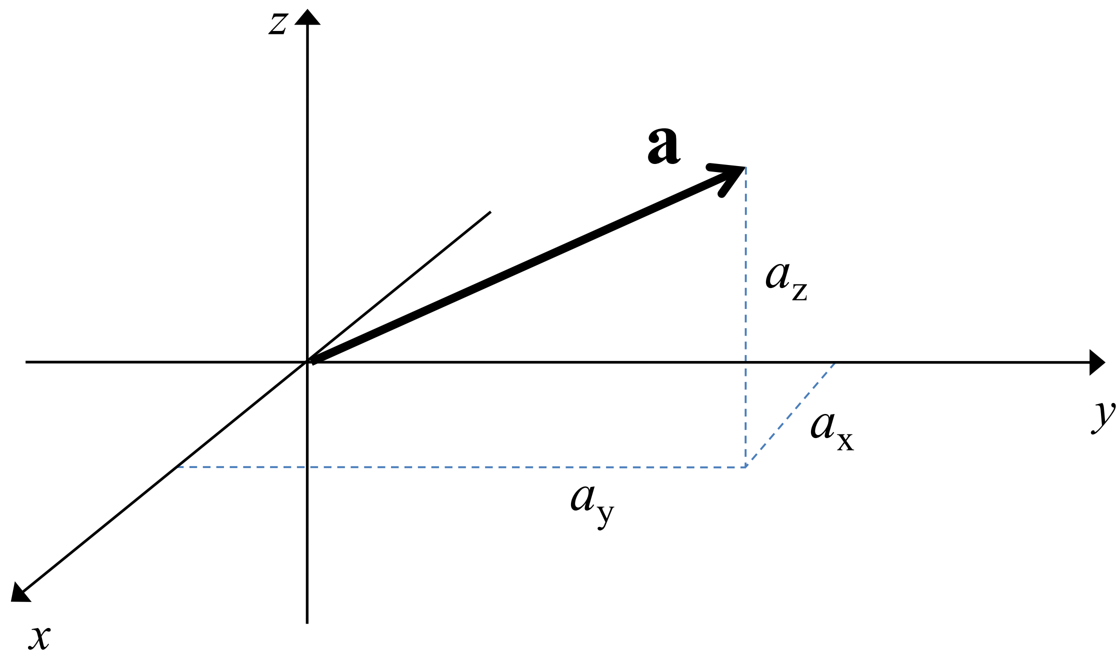
Review of vectors

A *scalar* quantity is completely specified by a single number

A *vector* is specified by a magnitude and a direction, and so is specified by two or more numbers



A vector can be written in terms of its components:



$$\mathbf{a} = \langle a_x, a_y, a_z \rangle$$

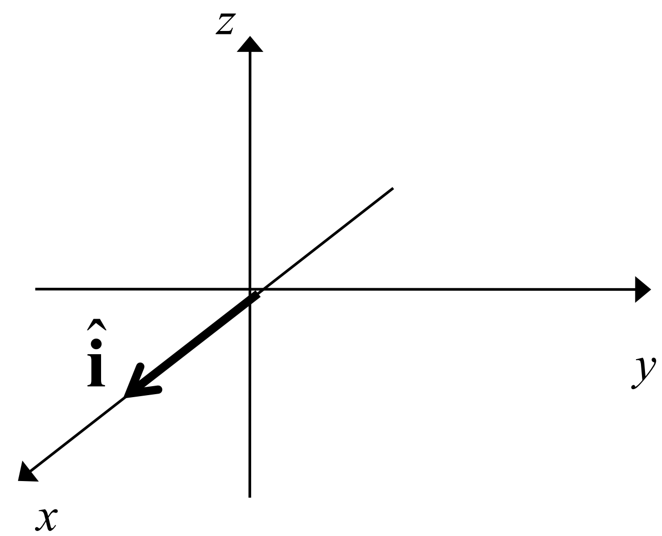
The *modulus*, or length of the vector is written

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

A *unit vector* is a vector of length 1. They are usually written with a "tilde"
Above the vector symbol.

eg: $$\hat{\mathbf{i}} = \langle 1,0,0 \rangle$$

is the unit vector in the x direction.



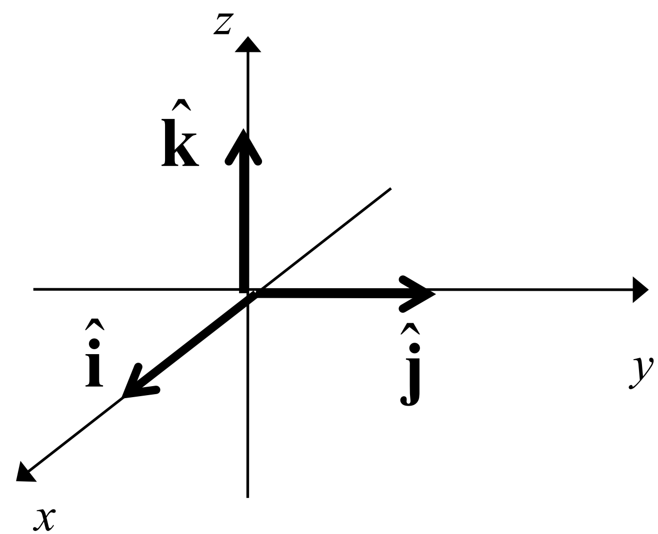
Any vector can be made into a unit vector by dividing it by its own length:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example:
Find the unit vector pointing in the same direction as $\mathbf{a} = (-1,2,3)$

The unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are known as the *coordinate axis vectors*:

$$\hat{\mathbf{i}} = \langle 1,0,0 \rangle$$
$$\hat{\mathbf{j}} = \langle 0,1,0 \rangle$$
$$\hat{\mathbf{k}} = \langle 0,0,1 \rangle$$



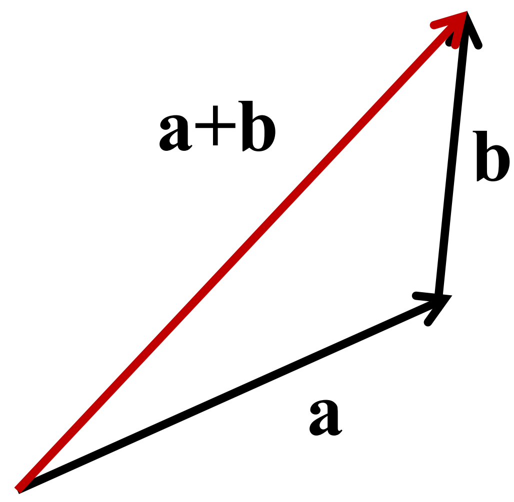
These vectors form a *complete basis*,
i.e. *any* vector $\mathbf{a}$ can be expressed in terms of $\mathbf{i}$ $\mathbf{j}$ and $\mathbf{k}$.

Vector Algebra

Vectors can be *added* component by component:

$$\mathbf{a} = (a_x , a_y , a_z)$$

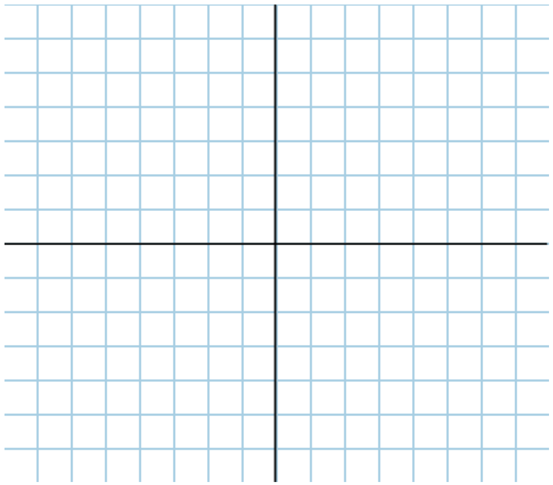
$$\mathbf{b} = (b_x , b_y , b_z)$$



$$\mathbf{a} + \mathbf{b} =$$

Example:

Find $\mathbf{c} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} = (2,1,0)$ and $\mathbf{b} = (1,2,3)$:

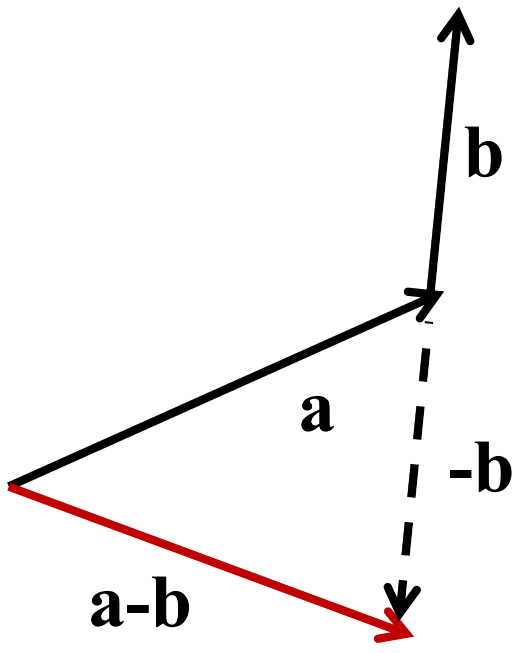


Vectors can be also be *subtracted*:

$$\mathbf{a} = (a_x , a_y , a_z)$$

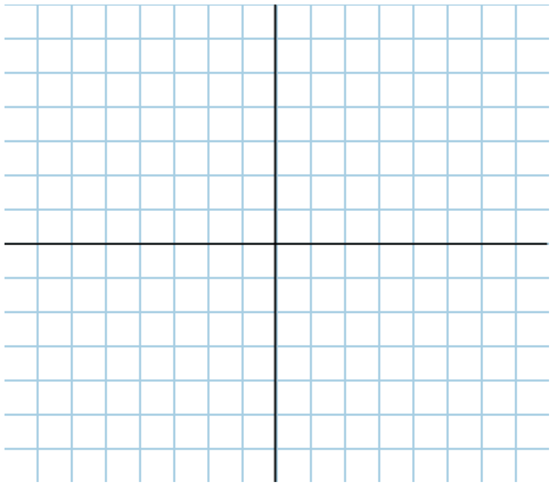
$$\mathbf{b} = (b_x , b_y , b_z)$$

$$\mathbf{a} - \mathbf{b} =$$



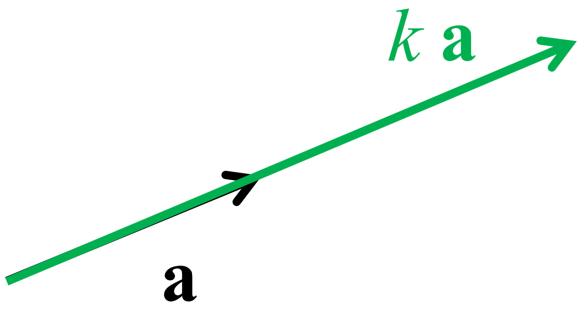
Example:

Find $\mathbf{c} = \mathbf{a} - \mathbf{b}$ where $\mathbf{a} = (2,1,2)$ and $\mathbf{b} = (1,2,3)$:



We can also multiply by a scalar:

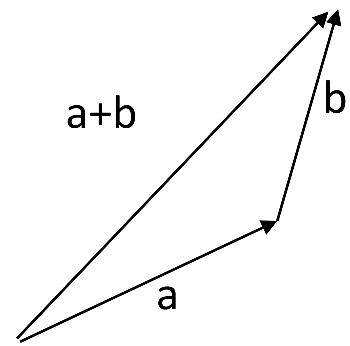
$$k\mathbf{a} = k(\,a_{\mathrm{x}}\,,\,a_{\mathrm{y}}\,,\,a_{\mathrm{z}}\,)$$
$$= (\,k\,a_{\mathrm{x}}\,,\,k\,a_{\mathrm{y}}\,,\,k\,a_{\mathrm{z}}\,)$$



This has the effect of stretching the vector (if k >1) or shrinking it (if k<1).

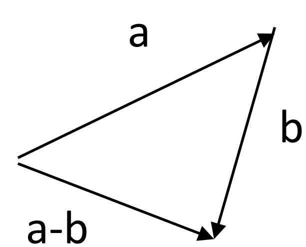
Vector operations:

Addition:



vector

Subtraction:

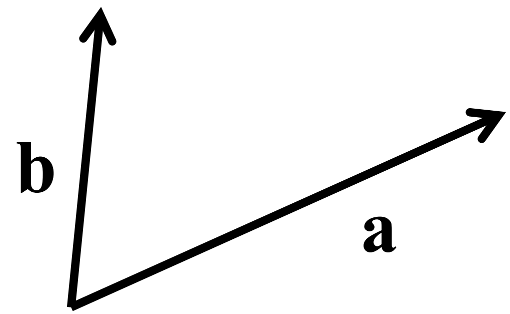


vector

Multiplication:

- Dot product → scalar
- Cross product → vector

The dot product



Definition:
The dot product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

The dot product is a *scalar* quantity,
and is sometimes called the *scalar product*.

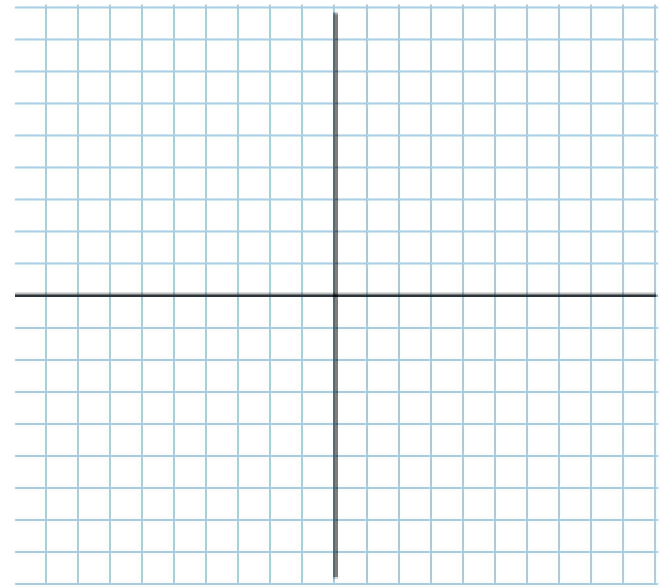
Two vectors → Scalar

The product rule is *distributive over vector addition*:

$$\mathbf{a}\cdot\left(\mathbf{b}+\mathbf{c}\right) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$$

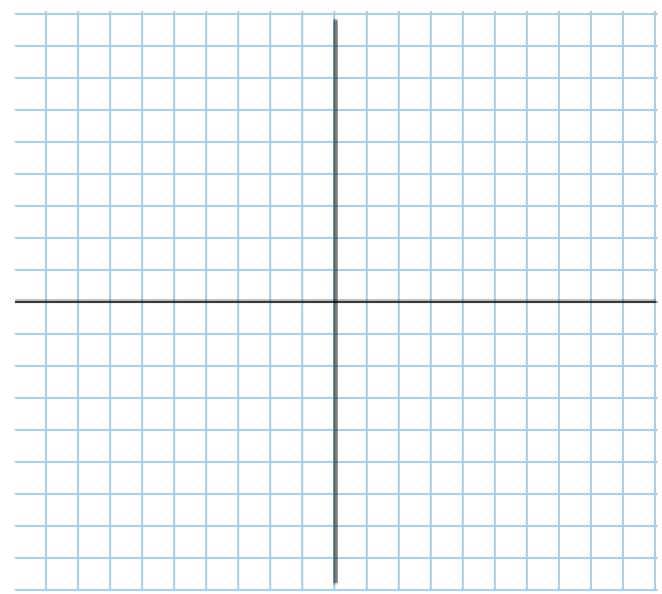
Example 1:

Find the dot product of **a** = (3,1,2) and **b** = (1,2,1)



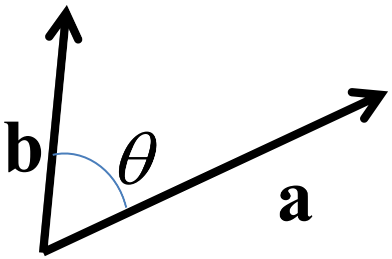
Example 2:

Find the dot product of **a** = (3,2,17) and **b** = (0,0,1)



Dot product

Two vectors → Scalar

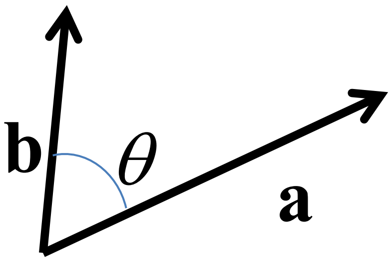


Tells how much two vectors are in the same direction

Cross product

Two vectors → Vector

Tells how much two vectors are perpendicular, and gives the normal to the two vectors



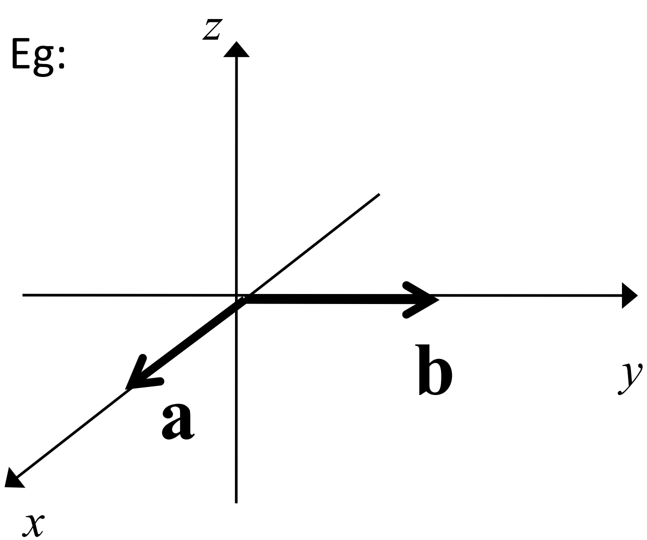
The cross product

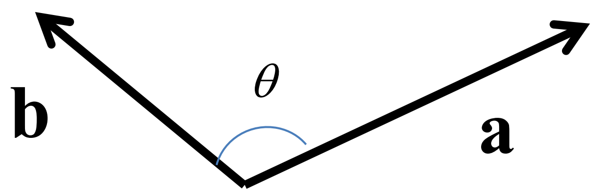
The cross product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is written

$$\mathbf{a} \times \mathbf{b}$$

The cross product is a vector which *always points in direction perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$

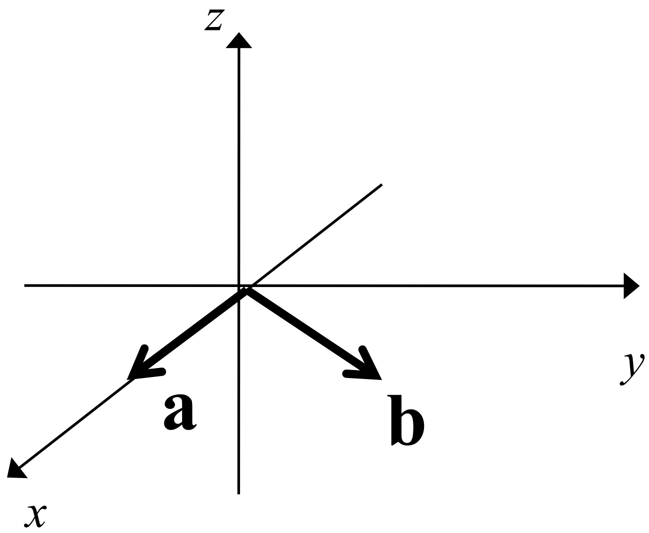
Eg:

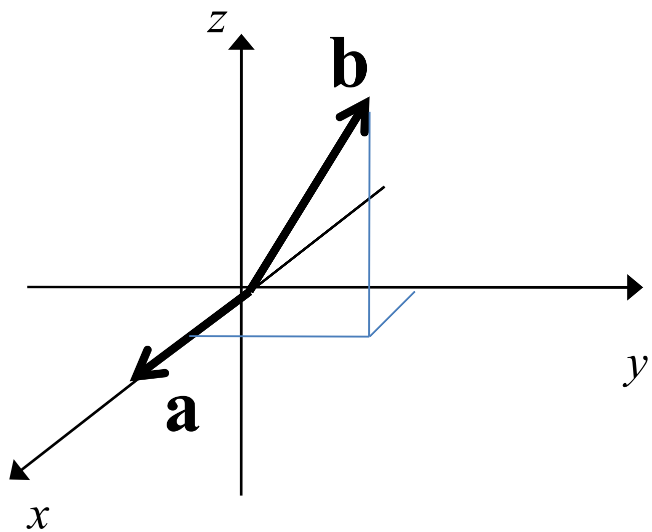




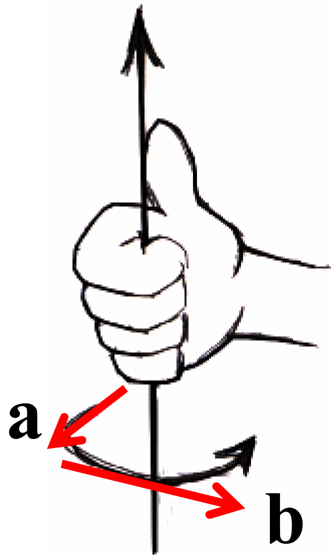
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$$

normal unit vector to both $\mathbf{a}$ and $\mathbf{b}$

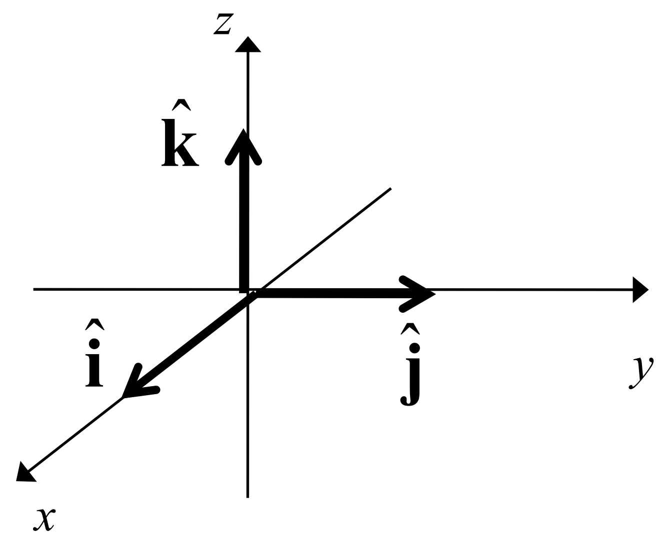


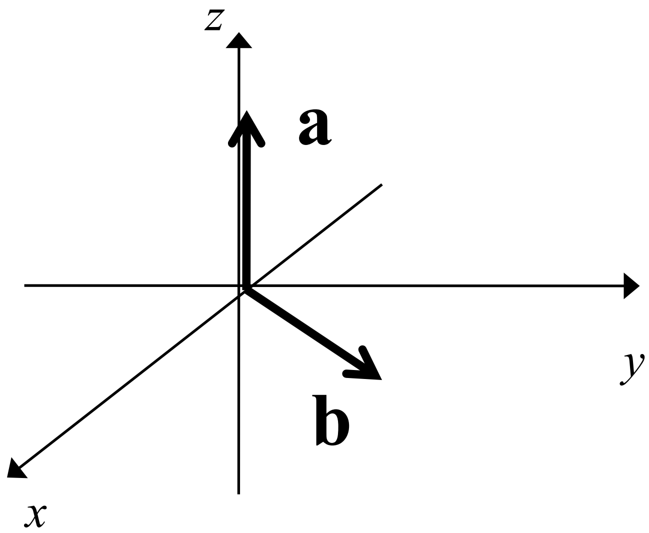


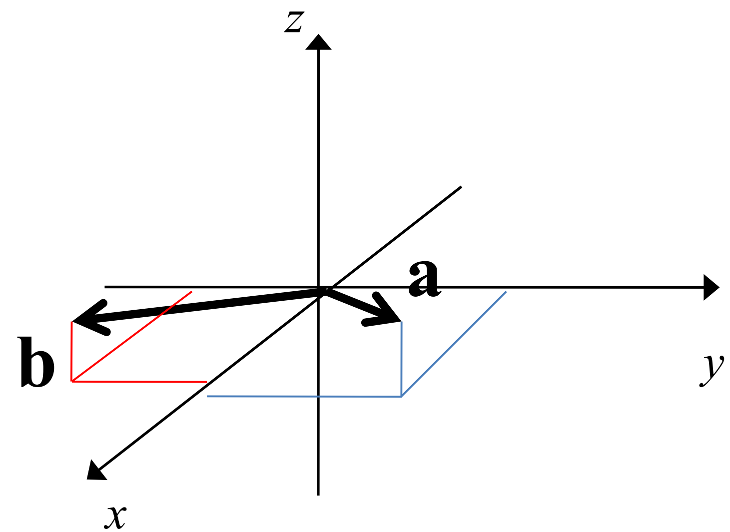
To avoid ambiguity we use the *right-hand rule:*

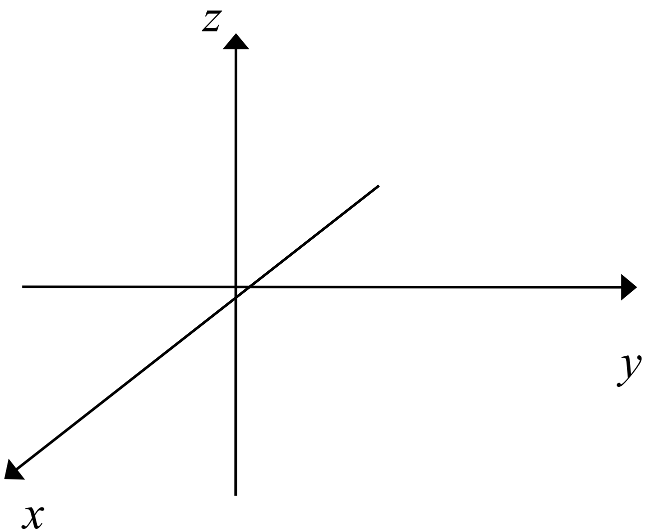


The unit vectors **i, j** and **k** form a *right-handed* coordinate system:




z
a
b
y
x


z
a
b
y
x

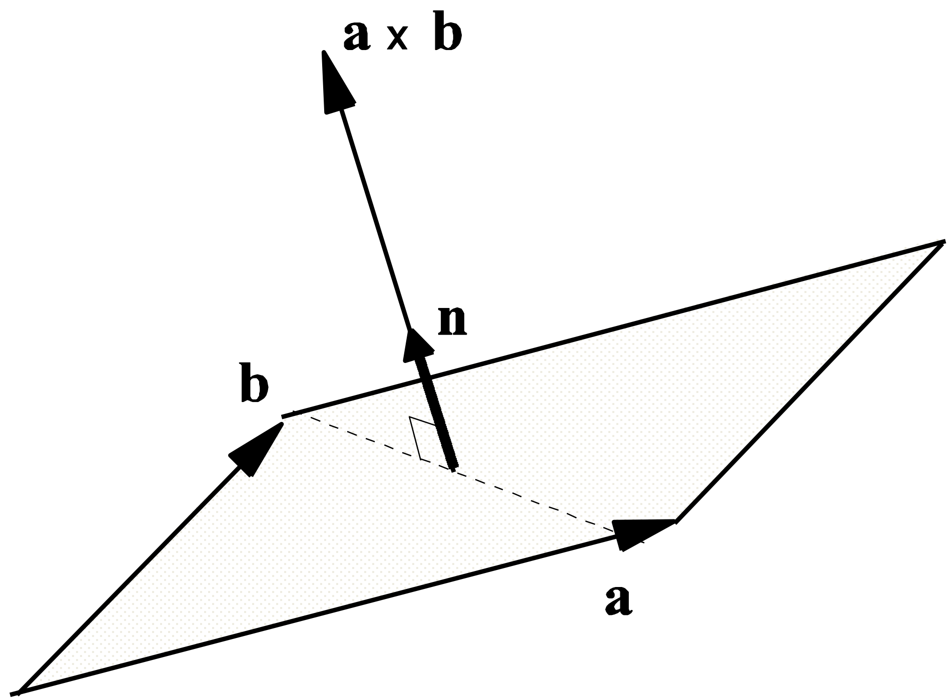

z
y
x

The cross product can be calculated using the determinant:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Example: Calculate the cross product between $\mathbf{a} = (1,2,0)$ and $\mathbf{b} = (0,3,1)$

$$\begin{array}{c} \\ \mathbf{a} \rightarrow \\ \mathbf{b} \rightarrow \end{array} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ & & \\ & & \end{vmatrix}$$



a x b
n
b
a

Vector triple products

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

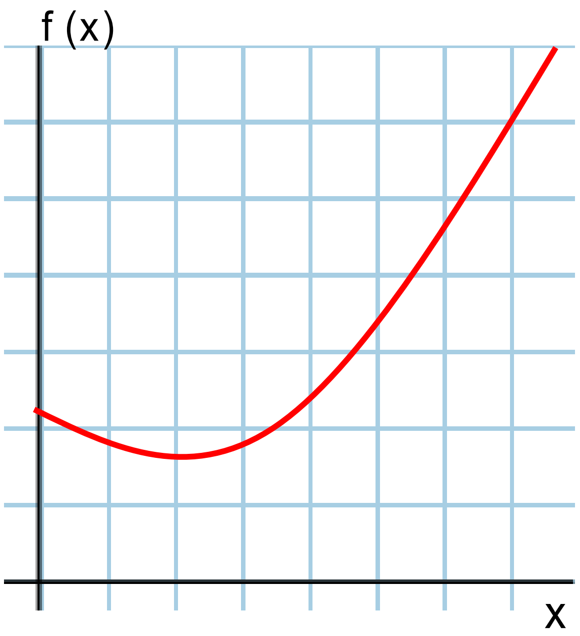
Review of partial derivatives and differentials

Recall: The **derivative** of a real function in 1D is defined as

rise

$$\frac{df}{dx} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

run



A function f is *differentiable* at the point x_0 if the derivative *exists*.

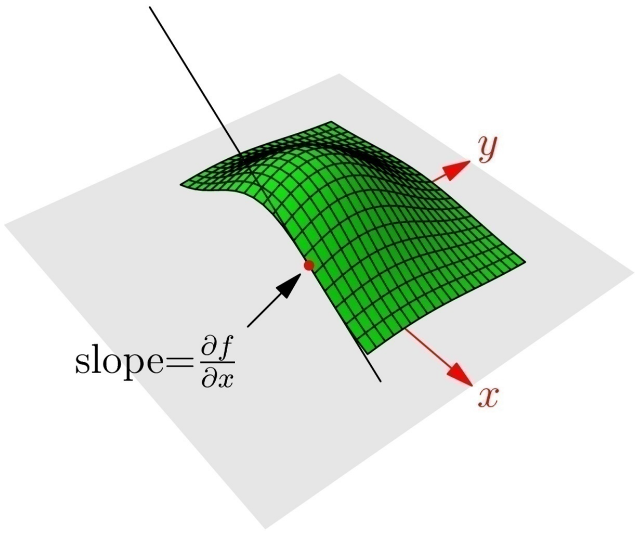
We often write the 1D derivative as $f'(x)$:

$$f'(x) \equiv \frac{df}{dx}$$

Definition of Partial Derivative

Let f be a function of *two* variables f(x,y). Then the partial derivatives of f at the point (x_0,y_0) are defined as

$$\frac{\partial f}{\partial x} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$



$$\frac{\partial f}{\partial y} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

We often write these as

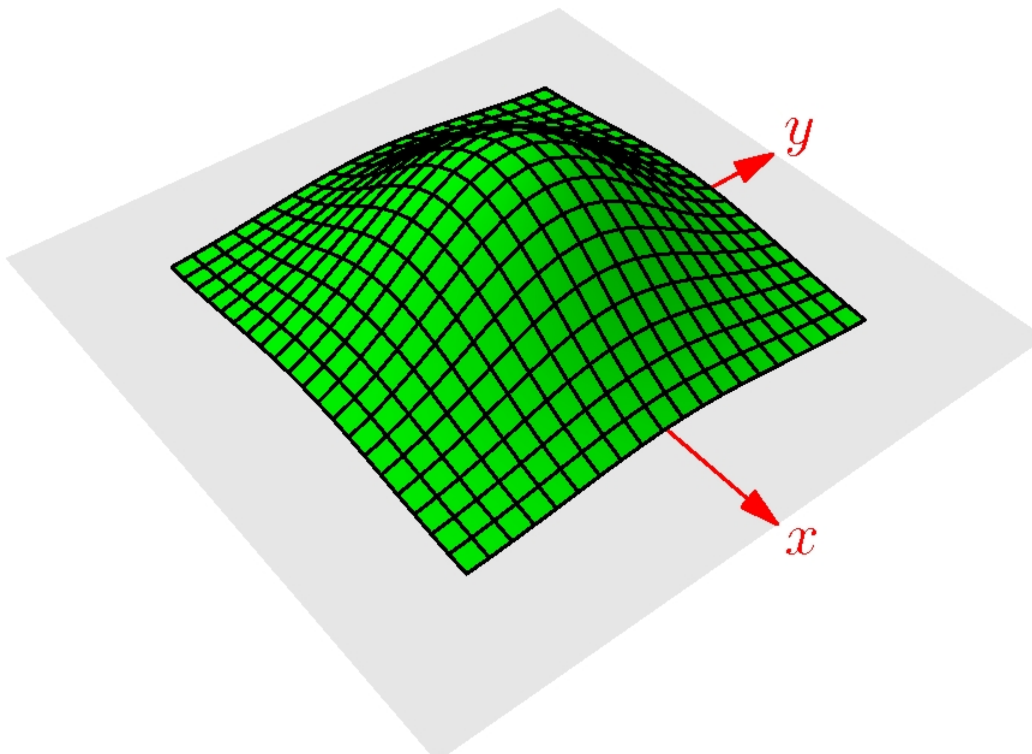
$$f_x \equiv \frac{\partial f}{\partial x} \qquad f_y \equiv \frac{\partial f}{\partial y}$$

The partial derivative is the derivative *with all the other variables held constant.*

NB: it is very important to remember to write it with the partial "∂" symbol.

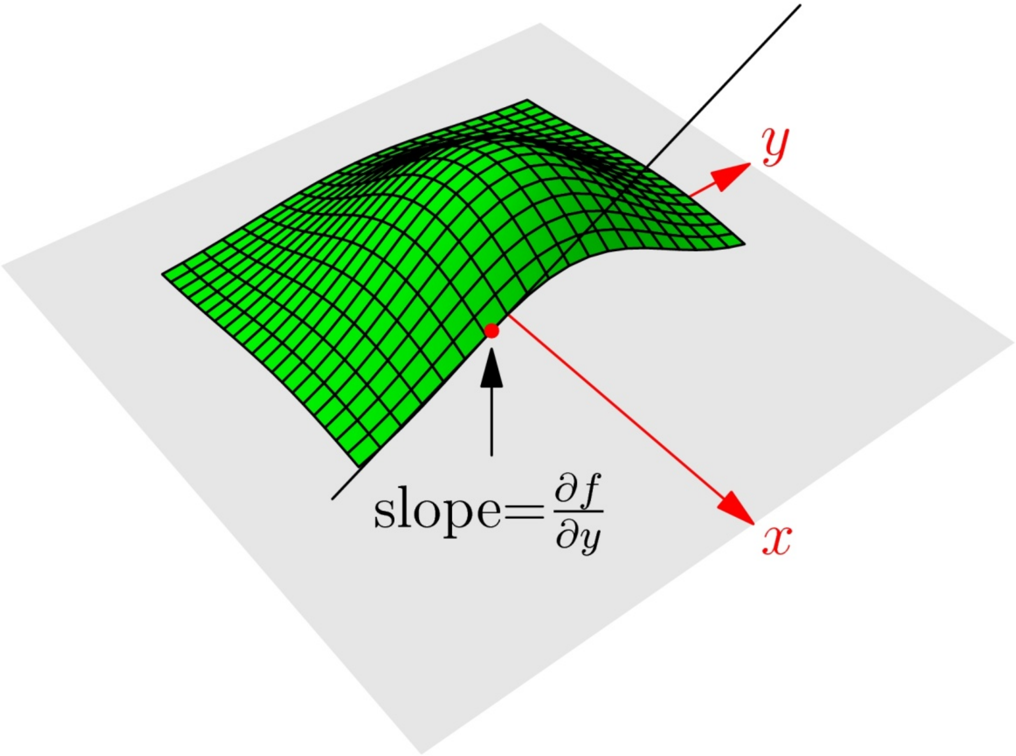
The partial derivative can be thought of as *the slope in a given direction*

Eg: $f(x,y) = e^{-(x^2+y^2)}$



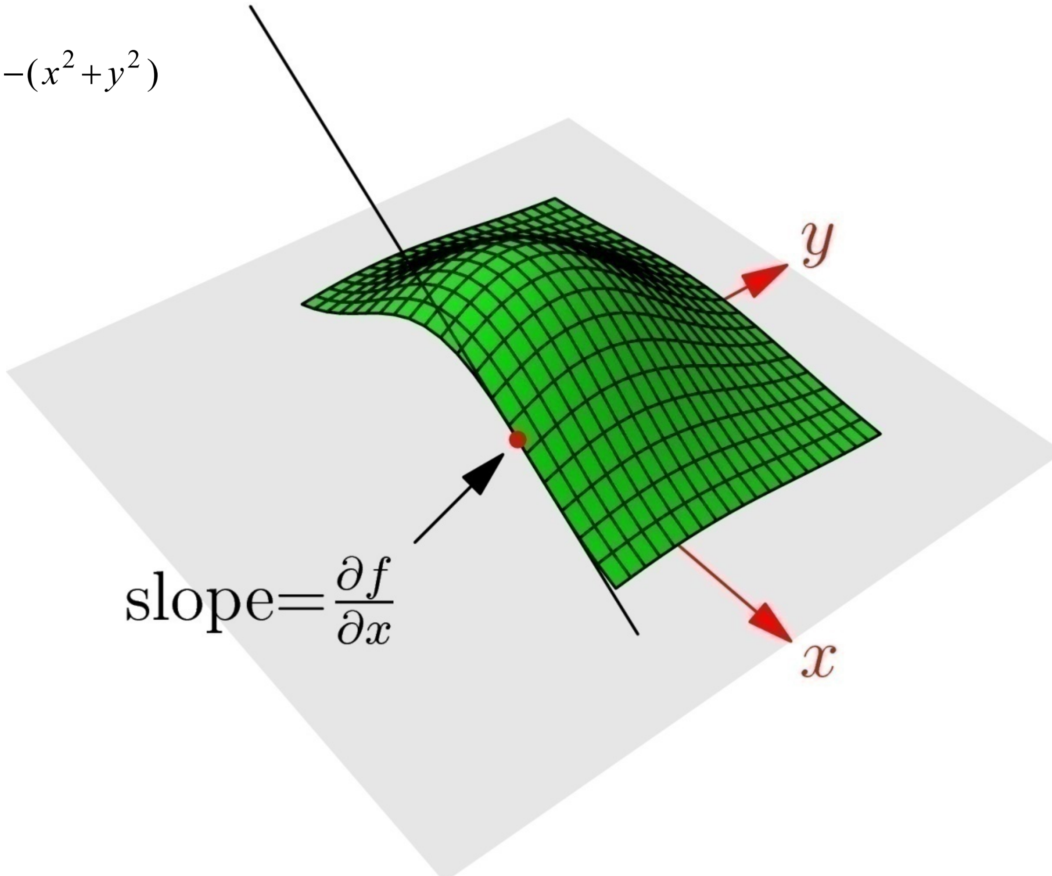
The partial derivative can be thought of as *the slope in a given direction*

Eg: $f(x, y) = e^{-(x^2+y^2)}$



The partial derivative can be thought of as *the slope in a given direction*

Eg: $f(x,y) = e^{-(x^2+y^2)}$



To compute the partial derivative we treat all other variables as constants:

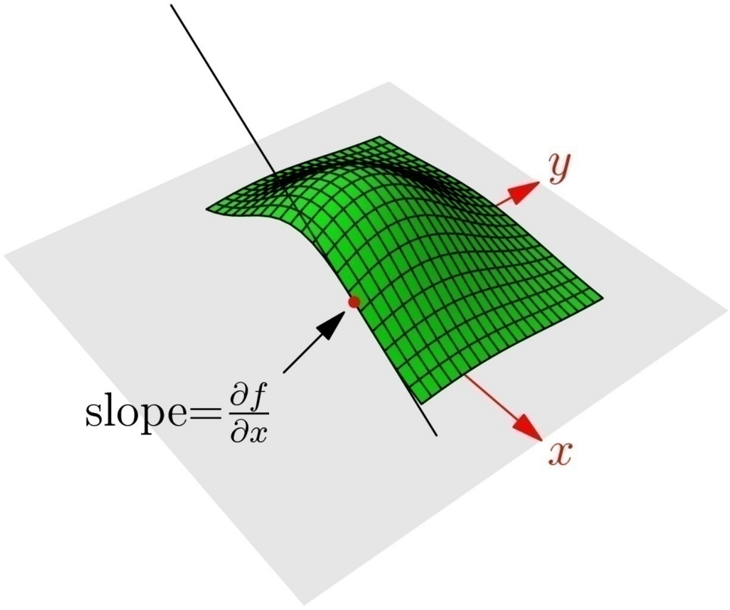
Example:
Let

$$f(x, y) = e^{-x^2 - y^2}$$

then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$



Example:

Compute f_x and f_y for

$$f(x,y) = 2x^2 + 2xy + 2y^2$$

Partial derivatives obey most of the usual rules of normal derivatives, such as the product rule, and the quotient rule.

However, two things are *different:*

1. It is *not* true that $\dfrac{\partial f}{\partial x} = 1/(\dfrac{\partial x}{\partial f})$
2. The *chain rule* is more complicated

Example:
Find f_x when:

$$f(x, y) = x \sin x \cos y$$

Example:

Find f_x and f_y of

$$f(x,y) = e^{-x^2}(xy^2+2)$$

Higher order partial derivatives:

The partial derivatives f_x and f_y are known as the *first partial derivatives* or *the partial derivatives of first order.*

By differentiating once again, we obtain four partial derivatives of *second order:*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

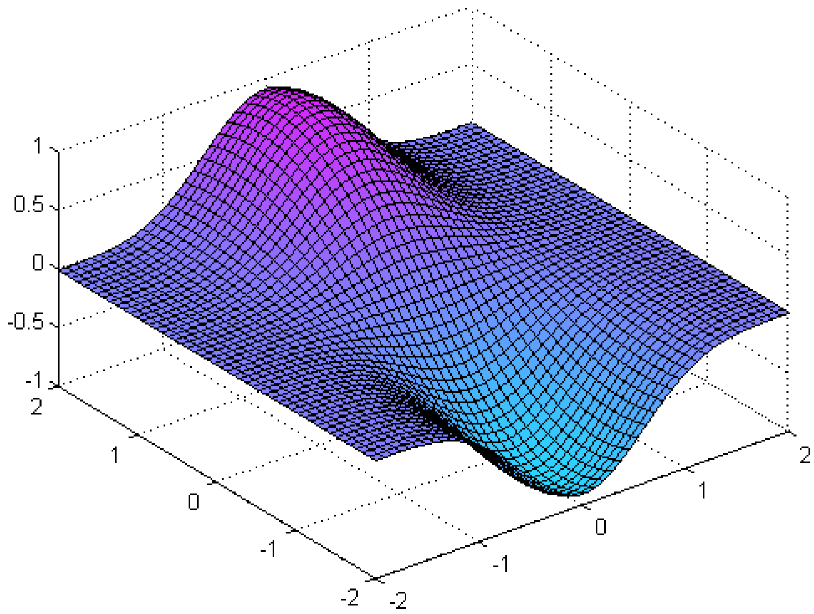
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

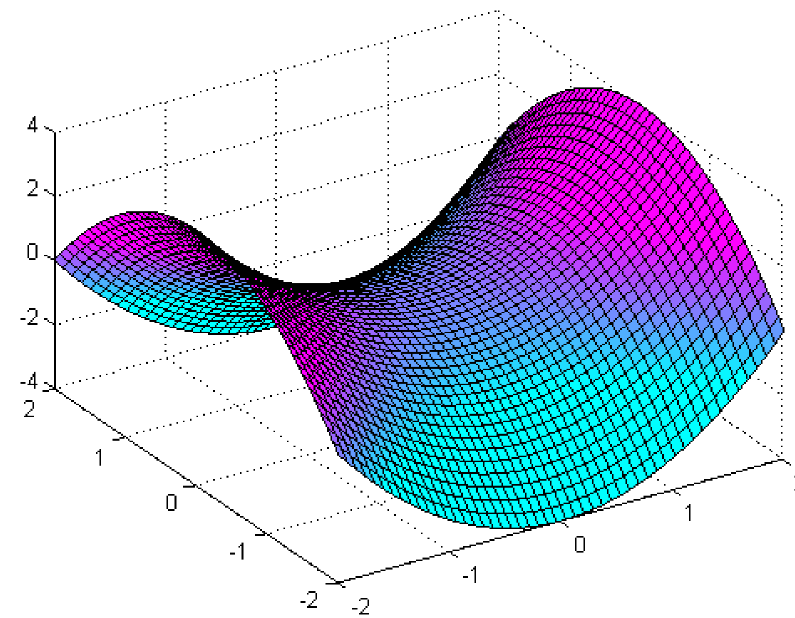
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

We can also form *third* partial derivatives and so on.

We can think of the second-order partial derivatives as representing *degree of curvature* in each of the different directions.





$f_{xx} > 0$

$f_{yy} > 0$

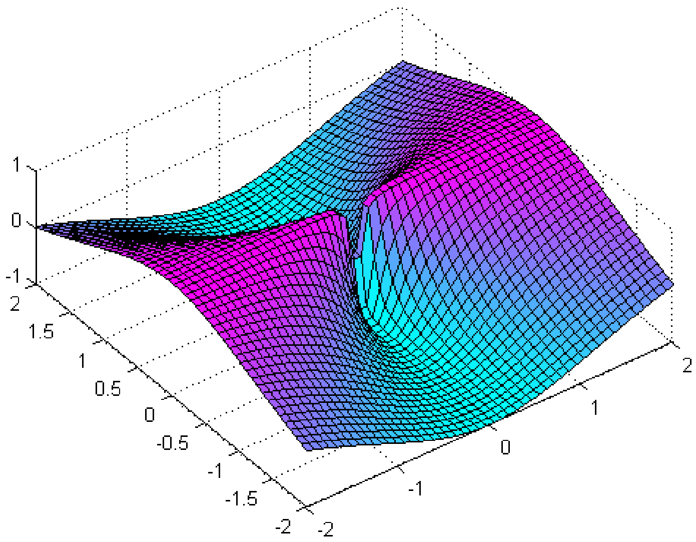
$f_{xx} < 0$

$f_{yy} < 0$

Mixed derivatives theorem: If both f_{xy} and f_{yx} are continuous on an open set surrounding (x_0,y_0), then

$$f_{xy} = f_{yx}$$

at the point (x_0,y_0).



Example:

Compute f_{xy} and f_{yx} for

$$f(x,y) = x^2 y + x \sin y$$

Div, Grad and Curl

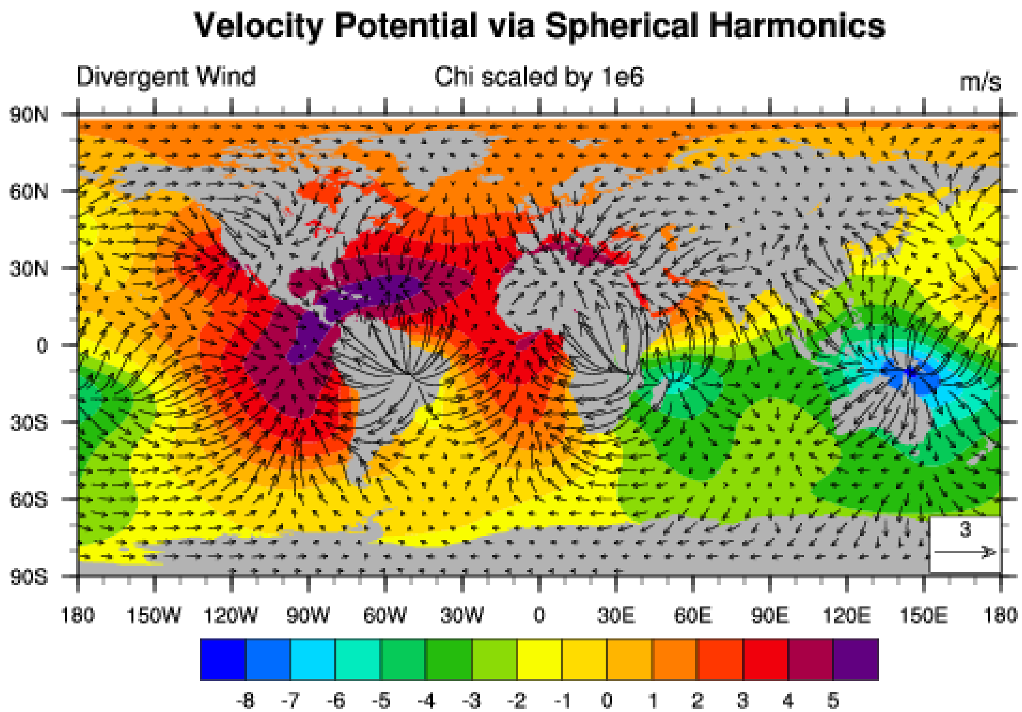
Vector field and scalar fields

A *vector field* is a vector function of position:

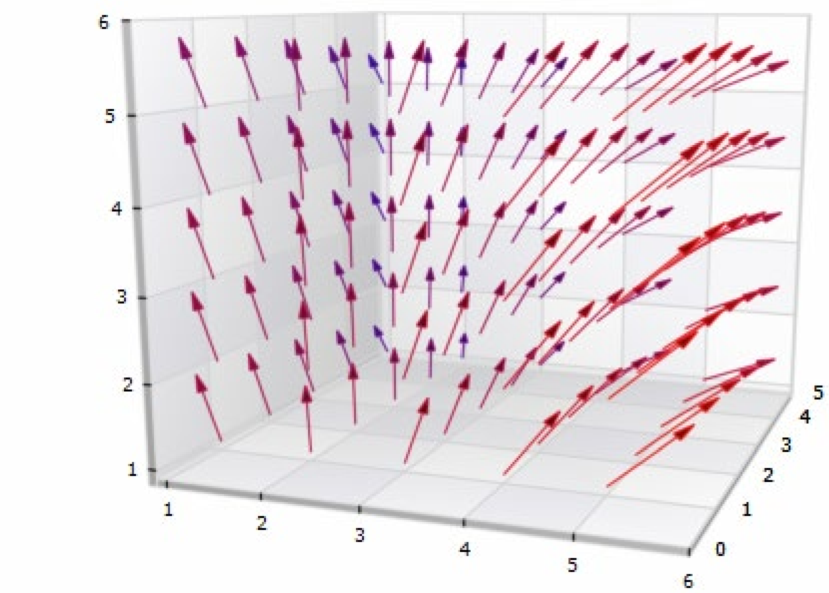
$$\mathbf{v} = \mathbf{v}(x, y, z)$$

A *scalar field* is a scalar function of position:

$$f = f(x, y, z)$$

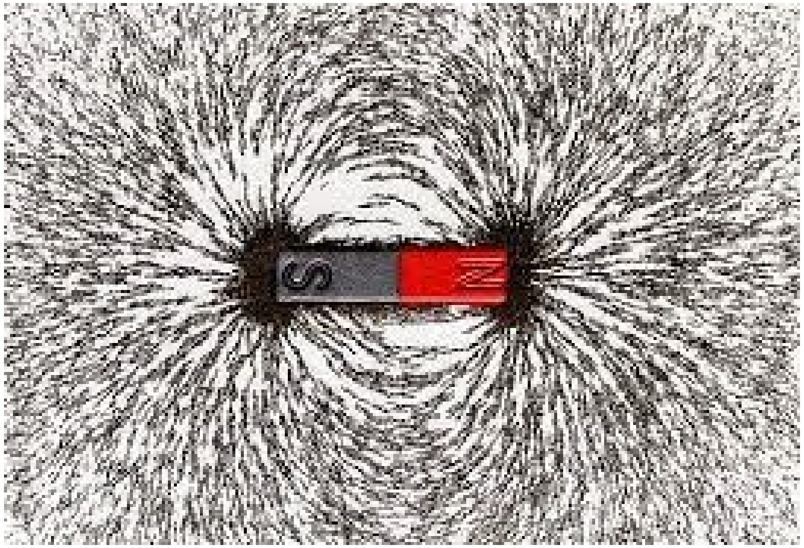


A vector field can be thought of as a group of vectors filling every point in space.



Examples of vector fields:

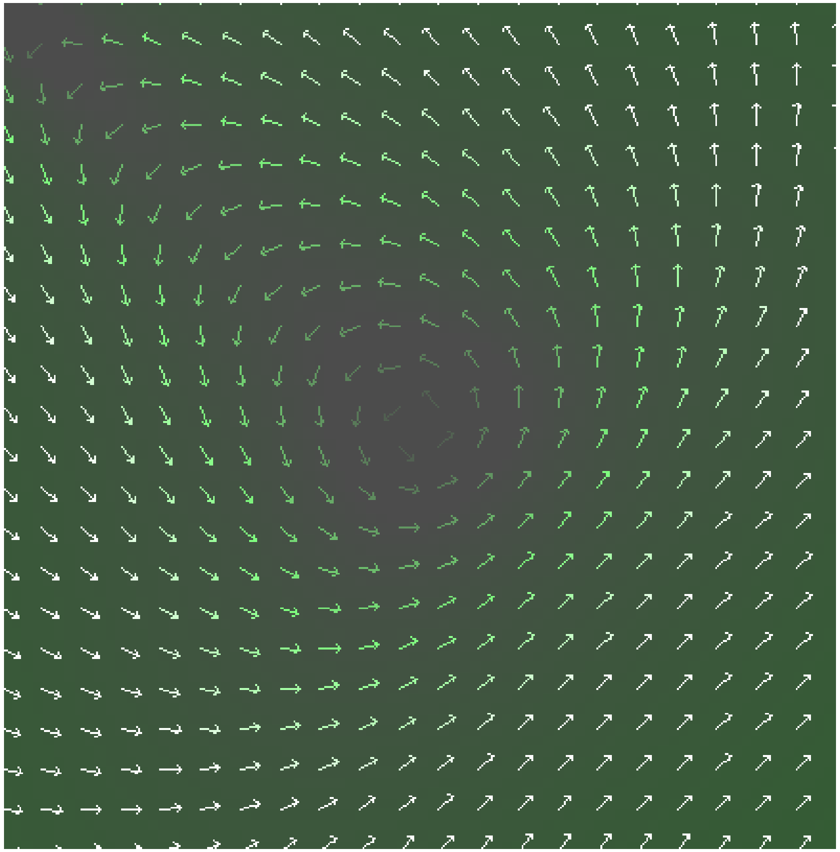
- Wind velocity
- Magnetic field
-



Examples of scalar fields:

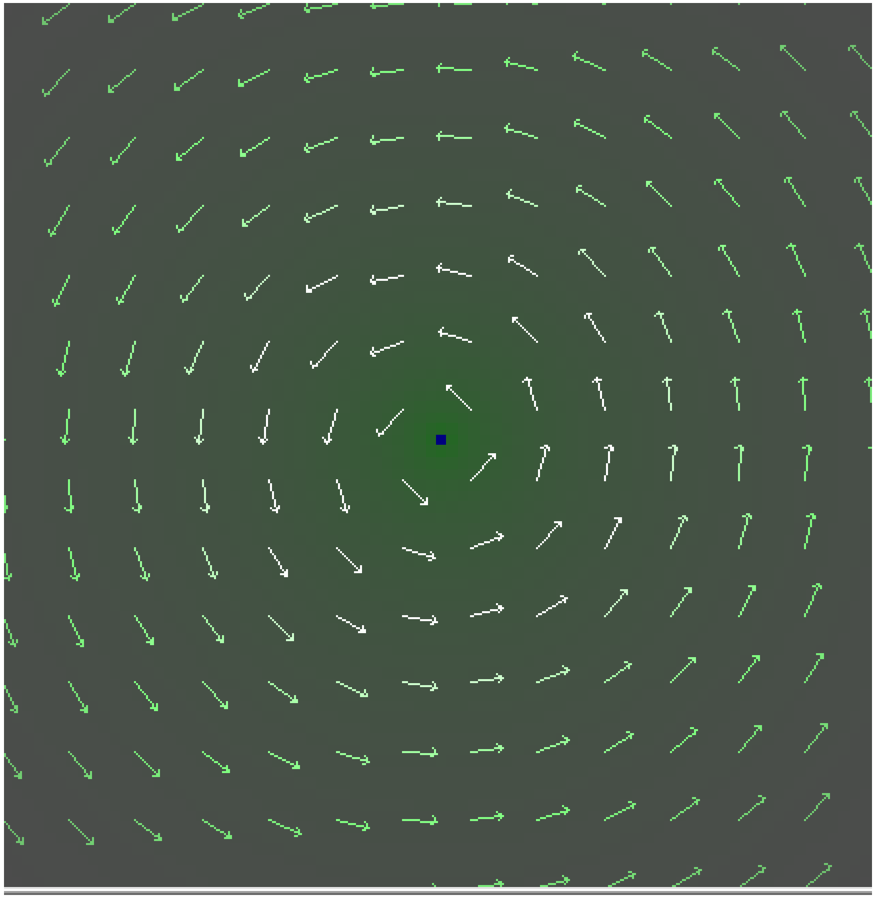
- temperature
-
-

It is important to remember that in a vector field, *every point in space has a vector associated with it*



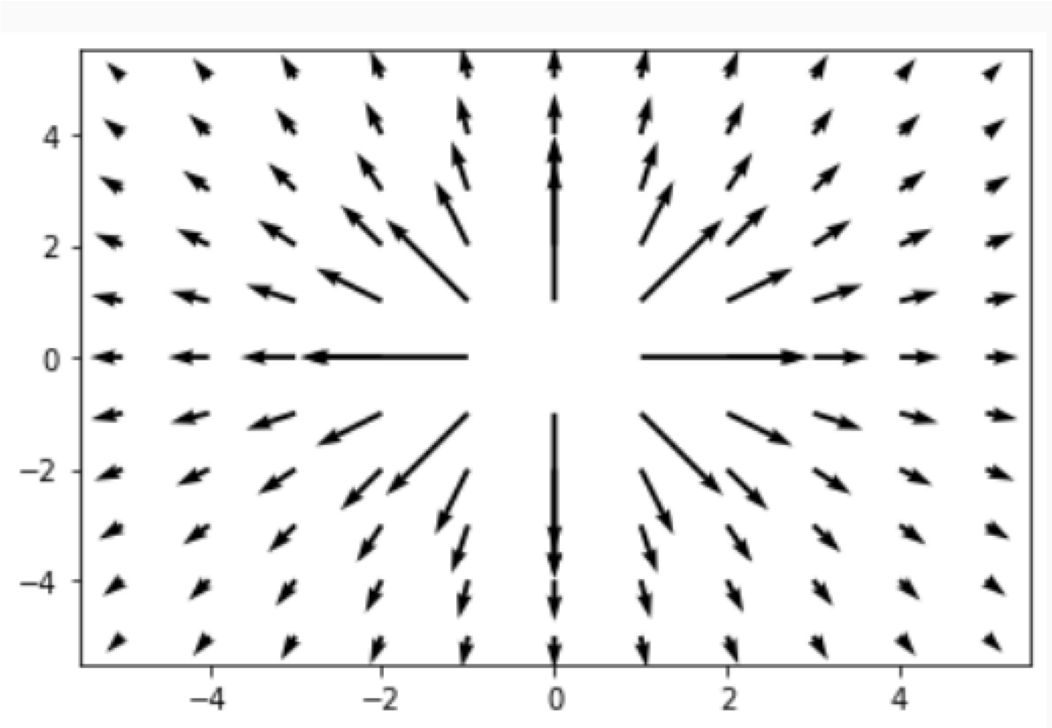
Alternatively, we can think of a *particular* vector in a field as being “anchored” to a point.

Vector fields can behave in some commonly-occurring ways.
First, they can “rotate”, or “curl” around themselves:

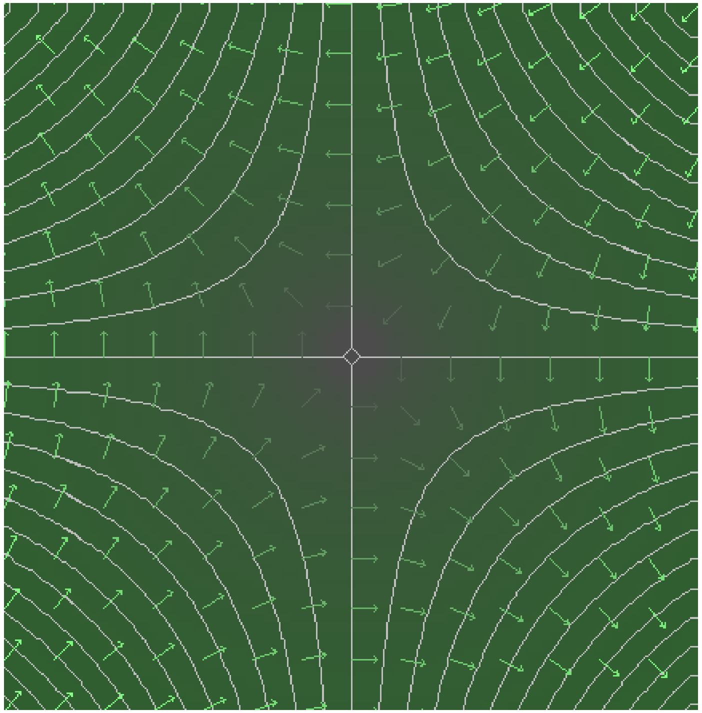


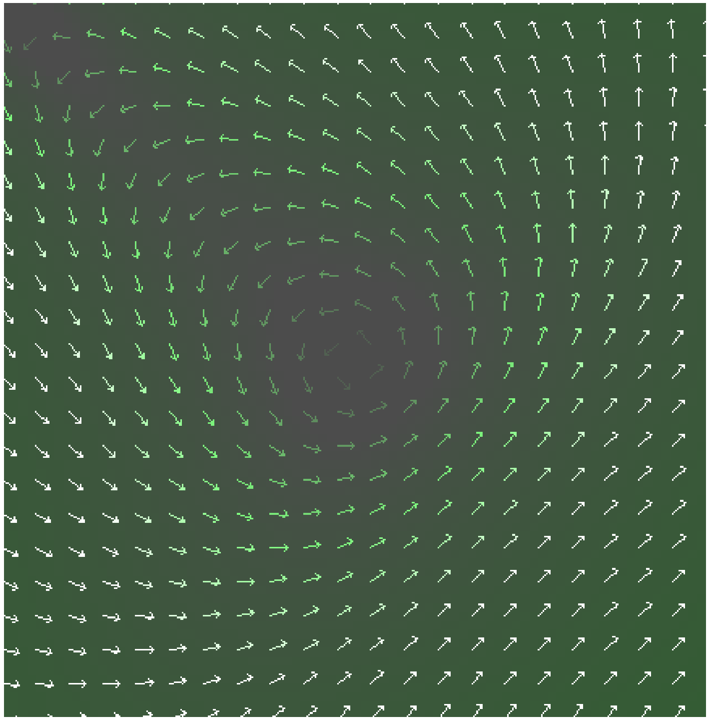
Note that while each vector is stationary, if we *follow the field around* then the direction of the vector changes.

Or they can *diverge* out from a point:



Usually it's a mixture of the two:





Having a mathematical formulation for the rotation ("curl") and the divergence ("div") of a vector field is a key goal of this lecture.

We will also need to describe a new vector field that describes the "vector slope", or "gradient" of scalar fields

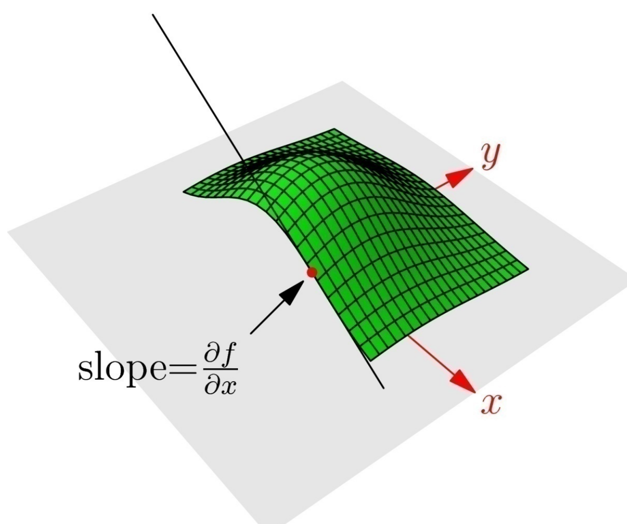
grad

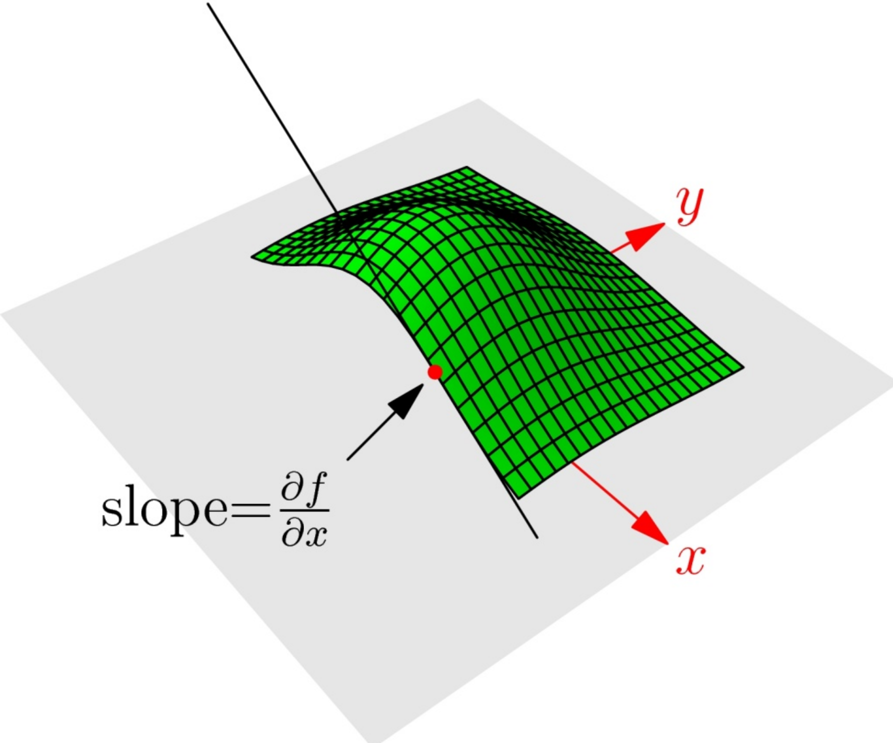
$$\nabla \phi$$

First Recall: Given a function of two variables f(x,y) *the partial derivative with respect to x* is the derivative when *y* is held constant:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

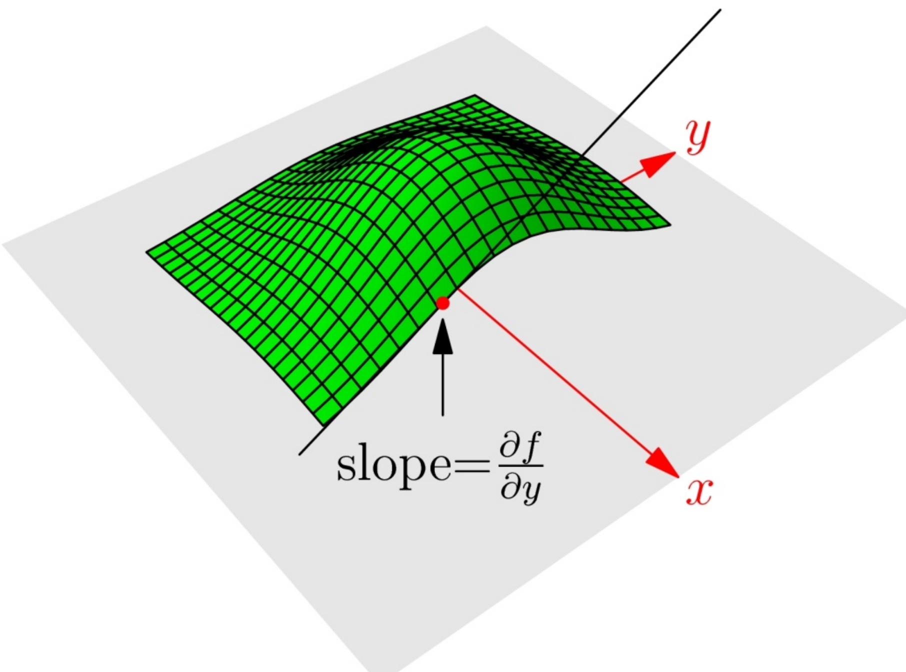
We can think of this quantity as being *the slope in the direction of x.*

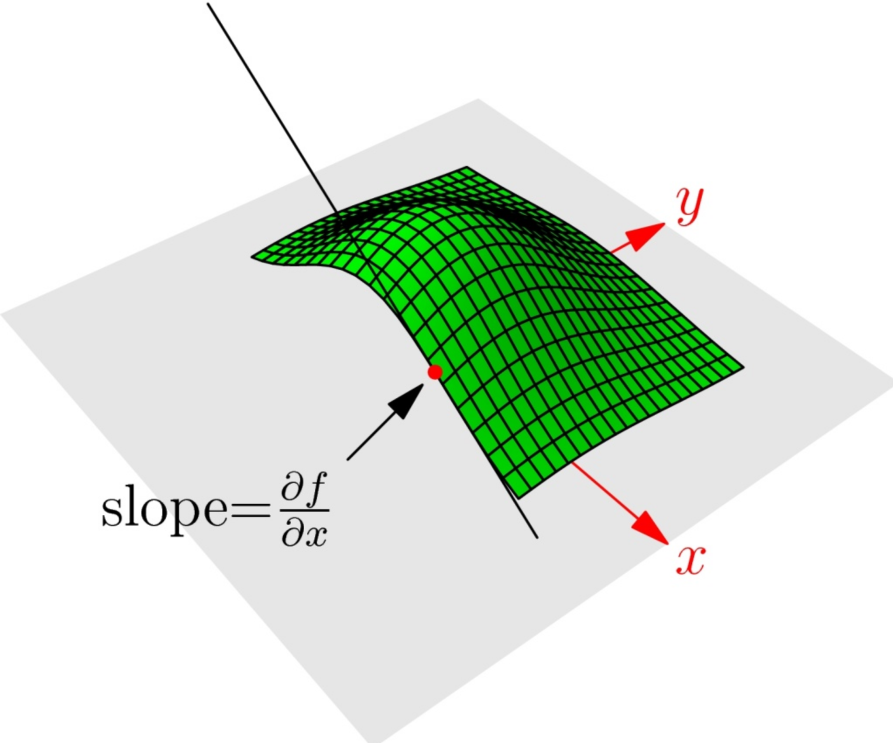


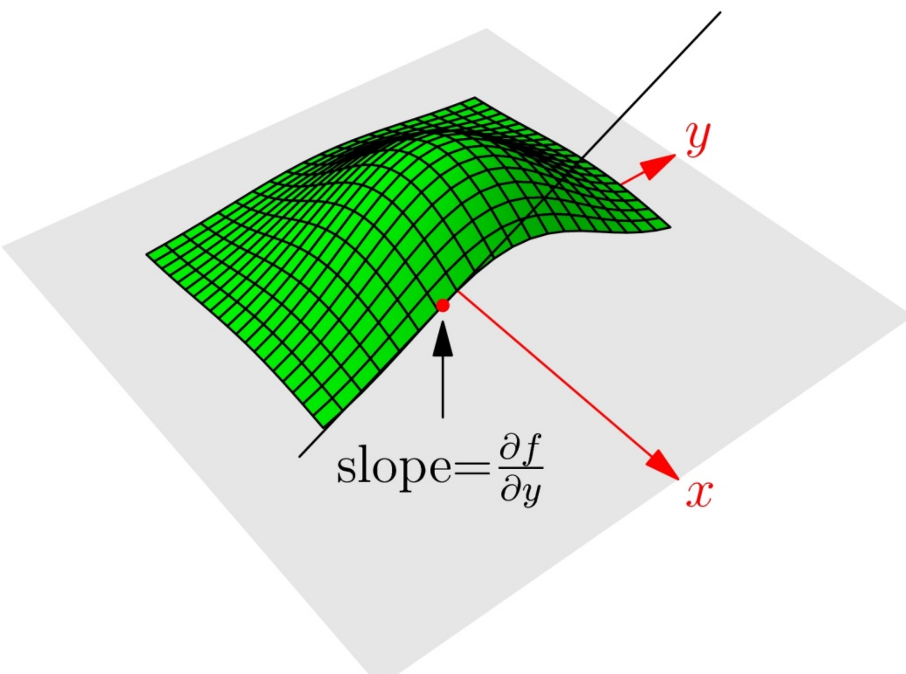


We can use this idea to define a new “vector slope“:

$$\left\langle \frac{\partial f}{\partial x}, \qquad \right\rangle$$







We can use this idea to define a new “vector slope“:

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Could also be written

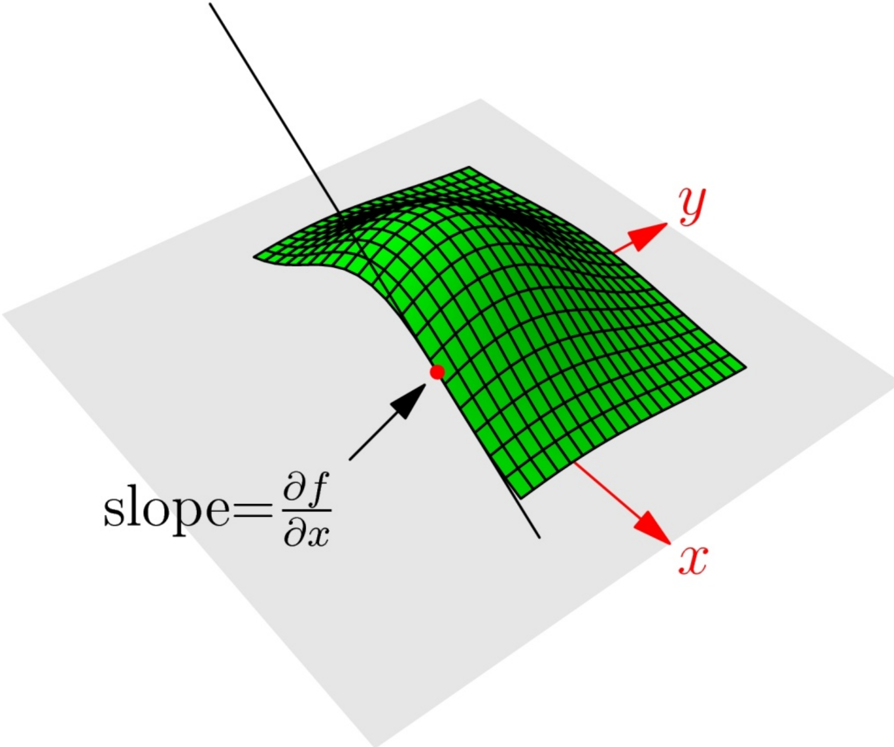
$$\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

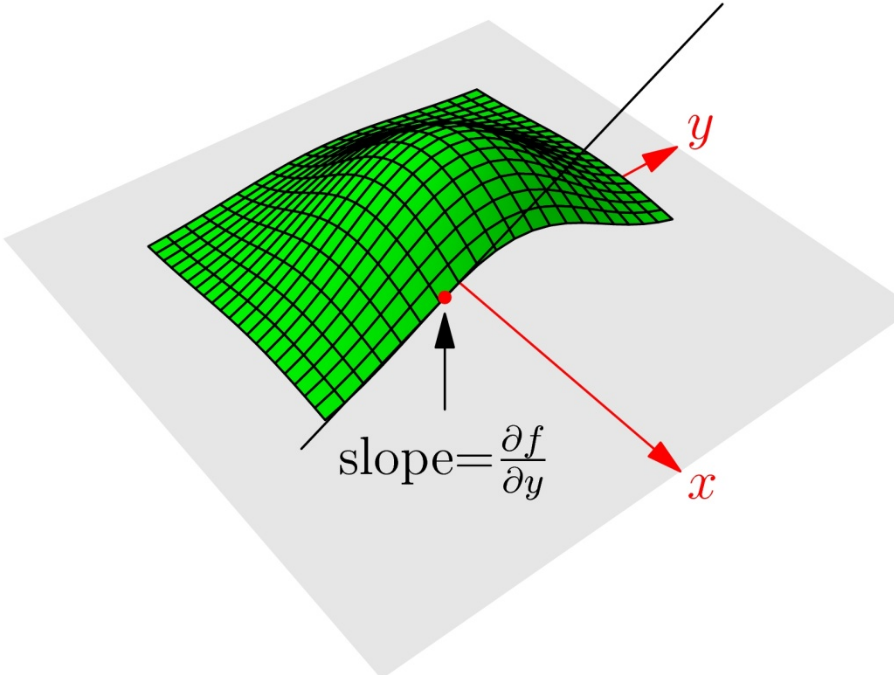
The gradient

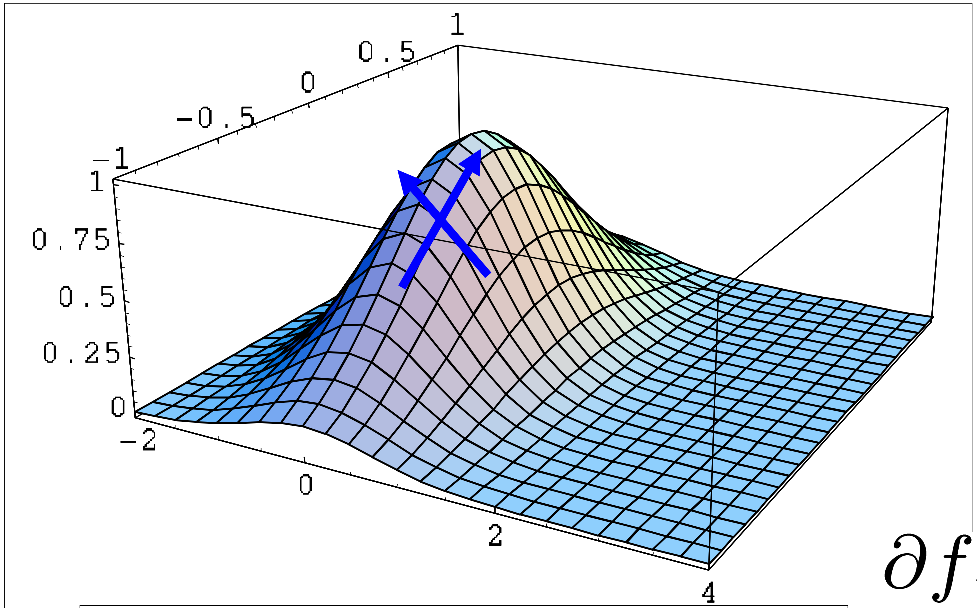
The gradient is a vector field that tells us *the direction of maximum increase of a function.*

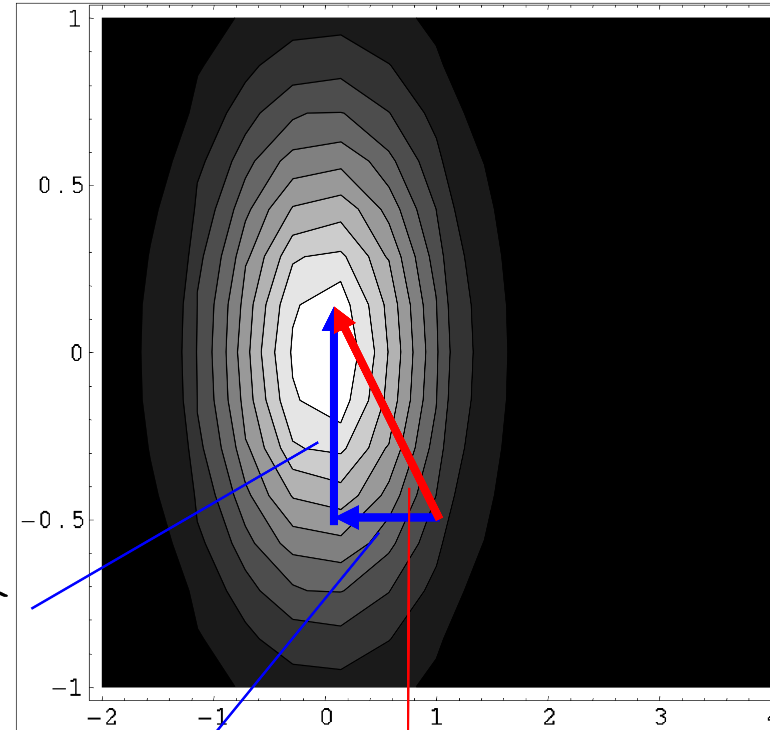
In 2D, the gradient of a scalar function f is defined as

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$





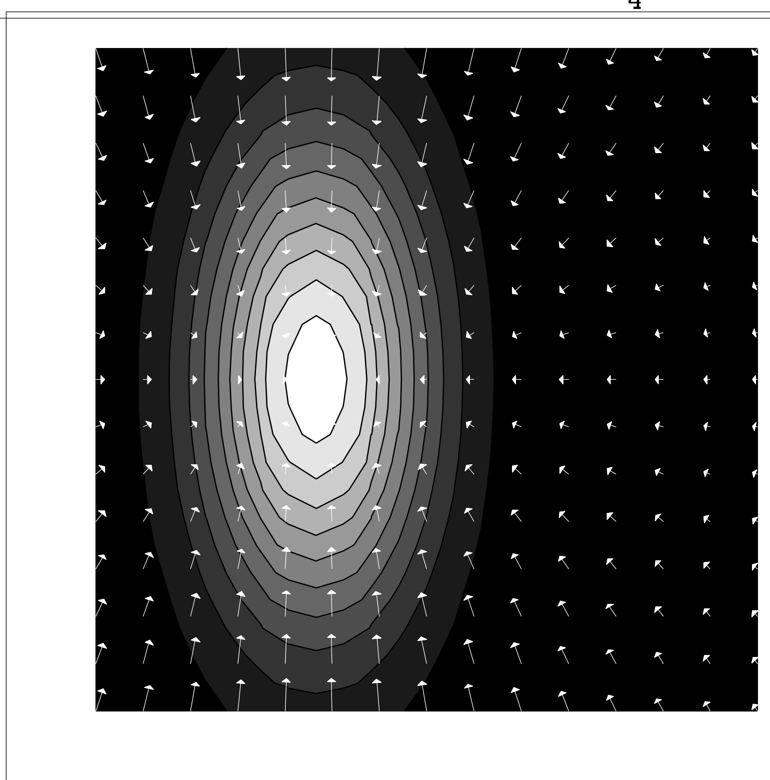




$$\frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

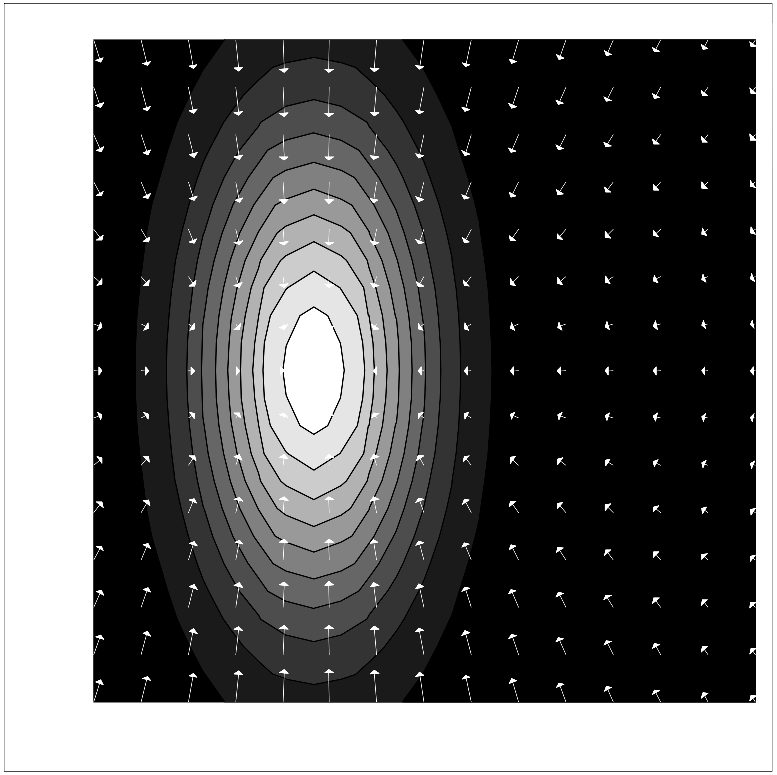
$$\frac{\partial f}{\partial x}\hat{\mathbf{i}}$$

$$\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$



The <u>gradient</u> of a 3D scalar function f(x,y,z) is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$



The gradient is a *vector* that points in the direction of maximum increase of f

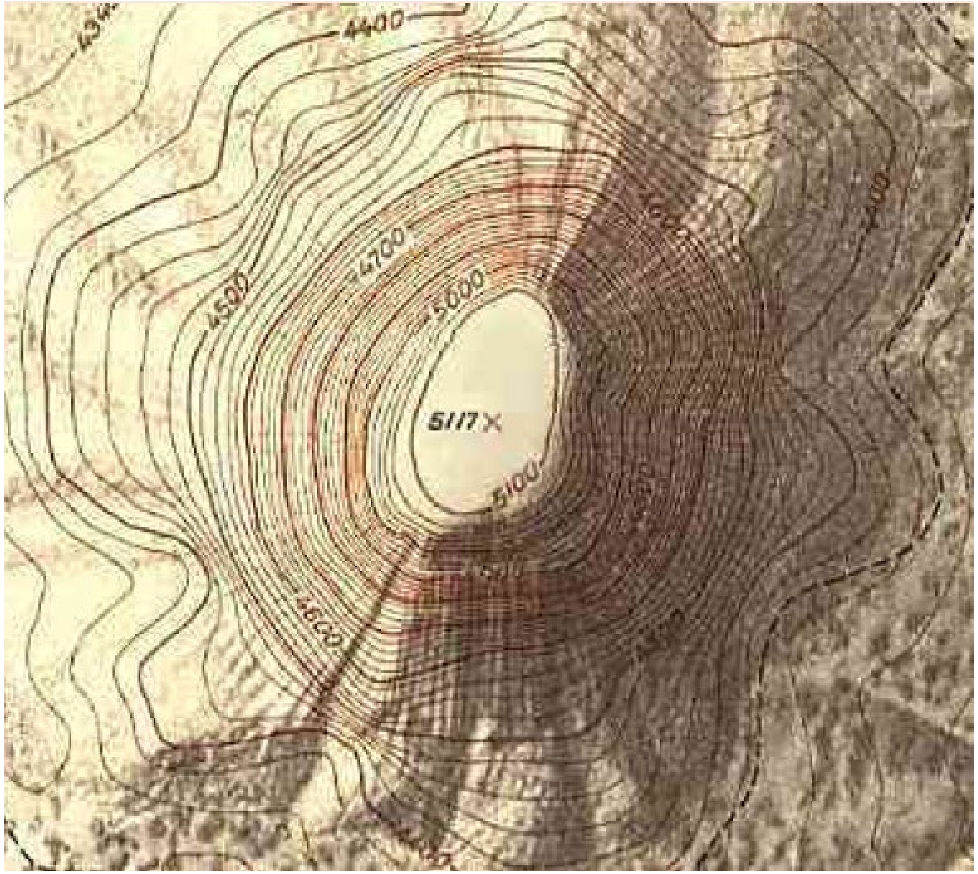
The symbol r is a differential operator that *acts on* the scalar function f.

$$\nabla := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

For a 2D function f(x,y), the gradient is the 2D vector

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

and always points "uphill".



Note that the gradient is always perpendicular to the level curves, which represent the lines of constant f.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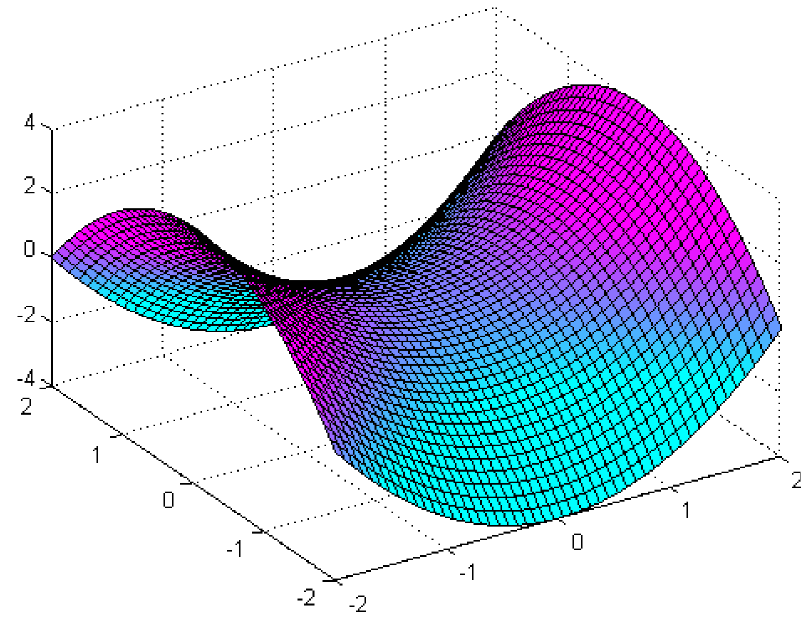
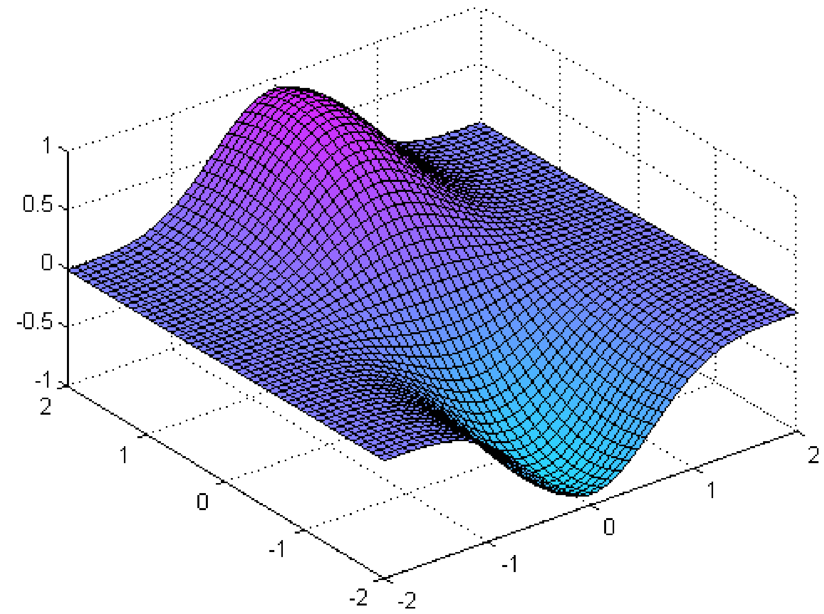
E.g. Calculate the gradient of

$$f(x, y) = x^2 - y^2$$



E.g. Calculate the gradient of

$$f(x, y) = e^{-x^2} \sin y$$

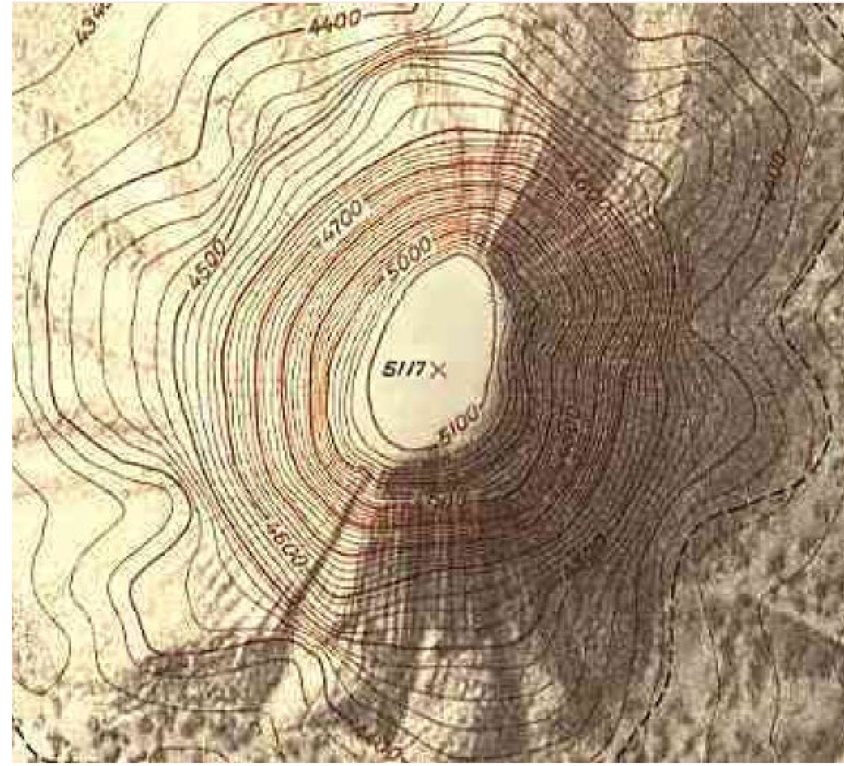


The gradient gives us a way to find the slope in *any direction.*

We define the *Directional Derivative* of a function *f* in the direction of **u** as

$$D_{\mathbf{u}} = \nabla f \cdot \hat{\mathbf{u}}$$

where **u** is a *unit vector.*



Because of how the dot product works, we find that:

1. the directional derivative is *zero* perpendicular to the gradient.
2. The slope is *maximum* in the direction of the gradient

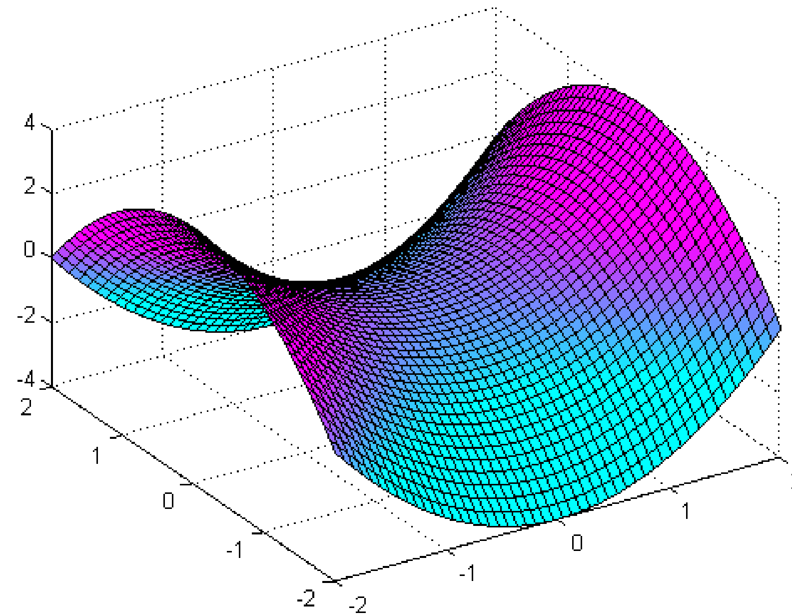
E.g. Calculate the slope of the function

$$f(x,y) = x^2 - y^2$$

in the direction of the vector

$$v = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

at the point (1,1).



The “del” operator

We define the differential operator r as

$$
\begin{aligned}
\nabla &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \rangle \\
&= \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}
\end{aligned}
$$

The gradient is given by “Multiplying” this operator by a scalar function $f(x,y,z)$

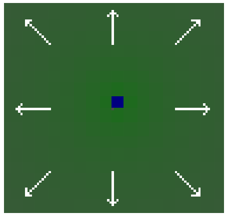
div

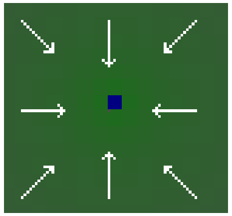
$$\nabla \cdot \mathbf{v}$$

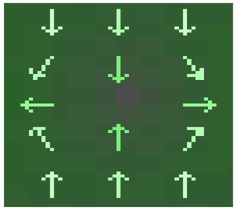
The divergence

Taking the dot product of r with a vector field $\mathbf{v}(x,y,z)$ gives the divergence of the vector field:

$$\nabla \cdot \mathbf{v} \quad = \quad \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \rangle \cdot \langle v_x, v_y, v_z \rangle$$



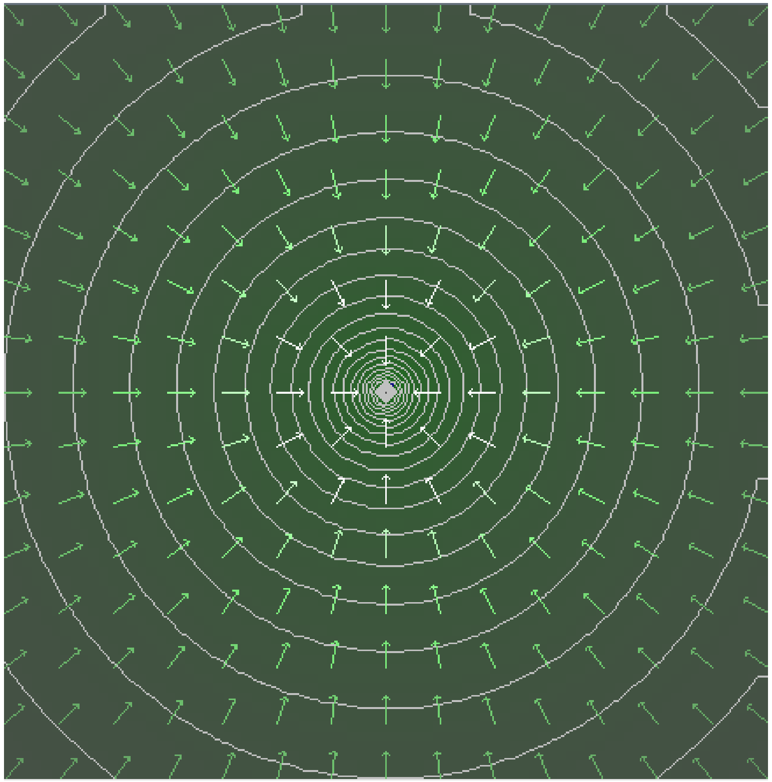


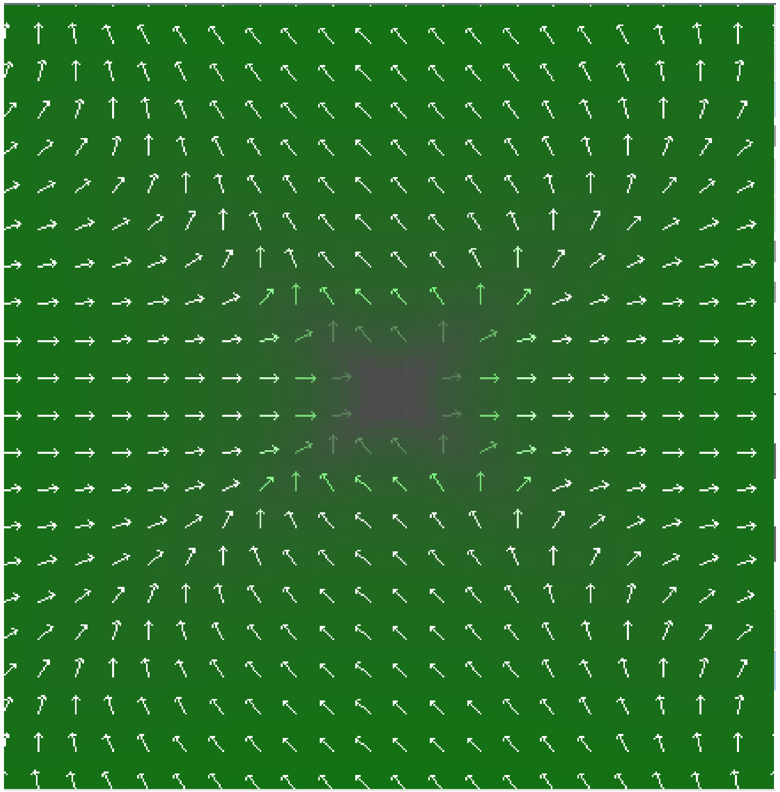


The divergence is a *scalar field.*

__What does the divergence mean?__

The divergence gives the amount that a vector field is “spreading out” at a particular point.



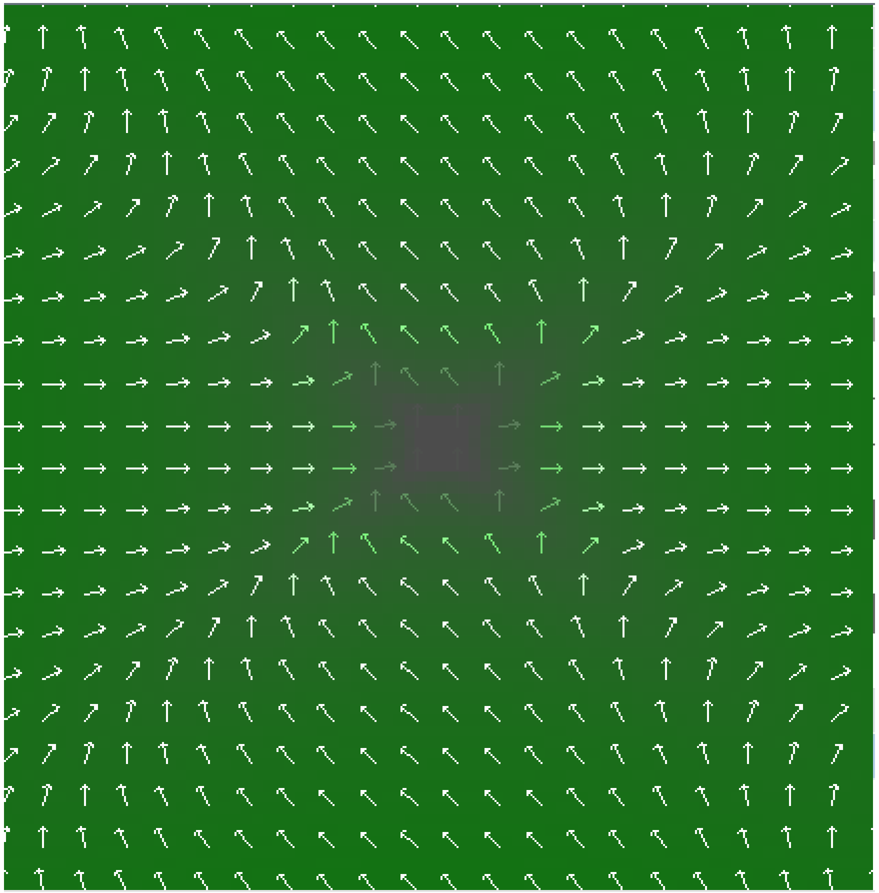


Points with positive divergence are known as __sources__.

Points with negative divergence are known as __sinks__.

Example: Find the divergence of

$$\mathbf{F}(x,y,z) = (x^2 - y^2)\hat{\mathbf{i}} + (y^2 - z^2)\hat{\mathbf{j}} + (z^2 - x^2)\hat{\mathbf{k}}\ .$$



curl

$$\nabla \times \mathbf{v}$$

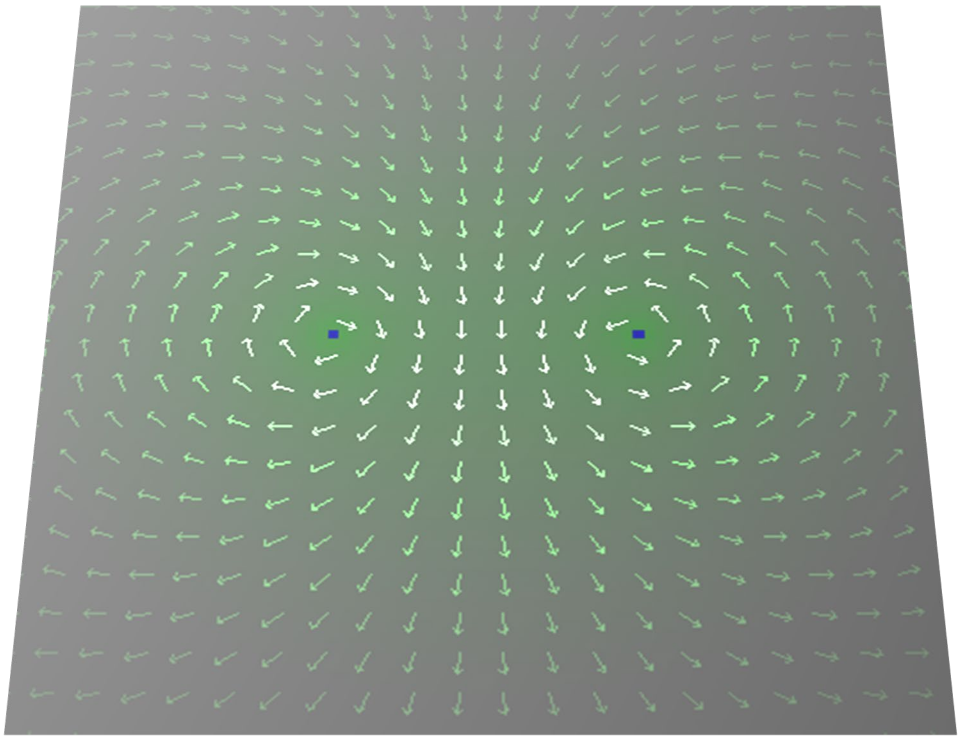
The curl

Taking the cross product of r with a vector field $\mathbf{v}(x,y,z)$ gives the curl of the vector field:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

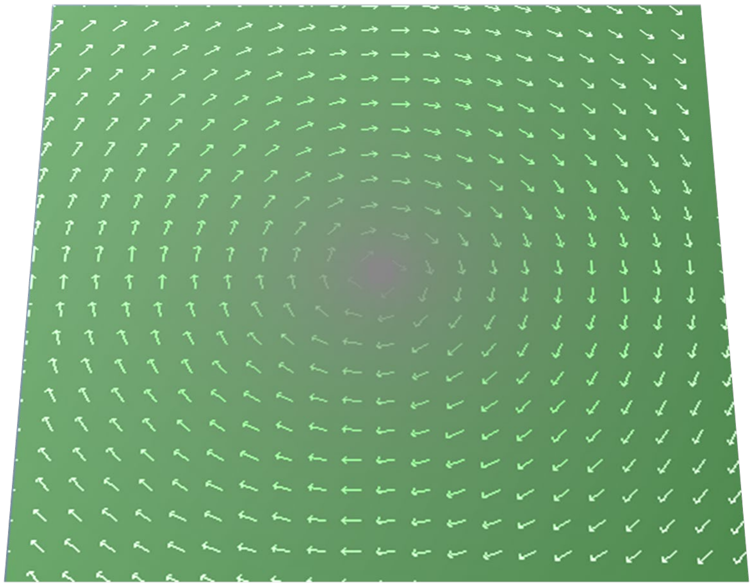
<u>What does the curl mean?</u>

The curl gives the amount that a vector field is rotating at each point,
as well as the direction in which it is rotating.



Example: find the curl of

$$\mathbf{v} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$



Example: Find the divergence and curl of

$$\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Example: Find the divergence and curl of

$$\mathbf{v} = x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + 3z^2\hat{\mathbf{k}}$$

Vector second derivatives

The *Laplacian operator* is formed by taking the divergence of a gradient:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

Heat/Diffusion equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Wave equation

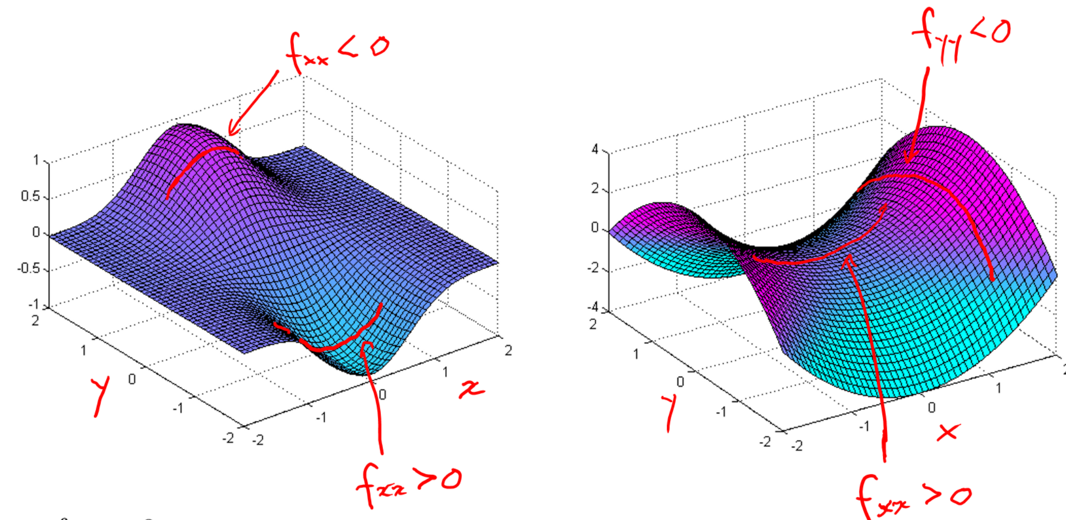
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y,z)\psi = E\psi$$

Schrodinger's equation

What does the Laplacian of a function mean?

Recall:

We can think of the second-order partial derivatives as representing *degree of curvature* in each of the different directions.



$f_{xx} > 0$ $\qquad$ $f_{yy} > 0$

$f_{xx} < 0$ $\qquad$ $f_{yy} < 0$

From the definition

$$\nabla^2 f = f_{xx} + f_{yy}$$

the Laplacian combines the curvatures in both directions, and so is a kind of *average curvature* at a point.

Other important 2nd derivatives:

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

A useful thing if you're stuck:
google: "vector calculus identities",
then click on images

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, provided $g \neq 0$.
5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times (\nabla f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$